Universal finite-size scaling functions for critical systems with tilted boundary conditions

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We calculate finite-size scaling functions (FSSF's) of Binder parameter g and magnetization distribution function $p(m)$ for the Ising model on $L_1 \times L_2$ square lattices with periodic boundary conditions in the horizontal L_1 direction and tilted boundary conditions in the vertical L_2 direction such that the ith site in the first row is connected with the mod$(i+cL_1,L_1)$th site in the L_2 row of the lattice, where $1 \leqslant i \leqslant L_1$. For fixed sets of (a,c) with $a=L_1/L_2$, the FSSF's of g and $p(m)$ are universal and in such cases $a/(c^2a^2+1)$ is an invariant. For percolation on lattices with fixed a, the FSSF of the existence probability (also called spanning probability) is not affected by c. [S1063-651X(99)13802-9]

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Finite-size scaling has been an active research subject in recent decades [1–15]. The quantities that have been analyzed by finite-size scaling include magnetization m [2,11,15], Binder parameter g [6], existence probability E_p [8–10] (also called crossing probability, see, e.g., [16]), percolation probability P [8–10], distribution of magnetization $p(m)$ [11], probability for the appearance of n percolating clusters W_n [12–14], and others. It has been found that the finite-size scaling functions (FSSF's) depend sensitively on aspect ratio and boundary conditions of the system [7–9,12] and by using appropriate aspect ratios [16] and nonuniversal metric factors [2], one may obtain universal finite-size scaling functions (UFSSF's) for percolation models [10,13,14] and Ising models [11,15] in some spatial dimensions. Using Monte Carlo methods, in this paper we calculate FSSF's for g and $p(m)$ for the Ising model on $L_1 \times L_2$ square (sq) lattices with periodic boundary conditions (pbc's) in the horizontal L_1 direction and tilted boundary conditions (tbc's) in the vertical L_2 direction such that the ith site in the first row is connected with the mod$(i+cL_1,L_1)$th site in the L_2 row of the lattice, where $1 \leqslant i \leqslant L_1$ [17]; see Fig. 1 for an example. We find that the FSSF's of g and $p(m)$ are universal for fixed sets of aspect ratio $a=L_1/L_2$ and tilt parameter c, and in such cases $a/(c^2a^2+1)$ is shown to be invariant. For percolation on lattices with fixed a, the FSSF of the existence probability is not affected by c.

We use the metropolis Monte Carlo simulation method [18] to simulate the Ising model on $L_1 \times L_2$ sq lattices with different values of L_1, L_2, and tilt parameter c. For each system, we calculate the magnetization distribution function $p(m)$ at T_c and the Binder parameter g near the critical temperature T_c, where

$$g=\frac{1}{2}\left(3-\frac{\langle m^4\rangle}{\langle m^2\rangle^2}\right). \tag{1}$$

Typical results for g as a function of $(T-T_c)L^{1/\nu}$ and $p(m)L^{-\beta/\nu}$ as a function of $mL^{\beta/\nu}$ are presented in Figs. 2(a) and 2(b) which show that g and $p(m)$ have very good finite-size scaling behavior. Moreover, FSSF's of g and $p(m)$ depend strongly on the tilt parameter c. Here, $L=(L_1L_2)^{1/2}$; $\nu=1$ and $\beta=1/8$ [19].

In Figs. 3(a) and 3(b), we show the data for g and $p(m)$ for (a,c) of (5,0.1), (4,0), (5,0.4), and (1,0) together showing that the pair (5,0.1) and (4,0) and the pair (5,0.4) and (1,0) share UFSSF's. These pairs are just examples. There are many such combinations that share UFSSF's.

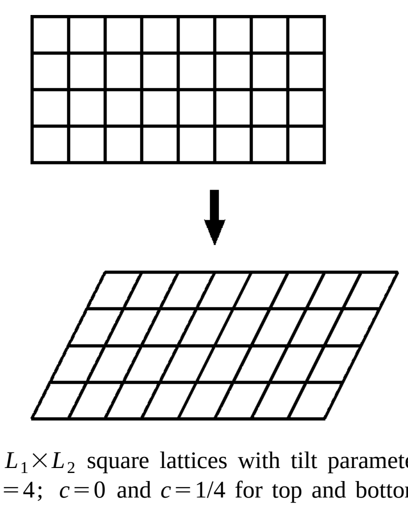

FIG. 1. $L_1 \times L_2$ square lattices with tilt parameter c. Here $L_1=8$ and $L_2=4$; $c=0$ and $c=1/4$ for top and bottom lattices, respectively. Note that in the top lattice the ith site, $1 \leqslant i \leqslant L_1$, of the first row is identical to the ith site in the last row and in the bottom lattice the ith site of the first row is identical to the mod$(i+cL_1,L_1)$th site in the last row. In both lattices, the leftmost site and the rightmost site on the same horizontal line are identical.

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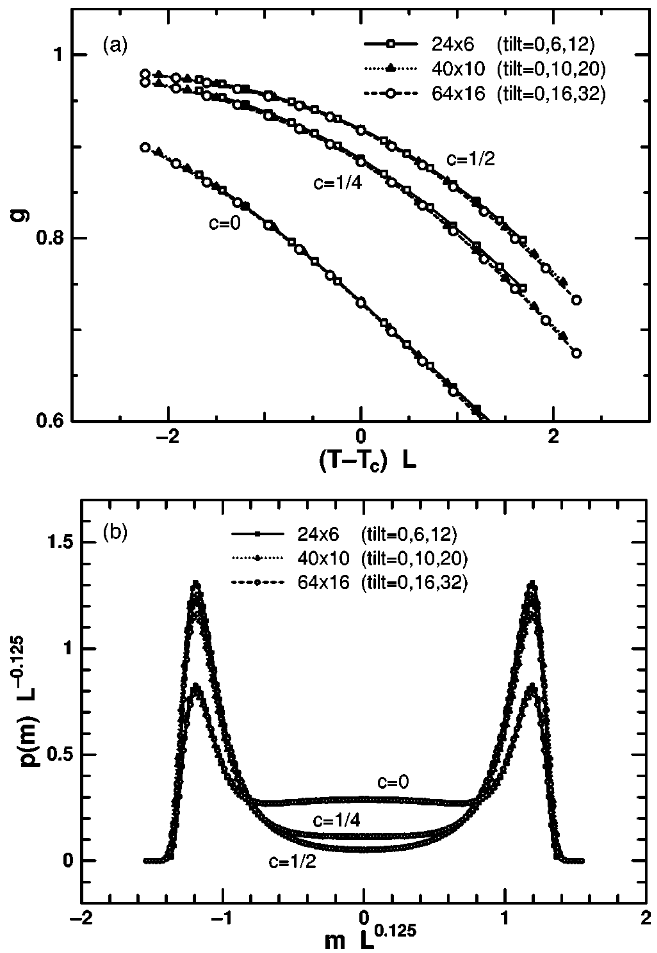

FIG. 2. (a) g as a function of $(T-T_c)L$ for several lattices with $a=4$ and $c=0$, 1/4, and 1/2. (b) $p(m)L^{-1/8}$ as a function of $mL^{1/8}$ for several lattices with $a=4$ and $c=0$, 1/4, and 1/2.

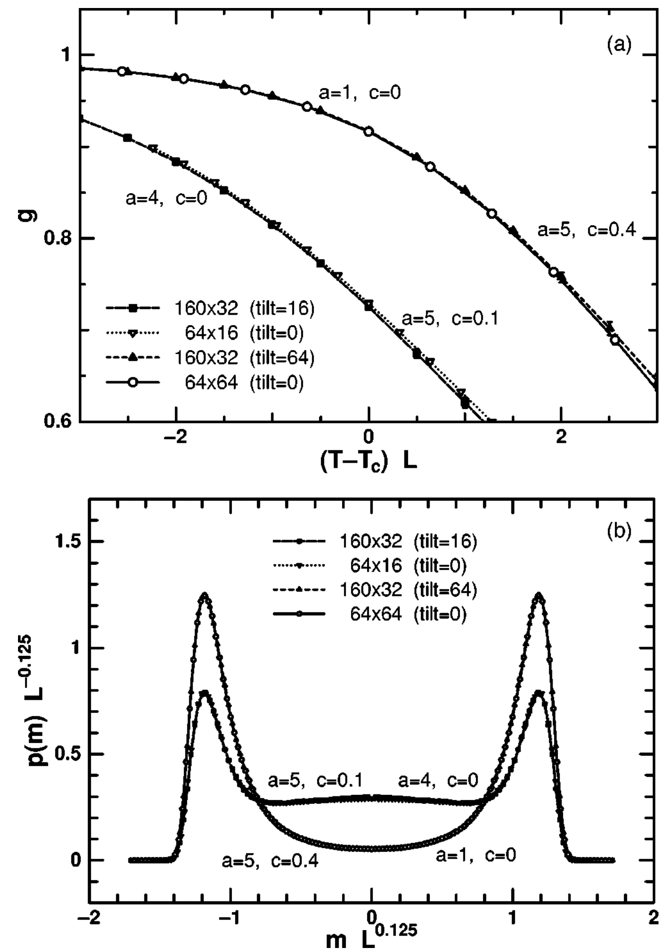

FIG. 3. (a) g as a function of $(T-T_c)L$ for $(a,c)=(5,0.1)$, $(4,0)$, $(5,0.4)$, and $(1,0)$. (b) $p(m)L^{-1/8}$ as a function of $mL^{1/8}$ for $(a,c)=(5,0.1)$, $(4,0)$, $(5,0.4)$, and $(1,0)$.

To understand the finite-size scaling behavior shown in Fig. 3, it is convenient to consider UFSSF's in the momentum space. Let $f(x,y)$ denote a quantity, e.g., the local magnetization, which depends on lattice coordinates x and y. The pbc in the horizontal direction and tbc with tilt parameter c in the vertical direction imply that

$$f(x+L_1,y)=f(x,y), \quad f(x+cL_1,y+L_2)=f(x,y), \tag{2}$$

where $x=0,\dots,L_1-1$, and $y=0,\dots,L_2-1$. By Fourier expansion, we have

$$f(x,y)=\sum_{k_x}\sum_{k_y} e^{ik_x x}e^{ik_y y}\hat{f}(k_x,k_y). \tag{3}$$

Equations (2) and (3) imply that $k_x=2\pi p/L_1$ and $k_y=2\pi(q-cp)/L_2$, where $p=0,\dots,L_1-1$, and $q=0,\dots,L_2-1$. The possible coordinates of k_x and k_y are plotted in Fig. 4(a), which shows that the lattice constant in the k_x direction is a times smaller than the lattice constant in the k_y direction, and the distance between the second point in the bottom line and the k_x axis is c times the lattice constant in the k_y direction, thus the values of a and c may be read from the figure. The two primitive vectors in the momentum space are then $\mathbf{k}_1=(2\pi/L_1,-c2\pi/L_2)$ and $\mathbf{k}_2=(0,2\pi/L_2)$ and here we consider $a\geqslant 1$, $0.5\geqslant c\geqslant 0$ with

$$|\mathbf{k}_1|\leqslant(k_x^2+k_y^2)^{1/2}, \tag{4}$$

for any (k_x,k_y) in Fig. 4(a).

Next consider a quantity X, which exhibits a critical anomaly. Since the finite-size scaling behavior is dominated by low-momentum excitations, we can regard X at the critical point as depending only on the primitive vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. Then under the scale transformation $\mathbf{k}\rightarrow b\mathbf{k}$, we can put the finite-size scaling *ansatz* for X in the following form:

$$X(\mathbf{k}_1,\mathbf{k}_2)=b^x X(b\mathbf{k}_1,b\mathbf{k}_2), \tag{5}$$

with an appropriate scaling exponent x. It is natural to suppose that the scaling function depends on the absolute values of vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, and the angle between the two vectors, ω. Then we have

$$X(\mathbf{k}_1,\mathbf{k}_2)=b^x X_2(b|\mathbf{k}_1|,b|\mathbf{k}_2|,\omega). \tag{6}$$

If we choose $b=1/|\mathbf{k}_2|$, then we have

$$X(\mathbf{k}_1,\mathbf{k}_2)=|\mathbf{k}_2|^{-x}X_2(|\mathbf{k}_1|/|\mathbf{k}_2|,1,\omega). \tag{7}$$

Since $|\mathbf{k}_2|\propto L_2^{-1}\propto L^{-1}$, we get

$$X(\mathbf{k}_1,\mathbf{k}_2)=L^x\hat{X}(|\mathbf{k}_1|/|\mathbf{k}_2|,\omega). \tag{8}$$

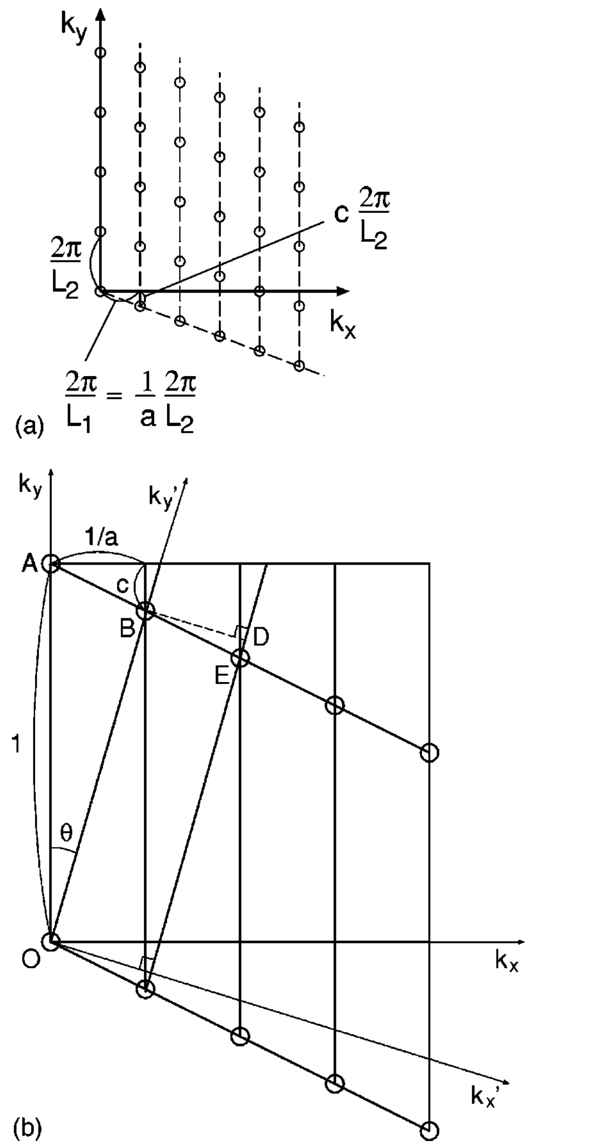

FIG. 4. (a) Possible values of k_x and k_y in (k_x,k_y) space. (b) Transformation from (k_x,k_y) space to (k_x',k_y') space.

It should be noted that the scaling function $\hat{X}$ is a function only of the ratio of the length of the primitive vectors and their relative angle, so that their absolute orientations are irrelevant.

Now we consider a rotation from (k_x,k_y) to (k_x',k_y') as shown in Fig. 4(b), where the unit of measurement is $2\pi/L_2$. The line OB is chosen to be the k_y' axis. The straight line that goes through O and is perpendicular to OB is chosen to be the k_x' axis. The angle between OA and OB is denoted by θ. Since $\overline{OB}=(a\sin\theta)^{-1}$, $\overline{BD}=\sin\theta$, $\overline{DE}=\cos\theta-\overline{OB}$, and $\tan\theta=[a(1-c)]^{-1}$, it is easy to show that for the (k_x',k_y') system, the aspect ratio a' and the tilt parameter c' are given by $a'=\overline{OB}/\overline{BD}=(a\sin^2\theta)^{-1}$, $c'=\overline{DE}/\overline{OB}=1-a\sin\theta\cos\theta$ and we have

$$a'/(c'^2a'^2+1)=a/(c^2a^2+1), \tag{9}$$

i.e., $A=a/(c^2a^2+1)$ is an invariant under transformation from (k_x,k_y) to (k_x',k_y') and can be regarded as the effective aspect ratio. Since the FSSF $\hat{X}$ depends only on geometrical parameters, we conclude that the FSSF's for systems which are related to each other by a rotation in momentum space are identical. It is easy to check that the pairs of (a,c), which have the UFSSF's shown in Fig. 3, satisfy Eqs. (4) and (9) [20].

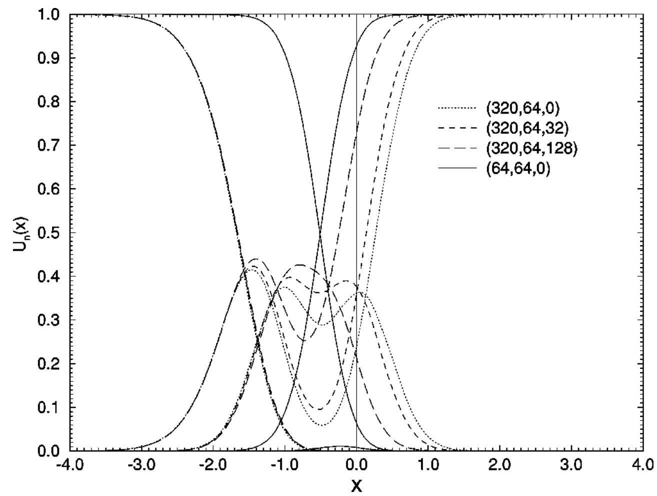

FIG. 5. W_n as a function of $x=(p-p_c)L^{1/\nu}$ for bond percolation on a 320×64 lattice with $c=0$, 0.1, and 0.4 and on a 64×64 lattice with $c=0$. The $L_1\times L_2$ lattice with tilt parameter c is denoted by (L_1,L_2,cL_1). The scaling function of W_n is denoted by U_n. The monotonic decreasing functions are for U_0; on the left data for U_0 for a 320×64 lattice with $c=0$, 0.1, and 0.4 collapse into one curve. The functions that approach 1 for large x are for U_1. Two curves of U_2 for (320,64,0) and (320,64,32) have M shapes.

The results for g and $p(m)$ presented above suggest that for fixed a, when c is increased from 0, the effective aspect ratio of the system decreases. For example, for systems with $a=5$, when c is increased from 0 to 0.4 the effective aspect ratio decreases from 5 to 1. However, such results are not always true for other quantities. In Fig. 5, we plot W_n, the probability of the appearance of n percolating clusters [12,13], as a function of $x=(p-p_c)L^{1/\nu}$ for bond percolation on 320×64 square lattices with c being 0, 0.1, and 0.4 and on 64×64 lattice with $c=0$; $\nu=4/3$ for two dimensional percolation [5]. A cluster is percolating if every horizontal line contains at least one site of that cluster [21]. The scaling function of W_n is denoted by U_n, i.e. $U_n(x)=W_n(p)$ with $x=(p-p_c)L^{1/\nu}$. It should be noted that $E_p=\Sigma_{n=1}^{\infty}W_n$ and $W_0=1-E_p$. Figure 5 shows that for systems with $a=5$, when c increases from 0 to 0.1 and 0.4, $U_1\sim U_2$ depend on c (e.g., as c increases, U_1 increases), whereas U_0 (hence E_p) does not change. Figure 5 also shows that the lattices with (a,c) of (5,0.4) and (1,0), which have UFSSF's for g and $p(m)$ shown in Fig. 3, have quite different results for U_n.

The results of Fig. 5 can be understood as follows. Since values of U_n represent a global property of the system, the arguments from Eq. (2) to Eq. (6) do not apply to U_n and systems that are related to each other by a rotation in the momentum space do not have UFSSF's for U_n. As c increases, two or more percolating clusters in the original system merge into one percolating cluster, thus U_1 increases.

However, increasing c does not affect the percolating property of the systems under the boundary conditions considered in this paper, and thus does not affect E_p.

In this paper, we have discussed UFSSF's for two-dimensional systems with tilted boundary conditions. Conformal invariance plays a role in two-dimensional systems [22], but the argument developed here is a general one. The extension to three-dimensional systems is now in progress. With rapid progress of computing and experimental facilities, the UFSSF's and tilted bc discussed in this paper may be studied for many other physical quantities or critical systems [23]. The related mathematical problems are also of interest to mathematical physicists.

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