Velocity distribution for strings in phase-ordering kinetics

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The continuity equations expressing conservation of string defect charge can be used to find an explicit expression for the string velocity field in terms of the order parameter in the case of an $O(n)$ symmetric time-dependent Ginzburg-Landau model. This expression for the velocity is used to find the string velocity probability distribution in the case of phase-ordering kinetics for a nonconserved order parameter. For long times t after the quench, velocities scale as $t^{-1/2}$. There is a large velocity tail in the distribution corresponding to annihilation of defects which goes as $V^{-(2d+2-n)}$ for both point and string defects in d spatial dimensions. [S1063-651X(99)13002-2]

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I. INTRODUCTION

In recent work [1] we discussed how one could use conservation of topological charge to study the statistics of velocities of point defects in phase-ordering systems. We were able to identify the appropriate point-defect velocity field in terms of the order-parameter field in the context of a d-dimensional $O(n)$ symmetric time-dependent Ginzburg-Landau (TDGL) model. Using this expression for the velocity field, for point particles ($n=d$), the probability distribution for defect velocities was determined in the case of the late-state phase ordering using the lowest-order approximation in the perturbation expansion method developed in Ref. [2]. This analysis is extended here to the case of string defects where $n=d-1$. The velocity probability distribution is worked out explicitly for $n=d-1=1$ and 2 and for the wall case $n=d-2=1$. These results and the results for all $n=d$ can all be written in the form

$$P[\mathbf{V}]=\left(\frac{1}{\pi\overline{v}^2}\right)^{d/2}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{(d-n)}{2}+1\right)}\times\left(\frac{\overline{v}}{V}\right)^{d-n}(1+V^2/\overline{v}^2)^{-(d+2)/2}, \quad (1)$$

where the velocities scale with a factor $\overline{v}\approx L(t)^{-1}$, where $L(t)\approx t^{1/2}$ is the characteristic scaling length which grows with time t after the quench. The result for $P[\mathbf{V}]$ indicates that the probability of finding a defect with a large velocity decreases with time. There is a high-velocity tail $V^{-(2d+2-n)}$ which corresponds to the annihilation of defects and defect loops.

Bray [3] has used scaling arguments to obtain estimates for the exponents governing the large velocity tails in this problem. From his Eqs. (20) and (21) for the case of a nonconserved order parameter ($z=2$ in his notation), one is led to the result, in our notation, $P[\mathbf{V}]\approx 1/V^{p+d-1}$, where $p=2+d+1-n$. The term $d-1$ added to p in the exponent comes about because Bray uses the normalization $\int_0^\infty dV P_{\text{Bray}}[V]=1$, while we use here $\int d^dV P[\mathbf{V}]=1$. Thus we find the encouraging result that Bray obtains the same tail exponent $2d+2-n$ as obtained here. A difference is that he finds these results for all $n\leqslant d$. Our results are restricted to the set described above.

The result obtained here for $P[\mathbf{V}]$ seems very simple. Does it correspond to experimental observation or the results of numerical simulations? Thus far there have been no direct tests. It would seem worthwhile to check the range of validity of the defect velocity probability distribution given by Eq. (1).

II. PROBLEM SETUP

We study an n-component nonconserved order-parameter field $\psi_\alpha(\mathbf{R},t)$ in d-spatial dimensions which satisfies the TDGL equation

$$\partial_t\psi_\alpha(\mathbf{R},t)=-\Gamma\,\frac{\delta F[\vec{\psi}]}{\delta\psi_\alpha(\mathbf{R},t)}+\eta_\alpha(\mathbf{R},t), \quad (2)$$

where F is an effective free-energy functional and Γ is a constant kinetic coefficient. We assume F is of the $O(n)$ symmetric square-gradient form

$$F=\int d^dR\left[\frac{c}{2}(\nabla\vec{\psi})^2+V(\vec{\psi})\right], \quad (3)$$

where $c>0$ and $V(\vec{\psi})$ is chosen to be a degenerate double-well or wine-bottle potential. This model is to be supplemented by random, uncorrelated, initial conditions. We assume that there is a rapid temperature quench from a high temperature to zero temperature where the noise η_α in Eq. (2) can be set to zero. In the scalar case ($n=1$) such systems order through the growth of domains separated by sharp walls. As time evolves these domains coarsen and order grows to progressively longer length scales. In the case of systems with continuous symmetry ($n>1$) the disordering elements [4,5] will depend on n and spatial dimensionality d. Thus, for example, for $n=d$ one has point defects (vortices or monopoles) while for $n=d-1$ one has vortex lines or stringlike objects. For $n>d$ there are no stable singular topological objects.

The main physics in phase-ordering systems [6–8] is the interplay between two characteristic lengths, a characteristic domain size $L(t)$, which grows with time, and a defect dimension ξ (interfacial width, vortex core size, etc.). However, at long enough times the single length $L(t)$ dominates, $L(t)\gg\xi$, the morphological structure looks self-similar under the rescaling of space and time, and the order-parameter correlation function satisfies the scaling equation

$$C_{\psi}(\mathbf{R},t)\equiv\langle\vec{\psi}(\mathbf{R},t)\cdot\vec{\psi}(0,t)\rangle=\psi_0^2F(x), \tag{4}$$

where $x\equiv R/L(t)$ and ψ_0 is the magnitude of $\vec{\psi}$ in the ordered state. The structure factor, the Fourier transform of $C_{\psi}(\mathbf{R},t)$, satisfies $\tilde{C}_{\psi}(\mathbf{q},t)=L^d\psi_0^2\tilde{F}(Q)$, where $Q\equiv qL$ is a scaled wave number. For pure systems with short-range interactions and a nonconserved order parameter the growth law is given by the Lifshitz-Cahn-Allen result $L\approx t^{1/2}$ for all n. The two-dimensional XY model [9–13] and the one-dimensional scalar model [14,15] appear to be interesting exceptions. For large Q and $n\leqslant d$, due to defects, the structure factor obeys the generalized Porod's law [16–20], $\tilde{F}(Q)\sim Q^{-(n+d)}$. This reflects increasingly weaker singularities in $F(x)$ for small x as a function of n. In the opposite limit, it appears that the large x behavior can, with proper definition of x, be put in the form $F(x)\approx x^{-\nu}e^{-(1/2)x^2}$, where ν is a subdominant index [2]. The quantities discussed above are evaluated at equal times after the quench. In the two-time case one again has a scaling law [21–23] and the on-site correlation function has the form

$$\langle\vec{\psi}(\mathbf{R},\tau+t)\cdot\vec{\psi}(\mathbf{R},t)\rangle\approx L(\tau)^{-\lambda} \tag{5}$$

for $\tau\gg t$, where λ is a nontrivial exponent which has been determined numerically and theoretically [2] for a number of systems.

III. DEFECT DYNAMICS FOR POINT PARTICLES

Since a great deal is known about order-parameter correlations in phase-ordering systems, attention has turned toward the study [24,25] of the statistics and dynamics of the annihilating defects themselves. The basic idea is that the positions of defects are located by the zeros of the order-parameter field $\vec{\psi}$, therefore the charged or signed density for point defects is given by

$$\rho(\mathbf{R},t)=\delta(\vec{\psi}(\mathbf{R},t))\mathcal{D}(\mathbf{R},t), \tag{6}$$

where $\mathcal{D}$, associated with the change of variables from the set of vortex positions to the field $\vec{\psi}$, is defined by

$$\mathcal{D}=\frac{1}{n!}\epsilon_{\mu_1\mu_2\cdots\mu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\nabla_{\mu_1}\psi_{\nu_1}\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}, \tag{7}$$

where $\epsilon_{\mu_1\mu_2\cdots\mu_n}$ is the n-dimensional fully antisymmetric tensor and summation over repeated indices is implied here and below. The unsigned defect density, $n(\mathbf{R},t)$, is given by

$$n(\mathbf{R},t)=\delta(\vec{\psi}(\mathbf{R},t))|\mathcal{D}(\mathbf{R},t)|. \tag{8}$$

It was shown in Ref. [1] that the vortex charge density ρ for point defects, defined by Eq. (6), satisfies a continuity equation of the form

$$\dot{\rho}=-\nabla_{\mu_1}[\rho v_{\mu_1}], \tag{9}$$

where the defect velocity field v_{μ_1} is given by

$$\mathcal{D}v_{\mu_1}=-\frac{1}{(n-1)!}\epsilon_{\mu_1\mu_2\cdots\mu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\times\dot{\psi}_{\nu_1}\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}, \tag{10}$$

where $\mathcal{D}$ is defined by Eq. (7). Thus one has an explicit expression for the defect velocity field which can be expressed strictly in terms of the order parameter and its spatial derivatives. Remember that the TDGL equation of motion can be used to express $\dot{\psi}_{\nu_1}$ in terms of ψ and it spatial derivatives.

The expression given by Eq. (10) for the velocity is very useful because it avoids the problem of having to specify the positions of the defects explicitly. The positions are implicitly determined by the zeros of the order-parameter field. The practical usefulness of Eq. (10) can be seen by asking the following question: In the scaling regime of a phase-ordering system with point defects, what is the probability of finding a defect with a velocity $\mathbf{V}$? This probability distribution function is defined by

$$\langle n\rangle P[\mathbf{V}]\equiv\langle n\,\delta(\mathbf{V}-\mathbf{v}(\vec{\psi}))\rangle, \tag{11}$$

where $\mathbf{V}$ is a reference velocity and n is the unsigned defect density defined by Eq. (8). The calculated $P[\mathbf{V}]$ is given by Eq. (1) with $n=d$.

Going further along these lines [26] we considered the two-vortex velocity probability distribution, $P[\mathbf{V}_1,\mathbf{V}_2,\mathbf{R}]$, which gives one the probability of finding the velocity of one defect in the fixed presence of another defect a known distance away with a known velocity. Clearly $P[\mathbf{V}_1,\mathbf{V}_2,\mathbf{R}]$ contains a tremendous amount of information about the dynamics of point defects. The physical results from the calculation of this quantity, carried out in detail for $n=d=2$ in Ref. [26], are relatively simple to state. The probability distribution is a function only of the scaled velocities $\vec{u}_i=\vec{V}_i/\bar{v}$ for $i=1$ or 2, and the scaled separation $\vec{x}=\vec{R}/L(t)$. The characteristic velocity $\bar{v}$ is the same quantity that appears in $P[\mathbf{V}]$. For a given scaled separation x, the most probable configuration corresponds, as expected, to a state with zero total velocity and a nonzero relative velocity only along the axis connecting the vortices: $\mathbf{V}_1=-\mathbf{V}_2\equiv v\hat{x}$. Moreover, there is a definite most probable nonzero value for $v=v_{\max}$ for a given value of x. The most striking feature of these results is that for small x the most probable velocity goes as $v_{\max}=\kappa/R$, where R is the unscaled separation between the vortices and $\kappa=2.19$ in dimensionless units defined in Ref. [26]. The result giving $v_{\max}$ inversely proportional to R is consistent with overdamped dynamics where the relative velocity of the two vortices is proportional to the force which in turn is the derivative of a potential which is logarithmic in the separation distance. Since there is low probability [24,27] of finding like-signed vortices at short distances, our results giving the velocity as a function of separation distance should be interpreted in terms of annihilating vortex-antivortex pairs and is in agreement with the general scaling ideas proposed by Bray [3].

IV. CONTINUITY EQUATIONS

Let us investigate the existence of a local statement of topological charge in the general case of $n \leqslant d$. The first step is to introduce the appropriate density for topological charge. In the case of point particles the conserved density is the charge density given by Eq. (6). The next obvious extension [28] is to string defects where $n=d-1$ and the defect line density is given by

$$\rho_{s_1}=\delta(\vec{\psi})\mathcal{D}_{s_1}, \tag{12}$$

where

$$\mathcal{D}_{s_1}=\frac{1}{n!}\epsilon_{s_1\mu_1\mu_2\cdots\mu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\nabla_{\mu_1}\psi_{\nu_1}\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n} \tag{13}$$

and the s and μ range from 1 to d and the ν range from 1 to n and there is summation over repeated indices. The generalization of Eqs. (9) and (12) to all $n \leqslant d$ is given by

$$\rho_{s_1s_2\cdots s_{d-n}}=\delta(\vec{\psi})\mathcal{D}_{s_1s_2\cdots s_{d-n}}, \tag{14}$$

where

$$\mathcal{D}_{s_1s_2\cdots s_{d-n}}=\frac{1}{n!}\epsilon_{s_1s_2\cdots s_{d-n}\mu_1\mu_2\cdots\mu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\times\nabla_{\mu_1}\psi_{\nu_1}\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}. \tag{15}$$

It is then straightforward to show that $\mathcal{D}_{s_1s_2\cdots s_{d-n}}$ itself satisfies a continuity equation given by

$$\dot{\mathcal{D}}_{s_1s_2\cdots s_{d-n}}=\nabla_{\mu_1}J_{s_1s_2\cdots s_{d-n}\mu_1} \tag{16}$$

with the current defined by

$$J_{s_1s_2\cdots s_{d-n}\mu_1}=\epsilon_{s_1s_2\cdots s_{d-n}\mu_1\mu_2\cdots\mu_n}g_{\mu_2\mu_3\cdots\mu_n} \tag{17}$$

and

$$g_{\mu_2\mu_3\cdots\mu_n}=\frac{1}{(n-1)!}\epsilon_{\nu_1\nu_2\cdots\nu_n}\dot{\psi}_{\nu_1}\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}. \tag{18}$$

To obtain the continuity equation for $\rho_{s_1s_2\cdots s_{d-n}}$, we need a second identity. Consider the quantity

$$\begin{aligned}J_{s_1s_2\cdots s_{d-n}\mu_1}\nabla_{\mu_1}\psi_\nu &=\epsilon_{s_1s_2\cdots s_{d-n}\mu_1\mu_2\cdots\mu_n}g_{\mu_2\mu_3\cdots\mu_n}\nabla_{\mu_1}\psi_\nu\\ &=\epsilon_{s_1s_2\cdots s_{d-n}\mu_1\mu_2\cdots\mu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\nabla_{\mu_1}\psi_\nu\\ &\quad\times\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}.\end{aligned} \tag{19}$$

The key observation is that the right-hand side of Eq. (19) has a factor which can be written in the form

$$\begin{aligned}&\epsilon_{s_1s_2\cdots s_{d-n}\mu_1\mu_2\cdots\mu_n}\nabla_{\mu_1}\psi_\nu\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}\\ &=\epsilon_{\nu\nu_2\cdots\nu_n}Q_{s_1s_2\cdots s_{d-n}},\end{aligned} \tag{20}$$

where $Q_{s_1s_2\cdots s_{d-n}}$ is determined by multiplying Eq. (20) by $\epsilon_{\nu\nu_2\cdots\nu_n}$ and summing over all the ν's. We easily obtain, remembering Eq. (15), $n!\,Q_{s_1s_2\cdots s_{d-n}}=n!\,\mathcal{D}_{s_1s_2\cdots s_{d-n}}$. Putting this result back into Eq. (19) gives

$$\begin{aligned}J_{s_1s_2\cdots s_{d-n}\mu_1}\nabla_{\mu_1}\psi_\nu &=\frac{1}{(n-1)!}\dot{\psi}_{\nu_1}\epsilon_{\nu_1\nu_2\cdots\nu_n}\epsilon_{\nu\nu_2\cdots\nu_n}\mathcal{D}_{s_1s_2\cdots s_{d-n}}\\ &=\frac{1}{(n-1)!}\dot{\psi}_{\nu_1}\delta_{\nu_1,\nu}(n-1)!\,\mathcal{D}_{s_1s_2\cdots s_{d-n}}\\ &=\dot{\psi}_\nu\mathcal{D}_{s_1s_2\cdots s_{d-n}}.\end{aligned} \tag{21}$$

Taking the time derivative of $\rho_{s_1s_2\cdots s_{d-n}}$ gives, using Eqs. (10) and (21),

$$\begin{aligned}\dot{\rho}_{s_1s_2\cdots s_{d-n}} &=\frac{\partial\delta(\vec{\psi})}{\partial\psi_\nu}\dot{\psi}_\nu\mathcal{D}_{s_1s_2\cdots s_{d-n}}+\delta(\vec{\psi})\dot{\mathcal{D}}_{s_1s_2\cdots s_{d-n}}\\ &=\frac{\partial\delta(\vec{\psi})}{\partial\psi_\nu}J_{s_1s_2\cdots s_{d-n}\mu_1}\nabla_{\mu_1}\psi_\nu\\ &\quad+\delta(\vec{\psi})\nabla_{\mu_1}J_{s_1s_2\cdots s_{d-n}\mu_1}\end{aligned}$$

and finally we obtain the desired continuity equation

$$\dot{\rho}_{s_1s_2\cdots s_{d-n}}=\nabla_{\mu_1}(\delta(\vec{\psi})J_{s_1s_2\cdots s_{d-n}\mu_1}). \tag{22}$$

For the simplest case of point defects $(n=d)$, Eq. (22) can be put into the conventional form given by Eq. (9) with

$$J_{\mu_1}=-v_{\mu_1}\mathcal{D} \tag{23}$$

and the velocity given by Eq. (10).

Let us turn next to the case of strings where the line density is a vector ρ_{μ_1} and the current is a two-component tensor, $J_{s_1\mu_1}=\epsilon_{s_1\mu_1\mu_2\cdots\mu_n}g_{\mu_2\mu_3\cdots\mu_n}$. Clearly $J_{s_1\mu_1}$ is antisymmetric in its subscripts. Since we expect the instantaneous velocity to be orthogonal to the local orientation of the string, we can define the velocity via

$$J_{\alpha\beta}=v_\alpha\mathcal{D}_\beta-v_\beta\mathcal{D}_\alpha. \tag{24}$$

Dotting the vector $\vec{\mathcal{D}}$ into this expression gives the result

$$v_\alpha=\frac{1}{\vec{\mathcal{D}}^2}J_{\alpha\beta}\mathcal{D}_\beta, \tag{25}$$

where we have taken advantage of the fact that $\vec{v}$ and $\vec{\mathcal{D}}$ are orthogonal: $\vec{\mathcal{D}}\cdot\vec{v}=\mathcal{D}_\alpha(1/\vec{\mathcal{D}}^2)J_{\alpha\beta}\mathcal{D}_\beta=0$. The velocity field for strings can be written in the form

$$v_{s_1}=\frac{1}{\vec{\mathcal{D}}^2}\mathcal{D}_{\mu_1}\epsilon_{s_1\mu_1\mu_2\cdots\mu_n}g_{\mu_2\mu_3\cdots\mu_n} \tag{26}$$

for general n.

Let us check this result and its sign for the simplest case of $n=1$, $d=2$. The vector $\vec{\mathcal{D}}$ in this case takes the simple form $\mathcal{D}_{\mu_1}=\epsilon_{\mu_1\mu_2}\nabla_{\mu_2}\psi$, $g=\dot{\psi}$, and

$$v_{s_1}=\frac{1}{(\nabla\psi)^2}\epsilon_{\mu_1\mu_2}\nabla_{\mu_2}\psi\epsilon_{s_1\mu_1}\dot{\psi}=-\frac{\nabla_{\mu_1}\psi}{(\nabla\psi)^2}\dot{\psi}. \tag{27}$$

Consider a circular loop of string of radius $R(t)$ where the order parameter near the interface formed by the loop is given in polar coordinates in the form

$$\psi(\vec{r})=A[r-R(t)], \tag{28}$$

where A is an overall constant amplitude. We then need the derivatives $\nabla_{\mu_1}\psi=A\hat{r}_{\mu_1}$ and $\dot{\psi}=-\dot{R}(t)A$ to obtain the velocity

$$v_{s_1}=\frac{1}{A^2}\dot{R}(t)\hat{r}_{s_1}A^2=\dot{R}(t)\hat{r}_{s_1} \tag{29}$$

as expected. Typically we will substitute for $\dot{\psi}$ using the equation of motion and use the defect locating δ function to set

$$\dot{\psi}=\Gamma c\nabla^2\psi \tag{30}$$

and

$$v_{s_1}=-\frac{(\nabla_{s_1}\psi)}{(\nabla\psi)^2}\Gamma c\nabla^2\psi. \tag{31}$$

Using the same ansatz given by Eq. (28) and remembering that the expression for the velocity is multiplied by a δ function setting $r=R(t)$ leads to the result $\nabla^2\psi=A/r=A/R(t)$, and we obtain the Lifshitz-Cahn-Allen [6–8] result,

$$v_{s_1}=-\Gamma c\frac{1}{R(t)}\hat{r}_{s_1}, \tag{32}$$

which tells us that the circular domain is shrinking and $R(t)\approx t^{1/2}$.

For $n=2$ and $d=3$ we have explicitly the results reported in Ref. [28]:

$$\vec{\mathcal{D}}=\frac{1}{2}\epsilon_{\nu_1\nu_2}(\vec{\nabla}\psi_{\nu_1}\times\vec{\nabla}\psi_{\nu_2}), \tag{33}$$

$$\vec{g}=\epsilon_{\nu_1\nu_2}\dot{\psi}_{\nu_1}\vec{\nabla}\psi_{\nu_2}, \tag{34}$$

and

$$\vec{v}=\frac{1}{\vec{\mathcal{D}}^2}\vec{\mathcal{D}}\times\vec{g}. \tag{35}$$

V. BRIEF DISCUSSION OF WALLS

Let us briefly discuss how things develop as one attempts to go further and increase $d-n$ to 2 and the case of walls. We must in this case deal with the quantities

$$\mathcal{D}_{s_1s_2}=\frac{1}{n!}\epsilon_{s_1s_2\mu_1\mu_2\cdots\mu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\nabla_{\mu_1}\psi_{\nu_1}\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n} \tag{36}$$

and

$$J_{s_1s_2\mu_1}=\epsilon_{s_1s_2\mu_1\mu_2\cdots\mu_n}g_{\mu_2\mu_3\cdots\mu_n}. \tag{37}$$

If we recall the definitions of the velocity field for point defects given by Eq. (23), and for string defects given by Eq. (24), then it is natural to write for walls

$$J_{s_1s_2\mu_1}=-\mathcal{D}_{s_1s_2}v_{\mu_1}-\mathcal{D}_{\mu_1s_1}v_{s_2}-\mathcal{D}_{s_2\mu_1}v_{s_1}. \tag{38}$$

This expression builds in the antisymmetry of $J_{s_1s_2\mu_1}$.

If one restricts the discussion to the physically most relevant case of $n=1$, $d=3$, then one can use the result $\mathcal{D}_{s_1s_2}=\epsilon_{s_1s_2\mu_3}\nabla_{\mu_3}\psi$ to show that the velocity field is given by

$$v_{\mu_1}=-\frac{1}{\mathcal{D}^2}\mathcal{D}_{s_1s_2}\epsilon_{s_1s_2\mu_3}\nabla_{\mu_1}\dot{\psi}, \tag{39}$$

where $\mathcal{D}^2=(\vec{\nabla}\psi)^2$. Further straightforward manipulation leads to the final result

$$v_{\mu}=-\frac{\nabla_{\mu}\psi}{(\vec{\nabla}\psi)^2}\dot{\psi}. \tag{40}$$

The key result here is that $\vec{v}$ is orthogonal to $\mathcal{D}_{s_1,s_2}$:

$$v_{\mu_1}\mathcal{D}_{\mu_1,s_3}=\frac{\dot{\psi}}{(\vec{\nabla}\psi)^2}\nabla_{\mu_1}\psi\epsilon_{\mu_1,s_3,\mu_3}\nabla_{\mu_3}\psi=0. \tag{41}$$

It remains to be seen if Eq. (38) serves as a useful definition of the velocity field for walls with $n>1$. Notice that the results for a scalar order parameter $(n=1)$ can all be written in the form of Eq. (40) for $d=1$, 2, and 3.

VI. EVALUATION OF $P[\mathbf{V}]$ FOR STRINGS

Our interest here is in determining the defect velocity probability distribution, $P[\mathbf{V}]$, for strings. Again we use the auxiliary field method [29–33] which has been successful in determining the scaling function for the order-parameter correlation function in a perturbation theory expansion. We evaluate P here to lowest order in this expansion where the auxiliary field can be treated as a Gaussian field. The first step in this theory is to express the order parameter in terms of an auxiliary field $\vec{m}$. For our purposes here the important result is that near a charge one vortex core the order parameter is linear in the auxiliary field $\vec{\psi}(\vec{m})=A\vec{m}+O(m^3)$. It is then easy to show that one can replace $\vec{\psi}$ by $\vec{m}$ in the expression for $\vec{v}$ given by Eq. (26) and in the expression for the string-charge density $\rho_\alpha(\vec{\psi})=\rho_\alpha(\vec{m})$. We then want to determine the string-velocity probability distribution

$$\langle|\vec{\rho}|\rangle P[\mathbf{V}] \equiv \langle|\vec{\rho}(\vec{\psi})|\,\delta(\mathbf{V}-\mathbf{v}(\vec{\psi}))\rangle = \langle|\vec{\rho}(\vec{m})|\,\delta(\mathbf{V}-\mathbf{v}(\vec{m}))\rangle. \tag{42}$$

One can determine $P[\mathbf{V}]$ by first evaluating the more general probability distribution

$$G(\xi,\vec{b}) = \langle \delta(\vec{m})\,\delta(\xi_\mu^\nu - \nabla_\mu m_\nu)\,\delta(\vec{b}-\nabla^2\vec{m})\rangle \tag{43}$$

since

$$n_0 P[\mathbf{V}] = \int d^n b \prod_{\mu=1}^{d}\prod_{\nu=1}^{n} d\xi_\mu^\nu |\vec{\mathcal{D}}(\xi)|\,\delta(\vec{V}-\vec{v}(\vec{b},\xi))G(\xi,\vec{b}), \tag{44}$$

where

$$v_\mu(\vec{b},\xi) = \frac{\Gamma c}{\vec{\mathcal{D}}^2}\frac{1}{(n-1)!}\epsilon_{\mu_1 s_1 \mu_2\cdots\mu_n} \times \mathcal{D}_{s_1}(\xi)\epsilon_{\nu_1\nu_2\cdots\nu_n} b_{\nu_1}\xi_{\mu_2}^{\nu_2}\cdots\xi_{\mu_n}^{\nu_n} \tag{45}$$

with

$$\mathcal{D}_{s_1}(\xi) = \frac{1}{n!}\epsilon_{s_1\mu_1\mu_2\cdots\mu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\xi_{\mu_1}^{\nu_1}\xi_{\mu_2}^{\nu_2}\cdots\xi_{\mu_n}^{\nu_n} \tag{46}$$

and $n_0 = \langle|\vec{\rho}(\vec{m})|\rangle$. We have assumed that the quench is to zero temperature so that the noise can be set to zero and we can use Eq. (30). The Gaussian average determining $G(\xi,\vec{b})$ is relatively straightforward to evaluate [34] and is given by

$$G(\xi,\vec{b}) = \frac{1}{(2\pi S_0)^{n/2}}\frac{e^{-(1/2\bar{S}_4)\vec{b}^2}}{(2\pi\bar{S}_4)^{n/2}}\frac{1}{(2\pi\bar{S}_2)^{nd/2}} \times \exp\left[-\frac{1}{2\bar{S}_2}\sum_{\mu,\nu}(\xi_\mu^\nu)^2\right], \tag{47}$$

where $S_0 = 1/n\langle\vec{m}^2\rangle \approx L^2$, $\bar{S}_2 = 1/dn\langle(\nabla\vec{m})^2\rangle \approx L^0$, and

$$\bar{S}_4 = \frac{1}{n}\langle(\nabla^2\vec{m})^2\rangle - \frac{(d\bar{S}_2)^2}{S_0} \approx L^{-2}. \tag{48}$$

The quantities $S_0,\bar{S}_2,\bar{S}_4$ are determined from the theory for the order-parameter correlation function and discussed further below.

The problem then reduces to evaluating the $\vec{b}$ and ξ integrations in the integral given by Eq. (44) using the result for $G(\xi,\vec{b})$ given by Eq. (47). We proceed by first doing the integration over $\vec{b}$. This is facilitated by first defining the matrix M_μ^ν via $v_\mu = \Gamma c M_\mu^\nu b_\nu$ and

$$M_\mu^\nu = \frac{1}{(n-1)!}\frac{1}{\vec{\mathcal{D}}^2(\xi)}\epsilon_{\mu s_1\mu_2\cdots\mu_n}\mathcal{D}_{s_1}(\xi)\epsilon_{\nu\nu_2\cdots\nu_n}\xi_{\mu_2}^{\nu_2}\cdots\xi_{\mu_n}^{\nu_n}. \tag{49}$$

Clearly the quantity $\vec{\mathcal{D}}^2$ is important in the development and is discussed in Appendix A. The matrix M_μ^ν is discussed in Appendix B. Then we use the integral representation for the δ function and find that we can evaluate the integral over $\vec{b}$ in terms of standard displaced Gaussian integrals with the results

$$n_0 P[\mathbf{V}] = \frac{1}{(2\pi S_0)^{n/2}}\int \prod_{\mu=1}^{d}\prod_{\nu=1}^{n} d\xi_\mu^\nu |\vec{\mathcal{D}}(\xi)|\frac{1}{(2\pi\bar{S}_2)^{nd/2}} \times \exp\left[-\frac{1}{2\bar{S}^{(2)}}\sum_{\mu,\nu}(\xi_\mu^\nu)^2\right]J(\xi), \tag{50}$$

where

$$J(\xi) = \int \frac{d^d k}{(2\pi)^d}e^{i\vec{k}\cdot\vec{V}}J_k(\xi) \tag{51}$$

and

$$J_k(\xi) = \int \frac{d^d b}{(2\pi\bar{S}_4)^{n/2}}e^{-i\vec{k}\cdot\vec{v}(\vec{b},\xi)}e^{-(1/2\bar{S}_4)\vec{b}^2} = \int \frac{d^n b}{(2\pi\bar{S}_4)^{n/2}}e^{-ik_\mu\Gamma c M_\mu^\nu b_\nu}e^{-(1/2\bar{S}_4)\vec{b}^2} = \exp\left[-\frac{1}{2}\bar{S}_4(\Gamma c)^2 k_\alpha k_\beta M_\alpha^\nu M_\beta^\nu\right]. \tag{52}$$

Sums over α and β range from 1 to d, those over ν's from 1 to n. We then have the apparently Gaussian integral over $\vec{k}$:

$$J(\xi) = \int \frac{d^d k}{(2\pi)^d}e^{i\vec{k}\cdot\vec{V}}\exp\left[-\frac{1}{2}\bar{S}_4(\Gamma c)^2 k_\alpha k_\beta \bar{M}_{\alpha\beta}\right], \tag{53}$$

where $\bar{M}_{\alpha\beta} = M_\alpha^\nu M_\beta^\nu$ is a symmetric $d\times d$ matrix. Since we have the property $\mathcal{D}_\alpha M_\alpha^\nu = 0$, we see that the matrix $\bar{M}_{\alpha\beta}$ has a zero eigenvalue corresponding to an eigenfunction in the $\vec{\mathcal{D}}$ direction: $\mathcal{D}_\alpha\bar{M}_{\alpha\beta} = 0$. In order to carry out the integral in Eq. (53), we must set up a coordinate system which singles out the $\vec{\mathcal{D}}$ direction and n orthogonal directions. Thus we construct an orthonormal set $(\hat{\mathcal{D}},\hat{\xi}^{(s)})$, for $s=1,2,\ldots,n$, which satisfy $\hat{\xi}_\alpha^{(s)}\hat{\mathcal{D}}_\alpha = 0$ and $\hat{\xi}_\alpha^{(s)}\hat{\xi}_\alpha^{(s')} = \delta_{ss'}$. In Appendix C we show that we can write $\hat{\xi}_\alpha^{(s)} = \Sigma_\nu A_{s\nu}\xi_\mu^\nu$ and it can be shown generally, see Appendix C, that

$$(\det A)(\det\tilde{A}) = \frac{1}{\det N} = \frac{1}{\vec{\mathcal{D}}^2}, \tag{54}$$

where N is the $n\times n$ matrix $N_{\nu\nu'} = \xi_\mu^\nu\xi_\mu^{\nu'}$. We then make the change of variables in the integral Eq. (53) from $\vec{k}$ to

$$k_\alpha = \sum_{\nu 1}^{n} t_\nu\hat{\xi}_\alpha^{(\nu)} + t_d\hat{\mathcal{D}}_\alpha \tag{55}$$

which clearly has a Jacobian of 1. The integral of interest is then given by

$$J(\xi)=\int \frac{d^d t}{(2\pi)^d} e^{it_d \hat{\mathcal{D}}\cdot\vec{V}} e^{it_\nu \hat{\xi}_\alpha^{(\nu)} V_\alpha} \times \exp\left[-\frac{\overline{S_4}}{2}(\Gamma c)^2 t_\nu t_{\nu'} Q_{\nu\nu'}\right], \tag{56}$$

where ν and ν' range from 1 to n. One can then do the t_d integration to obtain a δ function. The rest of the integrations are over Gaussian fields governed by the matrix $Q_{\nu\nu'} = \hat{\xi}_\alpha^{(\nu)} \hat{\xi}_\beta^{(\nu')} \overline{M}_{\alpha\beta}$ which does not possess any zero eigenvalues. Evaluating the standard Gaussian integral leads to the result

$$J(\xi)=\delta(\hat{\mathcal{D}}\cdot\vec{V}) \frac{1}{[2\pi\overline{S_4}(\Gamma c)^2]^{n/2}} \frac{1}{(\det Q)^{1/2}} \times \exp\left[-\frac{1}{2\overline{S_4}(\Gamma c)^2} V_\nu V_{\nu'} (Q^{-1})_{\nu\nu'}\right], \tag{57}$$

where the n vector V_ν is defined by

$$V_\nu = \hat{\xi}_\alpha^{(\nu)} V_\alpha . \tag{58}$$

Note, if we insert Eq. (58) into Eq. (57), we see that we need only the matrix

$$R_{\alpha\beta} = \hat{\xi}_\alpha^{(\nu)} \hat{\xi}_\beta^{(\nu')} (Q^{-1})_{\nu\nu'} \tag{59}$$

and

$$J(\xi)=\frac{\delta(\hat{\mathcal{D}}\cdot\vec{V})}{(\Gamma c)^n} \frac{1}{(2\pi\overline{S_4})^{n/2}} \frac{1}{(\det Q)^{1/2}} \times \exp\left[-\frac{1}{2\overline{S_4}(\Gamma c)^2} V_\alpha V_\beta R_{\alpha\beta}\right]. \tag{60}$$

We show in Appendix D that the matrix R, defined by Eq. (59), can be put into the very simple form $R_{\alpha\beta} = \xi_\alpha^\nu \xi_\beta^\nu$, and finally $J(\xi)$ is given by

$$J(\xi)=\frac{\delta(\hat{\mathcal{D}}\cdot\vec{V})}{(\Gamma c)^n} \frac{1}{(2\pi\overline{S_4})^{n/2}} |\vec{\mathcal{D}}| \times \exp\left[-\frac{1}{2\overline{S_4}(\Gamma c)^2} V_\alpha V_\beta \xi_\alpha^\nu \xi_\beta^\nu\right]. \tag{61}$$

Inserting this result into Eq. (50) gives

$$n_0 P[\mathbf{V}] = \frac{1}{(2\pi S_0 2\pi\overline{S_4})^{n/2}} \int \prod_{\mu=1}^{d} \prod_{\nu=1}^{n} \times d\xi_\mu^\nu \vec{\mathcal{D}}^2(\xi) \frac{\delta(\hat{\mathcal{D}}\cdot\vec{V})}{(\Gamma c)^n} \frac{1}{(2\pi\overline{S_2})^{nd/2}} e^{-(1/2)A(\xi)}, \tag{62}$$

where

$$A(\xi)=\frac{1}{\overline{S_2}} \sum_{\nu=1}^{n} \sum_{\alpha=1}^{d} (\xi_\alpha^\nu)^2 + \frac{1}{\overline{S_4}(\Gamma c)^2} \sum_{\nu=1}^{n} \sum_{\alpha,\beta=1}^{d} V_\alpha V_\beta \xi_\alpha^\nu \xi_\beta^\nu . \tag{63}$$

If we make the rescalings

$$\xi_\mu^\nu \to \sqrt{\overline{S_2}}\, \xi_\mu^\nu , \tag{64}$$

$$V_\alpha \to \overline{v} \tilde{V}_\alpha , \tag{65}$$

where the characteristic speed

$$\overline{v^2} = (\Gamma c)^2 \frac{\overline{S_4}}{\overline{S_2}} \tag{66}$$

is introduced, then $\vec{\mathcal{D}} \to (\overline{S_2})^{n/2} \vec{\mathcal{D}}$ and we obtain

$$n_0 P[\mathbf{V}] = \frac{1}{(2\pi)^n} \frac{1}{(\Gamma c)^d} \sqrt{\frac{\overline{S_2}}{\overline{S_4}}} \left(\frac{\overline{S_2^2}}{S_0 \overline{S_4}}\right)^{n/2} I(\tilde{V}), \tag{67}$$

where the dimensionless integral $I(\tilde{V})$ is defined by

$$I(\tilde{V}) = \int \prod_{\mu=1}^{d} \prod_{\nu=1}^{n} d\xi_\mu^\nu \vec{\mathcal{D}}^2(\xi)\, \delta(\hat{\mathcal{D}}(\xi)\cdot\vec{\tilde{V}}) \frac{e^{-(1/2)A_0(\xi)}}{(2\pi)^{nd/2}} \tag{68}$$

with

$$A_0(\xi) = \sum_{\nu=1}^{n} \left[\sum_{\alpha=1}^{d} (\xi_\alpha^\nu)^2 + \sum_{\alpha,\beta=1}^{d} \tilde{V}_\alpha \tilde{V}_\beta \xi_\alpha^\nu \xi_\beta^\nu\right]. \tag{69}$$

We can construct a form for the integral $I(\tilde{V})$ which does not involve a unit vector in the δ function via the following rearrangements:

$$\begin{aligned}\delta(\hat{\mathcal{D}}\cdot\vec{\tilde{V}}) &= |\vec{\mathcal{D}}|\, \delta(\vec{\mathcal{D}}\cdot\vec{\tilde{V}}) \\ &= \frac{\vec{\mathcal{D}}^2}{|\vec{\mathcal{D}}|} \delta(\vec{\mathcal{D}}\cdot\vec{\tilde{V}}) = \frac{\vec{\mathcal{D}}^2}{(\det N)^{1/2}} \delta(\vec{\mathcal{D}}\cdot\vec{\tilde{V}}) \\ &= \int \frac{d^n z}{(2\pi)^{n/2}} \vec{\mathcal{D}}^2\, \delta(\vec{\mathcal{D}}\cdot\vec{\tilde{V}}) e^{-(1/2) z_\nu N_{\nu,\nu'} z_{\nu'}}.\end{aligned} \tag{70}$$

Inserting this result into the integral $I[\tilde{V}]$ gives

$$I(\tilde{V}) = \int \prod_{\mu=1}^{d} \prod_{\nu=1}^{n} d\xi_\mu^\nu \int \frac{d^n z}{(2\pi)^{n/2}} \times \vec{\mathcal{D}}^4(\xi)\, \delta(\vec{\mathcal{D}}(\xi)\cdot\vec{\tilde{V}}) \frac{e^{-(1/2)A_0(\xi)}}{(2\pi)^{nd/2}}, \tag{71}$$

where

$$A_0(\xi,z) = (\xi_\alpha^\nu)^2 + \tilde{V}_\alpha \tilde{V}_\beta \xi_\alpha^\nu \xi_\beta^\nu + z_\nu \xi_\alpha^\nu \xi_\alpha^{\nu'} z_{\nu'} . \tag{72}$$

VII. CASE $n=1$ AND $d=2$

It is straightforward to work out $P[\mathbf{V}]$ and n_0 for the simplest case of defect lines in two dimensions. The key simplifying aspect in this example is that the matrix ξ_α^ν reduces to a vector ξ_α and $\mathcal{D}_{s_1}(\xi)=\epsilon_{s_1\mu_1}\xi_{\mu_1}$, with $\vec{\mathcal{D}}^2=\vec{\xi}^2$. We have from Eq. (67) that

$$n_0P[\mathbf{V}]=\frac{1}{(2\pi)}\frac{1}{(\Gamma c)^2}\sqrt{\frac{\overline{S}_2}{\overline{\overline{S}}_4}}\left(\frac{\overline{S}_2^2}{S_0\overline{\overline{S}}_4}\right)^{1/2}I(\tilde{V}), \tag{73}$$

$$I(\tilde{V})=\int d^2\xi\int\frac{dz}{(2\pi)^{1/2}}\vec{\xi}^4\delta(\vec{\mathcal{D}}(\xi)\cdot\vec{\tilde{V}})\frac{1}{(2\pi)}e^{-(1/2)A_0(\xi)}, \tag{74}$$

and

$$A_0(\xi,z)=\xi_\alpha^2+\tilde{V}_\alpha\tilde{V}_\beta\xi_\alpha\xi_\beta+z^2\xi_\alpha^2. \tag{75}$$

Then, since the integral is isotropic, we can pick $\mathbf{V}$ to be in the x direction and use $\vec{\mathcal{D}}(\xi)\cdot\vec{\tilde{V}}=V\xi_y$ in the δ function to do the integral over ξ_y and obtain

$$I(\tilde{V})=\frac{1}{(2\pi\tilde{V})}\int d\xi_x\int\frac{dz}{(2\pi)^{1/2}}\xi_x^4e^{-(1/2)\xi_x^2[1+z^2+\tilde{V}^2]}. \tag{76}$$

The remaining integrals are elementary and we obtain the results

$$I(\tilde{V})=\frac{2}{(\pi\tilde{V})}\frac{1}{(1+\tilde{V}^2)^2} \tag{77}$$

and

$$n_0P[\mathbf{V}]=\frac{1}{(2\pi)}\frac{1}{\Gamma c}\sqrt{\frac{\overline{S}_2}{\overline{\overline{S}}_4}}\left(\frac{\overline{S}_2^2}{S_0\overline{\overline{S}}_4}\right)^{1/2}\frac{2}{(\pi\tilde{V})}\frac{1}{(1+\tilde{V}^2)^2}. \tag{78}$$

We can then obtain the string density n_0 in two ways. We can compute it directly from

$$n_0=\langle|\vec{\mathcal{D}}(\vec{\psi})|\delta(\vec{\psi})\rangle=\int\prod_{\nu=1}^{n}d\xi_\mu^\nu|\vec{\mathcal{D}}(\xi)|G(\xi), \tag{79}$$

where $G(\xi)$ is the integral over $\vec{b}$ of Eq. (47) given by

$$G(\xi)=\frac{1}{(2\pi S_0)^{n/2}}\frac{1}{(2\pi\overline{S}_2)^{nd/2}}\exp\left[-\frac{1}{2\overline{S}^{(2)}}\sum_{\mu,\nu}(\xi_\mu^\nu)^2\right]. \tag{80}$$

Restricting the analysis to $n=1$ and $d=2$ gives

$$n_0=\int d^2\xi\frac{1}{\sqrt{2\pi S_0}}\frac{1}{2\pi\overline{S}_4}e^{-(1/2\overline{S}_2)\vec{\xi}^2}. \tag{81}$$

These integrations are elementary with the final result

$$n_0=\frac{1}{2}\left(\frac{\overline{S}_2}{S_0}\right)^{1/2}. \tag{82}$$

The second method for determining n_0, which serves as a check on intermediate steps in the calculation, is to integrate $n_0P[\mathbf{V}]$ over all $\mathbf{V}$ and use the fact that $P[\mathbf{V}]$ must be normalized. The exercise is straightforward and leads to the same result for n_0. We then have the final result for the probability distribution

$$P[\mathbf{V}]=\frac{2}{\pi^2}\frac{1}{V\overline{v}}\frac{1}{[1+(V/\overline{v})^2]^2} \tag{83}$$

and it is easy to see that this agrees with Eq. (1) for $n=1$ and $d=2$.

VIII. CASE $n=2$ AND $d=3$

We turn to the physically important case of strings in three spatial dimensions. In working out this case, the key observation is that

$$\mathcal{D}_\alpha(\xi)=\tfrac{1}{2}\epsilon_{\alpha,\mu_1,\mu_2}\epsilon_{\nu_1,\nu_2}\xi_{\mu_1}^{\nu_1}\xi_{\mu_2}^{\nu_2} \tag{84}$$

is a quadratic form in ξ. This suggests that we use the integral representation of the δ function,

$$\delta(\vec{\mathcal{D}}(\xi)\cdot\vec{\tilde{V}})=\int\frac{dk}{2\pi}e^{ik\vec{\mathcal{D}}(\xi)\cdot\vec{\tilde{V}}}, \tag{85}$$

to write the integral of interest, Eq. (71), in the form

$$I(\tilde{V})=\int\prod_{\mu=1}^{3}\prod_{\nu=1}^{2}\frac{d\xi_\mu^\nu}{(2\pi)^3}\int\frac{d^2z}{2\pi}\int\frac{dk}{2\pi}\vec{\mathcal{D}}^4(\xi)e^{-(1/2)\xi_\mu^\nu Q_{\mu\mu'}^{\nu\nu'}\xi_{\mu'}^{\nu'}}. \tag{86}$$

The matrix appearing in the Gaussian is given by

$$\begin{aligned}Q_{\mu\mu'}^{\nu\nu'}&=\delta_{\mu\mu'}\delta_{\nu\nu'}+\delta_{\nu\nu'}\tilde{V}_\mu\tilde{V}_{\mu'}\\&\quad+\delta_{\mu\mu'}z_\nu z_{\nu'}-ik_\alpha\epsilon_{\alpha,\mu,\mu'}\epsilon_{\nu,\nu'},\end{aligned} \tag{87}$$

where we have introduced $k_\alpha=k\tilde{V}_\alpha$ and have used the result

$$-\tfrac{1}{2}\xi_\mu^\nu(-ik_\alpha\epsilon_{\alpha,\mu,\mu'})\xi_{\mu'}^{\nu'}=i\vec{k}\cdot\vec{\mathcal{D}}(\xi). \tag{88}$$

We see that in principle we can carry out the ξ integration since it involves a product of polynomials times a Gaussian weight. Let us define the integral

$$L(\vec{k},z)=\int\prod_{\mu=1}^{3}\prod_{\nu=1}^{2}\frac{d\xi_\mu^\nu}{(2\pi)^3}\vec{\mathcal{D}}^4(\xi)e^{-(1/2)\xi_\mu^\nu Q_{\mu\mu'}^{\nu\nu'}\xi_{\mu'}^{\nu'}} \tag{89}$$

and

$$I(\tilde{V})=\int\frac{d^2z}{2\pi}\int\frac{dk}{2\pi}L(\vec{k},z). \tag{90}$$

A significant simplification occurs if we realize that gradients with respect to $\vec{k}$ pull down factors of $\vec{\mathcal{D}}$ and we can write

$$L(\vec{k},z)=\nabla_k^4 L_0(\vec{k},z), \tag{91}$$

where

$$L_0(\vec{k},z)=\int\prod_{\mu=1}^{3}\prod_{\nu=1}^{2}\frac{d\xi_\mu^\nu}{(2\pi)^3}e^{-(1/2)\xi_\mu^\nu Q_{\mu\mu'}^{\nu\nu'}\xi_{\mu'}^{\nu'}}. \tag{92}$$

Thus we are left with a set of Gaussian integrals.

The Gaussian integral can be carried out if we think of $Q_{\mu\mu'}^{\nu\nu'}$ as a 6×6 symmetric matrix. If we can find the eigenvalues λ_i, $i=1,2,\ldots,6$, of this matrix, then we have

$$L_0(\vec{k},z)=\frac{1}{\sqrt{\prod_{i=1}^{6}\lambda_i}}. \tag{93}$$

In Appendix E we discuss the relevant eigenvalue problem. The result of the analysis there is that the six-dimensional problem factorizes into a product of two three-dimensional problems. These three-dimensional eigenvalue problems reduce to cubic equations and the product of the three associated eigenvalues can be read off from the associated characteristic equation with the final result

$$L_0(\vec{k},z)=\frac{1}{\sqrt{(1+z^2)(1+z^2+\tilde{V}^2+k^2)+(\vec{k}\cdot\tilde{\vec{V}})^2}}\times\frac{1}{\sqrt{(1+z^2)(1+\tilde{V}^2)+k^2+(\vec{k}\cdot\tilde{\vec{V}})^2}}. \tag{94}$$

We must then apply ∇_k^4 to $L_0(\vec{k},z)$ and set $\vec{k}=k\tilde{\vec{V}}$. After a great deal of algebra one finds a complicated result for $I[\tilde{V}]$ which still requires integration over k and $\vec{z}$. It turns out that it is wise to first do the k integration. All of the contributions are proportional to integrals of the form

$$\int\frac{dk}{2\pi}\frac{(k\tilde{V})^{2p}}{(1+z^2+\tilde{V}^2k^2)^{p+3}}=\frac{\kappa_p}{2\pi\tilde{V}}\frac{1}{(1+z^2)^{5/2}}, \tag{95}$$

where p takes the values 0, 1, and 2 with $\kappa_0=3\pi/8$, $\kappa_1=\pi/16$, $\kappa_0=3\pi/8(16)$. The final integrals over $\vec{z}$ can all be expressed in terms of the integrals

$$J_{s_1,s_2}=\int\frac{d^2z}{2\pi}\frac{1}{(1+z^2+\tilde{V}^2)^{s_1/2}}\frac{1}{(1+z^2)^{s_2/2}}, \tag{96}$$

which, for integer s_1 and s_2, can be worked out analytically. After an enormous amount of additional algebra we obtain the very simple result

$$I[\tilde{V}]=\frac{3}{\tilde{V}}\frac{1}{(1+\tilde{V}^2)^{5/2}}. \tag{97}$$

This leads back to the result

$$n_0P[\mathbf{V}]=\frac{1}{(2\pi)^2}\frac{1}{(\Gamma c)^3}\sqrt{\frac{\overline{S}_2}{\overline{\overline{S}}_4}}\frac{\overline{S}_2^2}{S_0\overline{S}_4}\frac{3}{\tilde{V}}\frac{1}{(1+\tilde{V}^2)^{5/2}}. \tag{98}$$

Integrating over all $\mathbf{V}$ we easily obtain the density of strings

$$n_0=\frac{1}{\pi}\frac{\overline{S}_2}{S_0}. \tag{99}$$

Putting this result back into Eq. (98) gives the final result

$$P[\mathbf{V}]=\frac{3}{4\pi}\frac{1}{\overline{v}^3\tilde{V}}\frac{1}{(1+\tilde{V}^2)^{5/2}}, \tag{100}$$

which agrees with Eq. (1) for $n=2$ and $d=3$.

IX. WALLS IN THREE DIMENSIONS

Let us conclude with the example of walls in three dimensions ($n=1,d=3$). We have from Sec. V that $\mathcal{D}^2=(\vec{\nabla}\psi)^2$ and the velocity of the wall is given by

$$v_\mu=-\frac{\nabla_\mu\psi}{(\vec{\nabla}\psi)^2}\Gamma c\nabla^2\psi. \tag{101}$$

Following the same path as for the point and string defects we find that the velocity probability distribution is given by

$$n_0P[\mathbf{V}]=\int dbd^3\xi|\vec{\xi}|\delta(\vec{V}+\Gamma c\vec{\xi}b/\xi^2)G(\vec{\xi},b), \tag{102}$$

where $G(\vec{\xi},b)$ is given by Eq. (47) with $n=1$ and $d=3$ and

$$n_0P[\mathbf{V}]=\int dbd^3\xi|\vec{\xi}|\delta(\vec{V}+\Gamma c\vec{\xi}b/\xi^2)\times\frac{1}{(2\pi S_0)^{1/2}}\frac{e^{-(1/2\overline{S}_4)\vec{b}^2}}{(2\pi\overline{S}_4)^{1/2}}\frac{e^{-(1/2\overline{S}_2)\vec{\xi}^2}}{(2\pi\overline{S}_2)^{d/2}}. \tag{103}$$

Again inserting the integral representation for the δ function and doing the b integration, we obtain

$$n_0P[\mathbf{V}]=\frac{1}{\sqrt{2\pi S_0(2\pi\overline{S}_2)^d}}\int d^3\xi|\vec{\xi}|\exp\left[-\frac{1}{2\overline{S}_2}\vec{\xi}^2\right]J(\xi), \tag{104}$$

where

$$J(\xi)=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{V}}\exp\left[-\frac{1}{2}\overline{S}_4(\Gamma c)^2(\vec{k}\cdot\hat{\xi})^2/\xi^2\right]. \tag{105}$$

The integral in Eq. (105) can be carried out if we introduce the orthogonal set of coordinates $\hat{\xi}$, $\hat{b}^{(1)}$, $\hat{b}^{(2)}$ and we choose $\hat{\xi}=\hat{b}^{(1)}\times\hat{b}^{(2)}$. Then we obtain

$$J(\xi)=\delta(\vec{V}\cdot\hat{b}^{(1)})\,\delta(\vec{V}\cdot\hat{b}^{(2)})\frac{|\vec{\xi}|}{\sqrt{2\pi\overline{S}_4(\Gamma c)^2}}\times\exp\left[-\frac{1}{2\overline{S}_4(\Gamma c)^2}(\vec{V}\cdot\vec{\xi})^2\right]. \tag{106}$$

If we make the same change of variables given by Eqs. (64) and (65), we find the result

$$n_0P[\mathbf{V}]=\frac{1}{\overline{v}^3}\frac{1}{(2\pi)^{5/2}}\sqrt{\frac{\overline{S}_2}{\overline{S}_4}}I[\tilde{V}] \tag{107}$$

and

$$I[\tilde{V}]=\int d^3\xi\vec{\xi}^2\delta(\tilde{\vec{V}}\cdot\hat{b}^{(1)})\,\delta(\tilde{\vec{V}}\cdot\hat{b}^{(2)})e^{-(1/2)[\vec{\xi}^2+(\vec{\xi}\cdot\tilde{\vec{V}})^2]}. \tag{108}$$

If we now define $\vec{b}^{(1)}=\hat{b}^{(2)}\times\vec{\xi}$ and $\vec{b}^{(2)}=\vec{\xi}\times\hat{b}^{(2)}$, and assume that $\vec{V}$ is in the z direction, we obtain

$$I[\tilde{V}]=\int d^3\xi\vec{\xi}^4\delta(\tilde{V}b_z^{(1)})\,\delta(\tilde{V}b_z^{(2)})e^{-(1/2)[\vec{\xi}^2+\xi_z^2\tilde{V}^2]}. \tag{109}$$

The key point then is that

$$\delta(\tilde{V}b_z^{(1)})\,\delta(\tilde{V}b_z^{(2)})=\frac{1}{\tilde{V}^2}\delta(\xi_x)\,\delta(\xi_y) \tag{110}$$

and the integral reduces to an elementary one-dimensional integral

$$I[\tilde{V}]=\frac{1}{\tilde{V}^2}\int d\xi_z\xi_z^4e^{-(1/2)\vec{\xi}^2[1+\tilde{V}^2]}. \tag{111}$$

Inserting the result of doing this integral back into Eq. (107) leads immediately to the result that $P[\mathbf{V}]$ is again given by Eq. (1).

X. DETERMINATION OF $\overline{v}$

The determination of S_0, $\overline{S}_2$, $\overline{S}_4$, and $\overline{v}$ requires a theory for the auxiliary field correlation function

$$C_0(12)=\frac{1}{n}\langle\vec{m}(1)\cdot\vec{m}(2)\rangle. \tag{112}$$

If we follow the development in Refs. [2] and [28] we can write in the scaling regime,

$$C_0(12)=S_0e^{-\vec{r}^2/(2L^2)}, \tag{113}$$

where $\vec{r}=\vec{r}_1-\vec{r}_2$. We do not need S_0, $\overline{S}_2$, and $\overline{S}_4$ individually. We only need the combination given by Eq. (66). We easily find

$$\overline{v}^2=\frac{2d}{L^2}(\Gamma c)^2=\frac{\Gamma cd}{2t}. \tag{114}$$

XI. CONCLUSIONS AND QUESTIONS

The results of this work are simply summarized by Eq. (1). There remain several open questions. The most important question is whether Eq. (1) for $P[\mathbf{V}]$ corresponds to the results found in the real world. It is highly desirable to measure $P[\mathbf{V}]$ for the rather broad range of systems covered by Eq. (1).

How is this result for the defect velocity probability distribution changed at higher order in perturbation theory? So far, $P[\mathbf{V}]$ has only been calculated for the lowest order in the perturbation theory developed in Ref. [2]. Scaling arguments [3] would indicate that the large velocity tails will not be modified at higher orders in perturbation theory. This question will be addressed soon.

The results for $P[\mathbf{V}]$ given by Eq. (1) have only been proven for a subset of the range $n\leqslant d$. What about for $n=d-2=2$? This question is somewhat academic since it is outside the range of physically accessible spatial dimensions.

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APPENDIX A: $\vec{\mathcal{D}}^2$

Clearly one of the important quantities entering into the discussion of the string velocity probability distribution is the quantity $\mathcal{D}$ defined by Eq. (13) and its square:

$$\vec{\mathcal{D}}^2=\frac{1}{(n!)^2}\epsilon_{\mu\mu_1\mu_2\cdots\mu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\xi_{\mu_1}^{\nu_1}\xi_{\mu_2}^{\nu_2}\cdots\times\xi_{\mu_n}^{\nu_n}\epsilon_{\mu,\mu_1'\mu_2'\cdots\mu_n'}\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}\xi_{\mu_1'}^{\nu_1'}\xi_{\mu_2'}^{\nu_2'}\cdots\xi_{\mu_n'}^{\nu_n'}. \tag{A1}$$

This quantity can be written in a simplified form if we realize that

$$\epsilon_{\mu\mu_1\mu_2\cdots\mu_n}\epsilon_{\mu\mu_1'\mu_2'\cdots\mu_n'}=I[\mu_1\mu_2\cdots\mu_n;\mu_1'\mu_2'\cdots\mu_n'], \tag{A2}$$

where I is a set of products of δ functions giving all matched pairs between the unprimed and primed sets. There are minus signs if the matched pairs are an odd number of permutations of the labels in order to return to the order 1,2, . . . ,n. We have then

$$\begin{aligned}\vec{\mathcal{D}}^2&=\frac{1}{(n!)^2}I[\mu_1\mu_2\cdots\mu_n;\mu_1'\mu_2'\cdots\mu_n']\\&\quad\times\epsilon_{\nu_1\nu_2\cdots\nu_n}\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}\xi_{\mu_1}^{\nu_1}\xi_{\mu_2}^{\nu_2}\cdots\xi_{\mu_n}^{\nu_n}\xi_{\mu_1'}^{\nu_1'}\xi_{\mu_2'}^{\nu_2'}\cdots\xi_{\mu_n'}^{\nu_n'}\\&=\frac{1}{n!}\epsilon_{\nu_1\nu_2\cdots\nu_n}\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}N_{\nu_1\nu_1'}N_{\nu_2\nu_2'}\cdots N_{\nu_n\nu_n'}\\&=\frac{1}{n!}\epsilon_{\nu_1\nu_2\cdots\nu_n}\epsilon_{\nu_1\nu_2\cdots\nu_n}\det(N)=\det(N).\end{aligned} \tag{A3}$$

APPENDIX B: MATRIX M_μ^ν

The matrix M_μ^ν defined by Eq. (49) can be put into another useful form by inserting the explicit form for $\vec{\mathcal{D}}$ given by Eq. (46) and using the result given by Eq. (A2). An intermediate step gives

$$\begin{aligned}
&\epsilon_{\mu\mu_1\mu_2\cdots\mu_n}\mathcal{D}_{\mu_1}\\
&\quad=\epsilon_{\mu\mu_1\mu_2\cdots\mu_n}\frac{1}{n!}\epsilon_{\mu_1\mu_1'\mu_2'\cdots\mu_n'}\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}\xi_{\mu_1'}^{\nu_1'}\xi_{\mu_2'}^{\nu_2'}\cdots\xi_{\mu_n'}^{\nu_n'}\\
&\quad=-I[\mu,\mu_2\ldots\mu_n;\mu_1'\mu_2'\cdots\mu_n']\\
&\qquad\times\frac{1}{n!}\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}\xi_{\mu_1'}^{\nu_1'}\xi_{\mu_2'}^{\nu_2'}\cdots\xi_{\mu_n'}^{\nu_n'}\\
&\quad=-\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}\xi_{\mu}^{\nu_1'}\xi_{\mu_2}^{\nu_2'}\cdots\xi_{\mu_n}^{\nu_n'}. \qquad \text{(B1)}
\end{aligned}$$

Inserting this into the definition of M_μ^ν gives

$$\begin{aligned}
M_\mu^\nu&=-\frac{1}{\vec{\mathcal{D}}^2}\frac{1}{(n-1)!}\epsilon_{\nu\nu_2\cdots\nu_n}\xi_{\mu_2}^{\nu_2}\cdots\xi_{\mu_n}^{\nu_n}\\
&\quad\times\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}\xi_{\mu_1'}^{\nu_1'}\xi_{\mu_2'}^{\nu_2'}\cdots\xi_{\mu_n'}^{\nu_n'}\\
&=-\frac{1}{\vec{\mathcal{D}}^2}\frac{1}{(n-1)!}\epsilon_{\nu\nu_2\cdots\nu_n}\xi_{\mu}^{\nu_1'}\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}\\
&\quad\times N_{\nu_2\nu_2'}N_{\nu_3\nu_3'}\cdots N_{\nu_n\nu_n'}. \qquad \text{(B2)}
\end{aligned}$$

This expression is particularly useful if we consider the quantity

$$\begin{aligned}
\xi_\mu^{\nu'}M_\mu^\nu&=-\frac{1}{\vec{\mathcal{D}}^2}\frac{1}{(n-1)!}\\
&\quad\times\epsilon_{\nu,\nu_2\cdots\nu_n}\epsilon_{\nu_1'\nu_2'\cdots\nu_n'}N_{\nu'\nu_1'}N_{\nu_2\nu_2'}\cdots N_{\nu_n\nu_n'}\\
&=-\frac{1}{\vec{\mathcal{D}}^2}\frac{1}{(n-1)!}\epsilon_{\nu,\nu_2\cdots\nu_n}\epsilon_{\nu',\nu_2\cdots\nu_n}\det N=-\delta_{\nu,\nu'}. \qquad \text{(B3)}
\end{aligned}$$

APPENDIX C: ORTHOGONAL COORDINATE SYSTEM

We will construct an orthonormal coordinate system with the basis vectors

$$\hat{\xi}_\alpha^{(s)}=\sum_{\nu=1}^{n}A_{s\nu}\xi_\alpha^\nu. \qquad \text{(C1)}$$

These basis vectors are orthogonal to $\vec{\mathcal{D}}$ since $\xi_\alpha^\nu\mathcal{D}_\alpha=0$. Since we require

$$\hat{\xi}_\alpha^{(s)}\hat{\xi}_\alpha^{(s')}=\delta_{ss'}, \qquad \text{(C2)}$$

we have immediately, on inserting Eq. (C1) into Eq. (C2), that $\delta_{ss'}=A_{s\nu}A_{s'\nu'}N_{\nu\nu'}$. If we take the determinant we obtain $1=\det A\det N\det\tilde{A}$ or, using Eq. (A3),

$$\det\tilde{A}\det A=\frac{1}{\det N}=\frac{1}{\vec{\mathcal{D}}^2}. \qquad \text{(C3)}$$

We can obtain a realization of the matrix $A_{s\nu}$ by constructing the $\hat{\xi}_\alpha^{(s)}$ directly using the basis set

$$\chi_\alpha^{(s)}=\sum_{\nu=1}^{n}e^{2\pi is\nu/n}\xi_\alpha^\nu \qquad \text{(C4)}$$

which are not orthonormal. However, they can be constructed to be orthonormal using the Gram-Schmidt orthogonalization process. Thus each $\hat{\xi}_\alpha^{(s)}$ is a linear combination of the $\chi_\alpha^{(s)}$ which is proportional to a linear combination of ξ_α^ν. Thus one can extract an explicit expression for the matrix A. This explicit expression is not needed here.

APPENDIX D: INVERSE OF MATRIX Q

We need the inverse of the matrix

$$Q_{\nu\nu'}=\hat{\xi}_\alpha^{(\nu)}\hat{\xi}_\beta^{(\nu')}\overline{M}_{\alpha\beta}. \qquad \text{(D1)}$$

If we use the expression given by Eq. (C1) for the basis vectors, we obtain

$$Q_{\nu\nu'}=A_{\nu\bar{\nu}}\xi_\alpha^{\bar{\nu}}A_{\nu'\bar{\nu}'}\xi_\beta^{\bar{\nu}'}M_\alpha^{\nu''}M_\beta^{\nu''}. \qquad \text{(D2)}$$

We then use the identity Eq. (B3) twice to obtain

$$Q_{\nu\nu'}=A_{\nu\bar{\nu}}A_{\nu'\bar{\nu}'}\delta_{\nu'',\bar{\nu}}\delta_{\nu'',\bar{\nu}'}=A_{\nu\bar{\nu}}A_{\nu'\bar{\nu}}. \qquad \text{(D3)}$$

This immediately tells us that

$$\det Q=\det A\det\tilde{A}=\frac{1}{\vec{\mathcal{D}}^2}. \qquad \text{(D4)}$$

We still need to construct the inverse of Q. It is easy to see that this is given by the product $Q^{-1}=\tilde{A}^{-1}A^{-1}$. However, we only need the elements

$$\begin{aligned}
R_{\alpha\beta}&=\hat{\xi}_\alpha^{(\nu)}(Q^{-1})_{\nu\nu'}\hat{\xi}_\beta^{(\nu')}\\
&=A_{\nu s}\xi_\alpha^s(\tilde{A}^{-1})_{\nu\bar{\nu}}(A^{-1})_{\bar{\nu}\nu'}A_{\nu's'}\xi_\beta^{s'}\\
&=\xi_\alpha^s\delta_{\bar{\nu},s}\delta_{\bar{\nu},s'}\xi_\beta^{s'}=\xi_\alpha^s\xi_\beta^s. \qquad \text{(D5)}
\end{aligned}$$

APPENDIX E: EIGENVALUE PROBLEM

We need to find the six eigenvalues for the matrix Q defined by Eq. (87). The analysis can be carried out by looking at the action of Q when acting on the six basis vectors: $\psi_1=\hat{V}_\mu\hat{z}_\nu$, $\psi_2=\hat{V}_\mu\hat{c}_\nu$, $\psi_3=\hat{b}_\mu\hat{z}_\nu$, $\psi_4=\hat{b}_\mu\hat{c}_\nu$, $\psi_5=\hat{k}_\mu z_\nu$, $\psi_6=\hat{k}_\mu\hat{c}_\nu$, where we have introduced $\hat{c}_\nu=\epsilon_{\nu\nu'}\hat{z}_{\nu'}$, and $\hat{b}=\hat{k}\times\hat{V}$. The action of Q acting on these states is given by

$$Q\psi_1=(1+z^2+\tilde{V}^2)\psi_1+ik\psi_4, \qquad \text{(E1)}$$

$$Q\psi_2=(1+\tilde{V}^2)\psi_2-ik\psi_3, \qquad \text{(E2)}$$

$$Q\psi_3=(1+z^2)\psi_3-ik\psi_2+i\vec{k}\cdot\hat{V}\psi_6, \qquad \text{(E3)}$$

$$Q\psi_4=\psi_4+ik\psi_1-i\vec{k}\cdot\hat{V}\psi_5, \tag{E4}$$

$$Q\psi_5=(1+z^2)\psi_5+\tilde{V}^2\hat{k}\cdot\hat{V}\psi_1, \tag{E5}$$

$$Q\psi_6=\psi_6+\tilde{V}^2\hat{k}\cdot\hat{V}\psi_2. \tag{E6}$$

We see that these equations decouple into two cubic systems (ψ_1,ψ_4,ψ_5) and (ψ_2,ψ_3,ψ_6). We then only need the products $\lambda_1\lambda_2\lambda_3$ and $\lambda_4,\lambda_5\lambda_6$. It is well known (see Ref. [35]) that if one has a cubic characteristic equation

$$\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0, \tag{E7}$$

then the product of the three roots is given by $\lambda_1\lambda_2\lambda_3=-a_0$. In the present case we easily find for the two sets

$$a_0^{(1)}=-(1+z^2)(1+z^2\tilde{V}^2+k^2)-(\vec{k}\cdot\vec{\tilde{V}})^2 \tag{E8}$$

and

$$a_0^{(2)}=-(1+z^2)(1+\tilde{V}^2)1s-+k^2-(\vec{k}\cdot\vec{\tilde{V}})^2. \tag{E9}$$

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