COMMENTS

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Comment on ''Universal formulas for percolation thresholds. II. Extension to anisotropic and aperiodic lattices''

F. Babalievski*
Institute for Computer Applications 1 (ICA1), University of Stuttgart, 70569 Stuttgart, Germany
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Recently S. Galam and A. Mauger [Phys. Rev. E **56**, 322 (1997)] proposed an approximant which relates the bond and the site percolation threshold for a particular lattice. Their formula is based on a fit to exact and simulation results obtained earlier for different periodic and aperiodic lattices. However, the numerical result for an aperiodic dodecagonal lattice does not agree well with the proposed formula. I present here new and more precise data for this and other aperiodic lattices. The previously published value for the dodecagonal lattice is confirmed. The reason for the deviation from the Galam and Mauger approximant is discussed. [S1063-651X(99)09101-1]

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In a recent paper [1], Galam and Mauger proposed a formula which relates the site and bond percolation thresholds for a given lattice. This formula is a modification of a previous one [2] designed to predict the percolation thresholds for a variety of periodic lattices. In the new Galam and Mauger approximant (NGMA) the (mean) coordination number of a lattice (z) is replaced by the effective one (z_{eff}):

$$p_c = p_0[(d-1)(z_{\mathrm{eff}}-1)]^{-a} d^{\,b},$$

where d is the spatial dimension, p_0 and a are free parameters, while b is either equal to a (for bond dilution) or $b=0$ (for site dilution). Distinct parameter sets for p_0, a divide lattices into separate classes. The formula provides a connection between the site and the bond percolation threshold in a particular class. So, if (say) the bond percolation threshold is known for a lattice (and one guesses somehow to which class this lattice belongs), then one can estimate z_{eff}. After replacing p_0 and a with their values for site percolation (within the same class) one could obtain a value for the site percolation threshold. In the same manner one could start from a site percolation threshold and to get the bond threshold.

The NGMA was tested on simulation and exact values for 17 lattices and among them six lattices based on aperiodic tilings. The values for two variants of the octagonal and dodecagonal aperiodic lattices were extracted from my computer simulation results [3]. It appeared that some of these values do not fit well to the NGMA. The authors of [1] supposed that the value for bond percolation on the ferrovariant of the dodecagonal lattice was estimated numerically with rather low precision. Instead of my result $p_c=0.495$, they suggest that the correct value should be close to 0.475, so it would fit much better to the theoretical curve (the NGMA). Since the bond percolation results given in my paper were stated to be preliminary, it is reasonable to be suspicious about them.

Here I present new and more precise numerical data for these thresholds as well as for the bond percolation thresholds for the other aperiodic lattices used for testing the NGMA — see the left part of Table I.

These values were obtained by computer simulation analogous to that in [3]. The lattice sizes were much larger here: up to 500×500 bond lengths (the longer bond) and the computational efforts were between 100 and 1000 times larger for the different lattices. For this work I used other pseudorandom number generators (PRNG). For most of the simulation the `drand48()` PRNG from the standard GnuC distribution was used. I used also the `ran2` generator from [5] and the recently discussed [6] four tap shift-register gen-

TABLE I. New numerical results for the bond percolation thresholds (p_c) of questioned aperiodic lattices. The values on the right are the rigidity (bond) percolation thresholds (p_c^r) taken from [8]. The column with 4/z corresponds to the Maxwell approximant for the rigidity threshold. Some of the lattices are always floppy (a.f.) in view of the rigidity percolation model. The error estimates are put in parentheses and concern the underlined digit.

	p_c	NMGA	p_c^r	4/z
Penrose	0.4767(5)	0.4748	a.f.	1
Penrose *(f)*	0.429(1)	0.435 . . .	0.836(2)	0.840 . . .
Octagonal	0.478(3)	0.4773	a.f.	1
Octagonal *(f)*	0.402(5)	0.406 . . .	0.769(2)	0.774 . . .
Dodecagonal	0.538(1)	0.5419 . . .	a.f.	1.102 . . .
Dodecagonal *(f)*	0.4950(5)	0.475 . . .	0.938(1)	0.937 . . .

*Permanent address: Institute of General and Inorganic Chemistry, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria.

erator $R(471,1586,6988,9689)$. The results coincided within less than 1/10 of a percent in the mean of the cumulative Gaussian distribution which was used for a fitting function. The fits just with a polynomial of third degree gave almost the same results for the places (the values of *p)* where the spanning probability curve reaches a value of 1/2. One should notice also that the value for the nonmodified Penrose lattice confirms the earlier results.

So I could claim (within the rigor of a Monte Carlo simulation) that the bond percolation threshold for the ''ferro''modification of the dodecagonal aperiodic lattice is indeed 0.495 within error bars of ± 0.001. In fact, a more precise value could be extracted from the data of Fig. 1, $p_c = 0.4950 \pm 0.0005$. The error bars would be even smaller if one disregarded the shift in the spanning probability curve for $L = 300$.

Now one could draw the conclusion that deviations of this percolation threshold from the NGMA do exist and one has either to accept that the approximant is good within a larger interval of deviations, $|p_c - p_c^e| = \Delta < 0.014$, or to search for a way to improve the approximant itself.

The reason for the discrepancy could be suggested in the line of thinking that van der Marck has presented recently [4] for site percolation. He paid attention to certain lattices for which there are sites positioned on specific positions that do not contribute to percolation at all — e.g., in 2D these are the ''star sites'' inside a triangle.

Indeed for bond percolation such entirely irrelevant bonds are impossible. However, one can show that some bonds contribute much less to the short-range connectivity than the others. This is the case of the questioned aperiodic dodecagonal lattice with ''ferro''bonds. This lattice is a modification of the original dodecagonal lattice constructed by Socolar, which corresponds to an aperiodic tiling of a plane with a hexagon, a square, and a rhombus. The ''ferro''modifications consist in adding new bonds along the short diagonals of the rhombuses. Now one can show that these new bonds contribute less to the short-range percolative connectivity.

An estimation for this contribution could be done by stating a percolation problem for ''one rhombus system.'' We have five bonds arranged in a rhombus where the fifth bond is along the short diagonal. The question is what is the probability for a bond to be a ''cutting'' one for percolation between sites on the acute angles (i.e., what is the probability that the extraction of this bond destroys the connection between the two sites). One can easily see that this probability is $2p^3(1-p)^2$ (the two zigzag paths) for the ferrobond, which gives $\approx 1/16$ for $p \approx 1/2$ (close to the questioned threshold value). The same value (near to $p = 1/2$) is $\approx 3/16$ for the other bonds $[4p^3(1-p)^2 + p^2(1-p)^3 + p^4(1-p)]$.

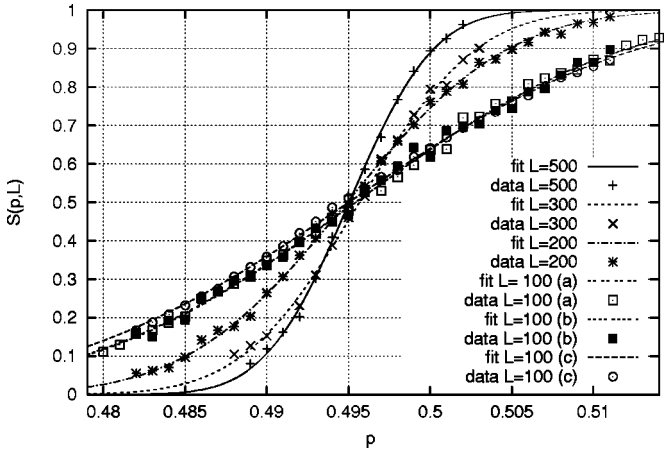

FIG. 1. The frequency of the occurrence of spanning cluster $S(p,L)$ for dodecagonal lattices with different size *(L)*. Each data point is averaged over 1000 realizations. The data sets for $L = 100$ are collected with the following PRNG: (a) `drand48()`, (b) `ran2()`, (c) $R(471,1586,6988,9689)$.

Of course, things change when one considers percolation between the sites connected directly with the ''ferro''bond. The cutting probability (again near $p = 1/2$) for this bond is already $\approx 9/32[p(1-p^2)^2]$ and the cutting probability of each of the other bonds is $\approx 3/32[p^2(1-p)^2(1+p)]$.

It appears that the chance to be a cutting bond is the same for the two types of bonds if both pairs of opposite vertices of the rhombus are considered . But one has to mention (e.g., on Fig. 1(g) in [1]) that there are only one or two outgoing bonds from the obtuse angle vertices, while the acute vortices have two to four outgoing bonds.

So if one considers the ''spreading of connectivity'' one more step beyond the two diagonals of the rhombus, the contribution of the ''ferro''bond is less significant (by a factor of 2 approximately).

In contrast to the above, the same analysis for central-force rigidity percolation [7] on the same lattices [8] shows that the ''ferro''bond is always a cutting bond within the questioned rhombus. There is a simple approximant for the rigidity thresholds — the so-called Maxwell approximant [7,8]. It is just twice the other mean-field-like approximant: the Scher and Zallen prediction for (connectivity) bond percolation thresholds in two dimensions ($p_c \approx 2/z$,) where z is the mean coordination number). One can see from Table I that the estimates for rigidity percolation models, where each bond has (almost) equal significance for spreading the rigidity, agree with the Maxwell approximant extremely well.

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