Generation of light spatiotemporal solitons from asymmetric pulses in saturating nonlinear media

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The dynamics of multidimensional spatiotemporal solitons, i.e., light bullets, in saturating nonlinear media is considered. The analytical approach based on the variation method and the numerical simulations show that light bullets can be generated for a large range of parameters corresponding to the initially asymmetric pulses. [S1063-651X(99)08801-7]

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Recently, growing attention has been devoted to the study of ultrashort-pulse propagation in nonlinear media [1]. In particular, the spatiotemporal dynamics of electromagnetic (em) pulses has been the subject of intense investigations. A stable solitonic pulse completely localized in space and time due to the nonlinear effects is called a light bullet [2]. The evolution of the light bullets is described by the $(3+1)$-dimensional nonlinear Schrödinger equation (NSE) containing three ''transverse'' dimensions and one propagation direction.

It has been shown by Vakhitov and Kolokolov that the NSE with the nonlinearity saturation admits three-dimensional soliton solutions that are stable for infinitely small perturbations [3]. Such solutions, i.e., the light bullets, are characterized by a balance of diffraction and dispersion effects, respectively, with nonlinear self-focusing in space and time. This balance implies that its spatial width R and its temporal duration T satisfy the condition $R=T(-kD)^{-1/2}$, where k is wave vector and $D=d^2k/d\omega^2$ is group velocity dispersion (GVD), which is assumed to be anomalous $D<0$. Recently, it has been shown numerically that under certain circumstances the light bullets are stable even for large perturbations and their dynamics resembles the dynamics of a soliton solution of an integrable system [4]. In our recent publication we considered the conditions for the light bullets generation in media with saturating nonlinearity [5]. If an initial profile of the pulse is close to the stable equilibrium one (the exact numerical solution), the pulse quickly attains the profile of the ground-state soliton, in agreement with the Vakhitov-Kolokolov theory. However, as predicted by our analytical approach and confirmed by numerical simulations due to the exceptional robustness, light bullets can be generated from a large range of initial parameters even far from stable equilibrium.

In Ref. [5] it is considered that the initial pulse parameters satisfy the ''symmetry'' condition $R_0=T_0(-kD)^{1/2}$, where R_0 and T_0 are the initial spatial width and temporal duration of a Gaussian input pulse. It is complicated to realize such a condition experimentally.

The purpose of this paper is to investigate the light bullets generation using an initial ''asymmetric'' pulse with the profile significantly different from the equilibrium one. For the NSE with a saturating nonlinearity the exact analytical methods to derive nonstationary solutions do not exist. One has to resort to computer simulations, even if the analytical approach is needed for a qualitative understanding of the numerical results obtained. Consequently, in order to treat the evolution of asymmetric pulses, we first generalize our analytical approach by removing the symmetry condition. This approach is then applied to a concrete saturating nonlinearity. Finally, guided by analytical results, the numerical simulations of the pulse dynamics are carried out.

The dynamics of an em pulse propagating in a nonlinear medium is based on the analysis of the NSE

$$2ik\left(\frac{\partial\mathcal{E}}{\partial z}+\frac{1}{v_g}\frac{\partial\mathcal{E}}{\partial t}\right)+\Delta_\perp\mathcal{E}-kD\frac{\partial^2\mathcal{E}}{\partial t^2}+2k^2\frac{\delta n(|\mathcal{E}|^2)}{n_0}\mathcal{E}=0, \tag{1}$$

where $\mathcal{E}$ is a slowly varying field envelope, v_g is the group velocity of the pulse propagating along the z axis, n_0 and $\delta n(|\mathcal{E}|^2)$ are, respectively, linear and nonlinear optical indices, and $\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the two-dimensional Laplacian describing beam diffraction. In dimensionless form Eq. (1) can be rewritten as

$$i\frac{\partial E}{\partial z}+s\frac{\partial^2 E}{\partial\tau^2}+\Delta_\perp E+f(|E|^2)E=0. \tag{2}$$

Taking into account that $\tau=t-z/v_g$ is a retarded time variable, the field envelope E is $\mathcal{E}$ redefined according to the nonlinearity under consideration and $s=\pm1$ corresponds respectively to the anomalous and normal GVD cases. The Lagrangian density corresponding to Eq. (2) is

$$L=\left|\frac{\partial E}{\partial x}\right|^2+\left|\frac{\partial E}{\partial y}\right|^2+s\left|\frac{\partial E}{\partial \tau}\right|^2+\frac{i}{2}\left(E\frac{\partial E^*}{\partial \tau}-E^*\frac{\partial E}{\partial \tau}\right)-F(|E|^2) \quad (3)$$

where the asterisk denotes a complex conjugate and $F(u)=\int_0^u f(u')du'$.

To analyze rather complex dynamical properties of the pulses governed by Eq. (2) we use a variational approach [6]. This approach determines the relations between the characteristic parameters of the localized solution approximated by a trial function. As the trial function we use Gaussian-shaped pulse

$$E=A(z)\exp\left(-\frac{x^2}{2X^2(z)}-\frac{y^2}{2Y^2(z)}-\frac{\tau^2}{2T^2(z)}+i\psi\right), \quad (4)$$

where $\psi=x^2b_x(z)+y^2b_y+\tau^2c(z)+\phi(z)$. The self-similar evolution of the pulse is parametrized by the z-dependent amplitude A, transverse widths X and Y, temporal duration T, and phase ϕ. The parameters b_x and b_y are the wave front curvatures and c is the chirp parameter (the ''temporal curvature'').

After substitution of Eq. (4) into Eq. (3) and integration over x, y, and τ, the average Lagrangian obtained depends only on optimizing the z-dependent parameters of the trial function. By demanding that the variation of the average Lagrangian with respect to each of these parameters be zero, the corresponding set of Euler-Lagrange equations is derived,

$$\frac{d^2\mathbf{R}}{dz^2}=-2\frac{\partial}{\partial \mathbf{R}}V(X,Y,T), \quad (5)$$

$$\frac{d^2T}{dz^2}=-2s\frac{\partial}{\partial T}V(X,Y,T), \quad (6)$$

where $\mathbf{R}=(X,Y)$ and the effective potential V has the form

$$V(X,Y,T)=\frac{1}{X^2}+\frac{1}{Y^2}+\frac{s}{T^2}-\frac{K(A^2)}{A^2}, \quad (7)$$

with the nonlinearity function

$$K(u)=\frac{8}{\sqrt{\pi}}\int_0^\infty dp\; p^2F(ue^{-p^2}). \quad (8)$$

During the pulse evolution the ''energy'' $N=A^2XYT$ is conserved. The wave front curvatures are $b_x=(1/4X)dX/dz$ and $b_y=(1/4Y)dY/dz$, while the chirp parameter is $c=(1/4sT)dT/dz$.

Equations (5)–(8) are equivalent to those describing the dynamics of a particle in a three-dimensional potential well. Using this analogy we can acquire a deeper physical understanding of the dynamics of light pulse. Let us first examine the possibility of steady self-trapping of light pulse corresponding to the situation when the nonlinearity exactly balances both diffraction and dispersion. Such an equilibrium corresponds to the absolute extremum of the potential [i.e., the right-hand sides of Eqs. (5) and (6) are zero]. It is obvious that in the case of normal GVD ($s=-1$) the absolute extremum does not exist. Consequently, a three-dimensional localization of the em pulse in the medium is not possible independently of the structure of saturating nonlinearity. A comprehensive treatment of the em pulse dynamics in Kerr media with normal GVD can be found in Ref. [7]. In what follows we concentrate on the case of anomalous GVD ($s=1$). The equilibrium parameters of the pulse are $X_e=Y_e=T_e=A_0/K(A_0^2)^{1/2}$, where A_0 is the initial amplitude of the pulse. It is straightforward to show that the absolute minimum of the potential exists if the field amplitude is larger than the critical one A_c, whose value depends on the chosen saturating nonlinearity. This stable and spherically symmetric solution is the light bullet.

In the general formalism presented above we did not use an explicit form for the nonlinear function $f(|E|^2)$. In order to investigate the dynamical properties of nonsteady solutions, in the subsequent analysis we will consider a nonlinear term of the form

$$f(|E|^2)=|E|^2-|E|^4. \quad (9)$$

This kind of nonlinearity has been widely applied in different domains of research [8]. Recent measurements of organic materials show that, for instance, polydiacetylene *para*-toluene sulfonate (PTS) exhibits this kind of saturation nonlinearity [9]. For PTS the second term in Eq. (9) can be of the same order as or even larger than the first one. The nonlinearity function K [see Eq. (8)] is now $K(A^2)=\alpha A^4-\beta A^6$, where $\alpha=2^{-3/2}$ and $\beta=2\times 3^{-5/2}$. Consequently, for the equilibrium radius of the pulse one gets

$$X_e^2=Y_e^2=T_e^2=\frac{2}{\alpha A_0^2-2\beta A_0^4}. \quad (10)$$

The equilibrium solution is stable against small perturbations if $A_0>A_c\approx 0.6$, in agreement with Ref. [5].

Now we examine the dynamical behavior of a pulse that is initially far from its symmetric equilibrium corresponding to the light bullet. Integration of Eqs. (5)–(7) gives

$$\frac{1}{4}\left(\frac{dX}{dz}\right)^2+\frac{1}{4}\left(\frac{dY}{dz}\right)^2+\frac{1}{4}\left(\frac{dT}{dz}\right)^2+V(X,Y,T)=V(X_0,Y_0,T_0), \quad (11)$$

where

$$V(X,Y,T)=\frac{1}{X^2}+\frac{1}{Y^2}+\frac{1}{T^2}-\frac{\alpha N}{XYT}+\frac{\beta N^2}{X^2Y^2T^2}. \quad (12)$$

The conservation of energy $N=A^2XYT=A_0^2X_0Y_0T_0$ is used in deriving Eqs. (11) and (12). For simplicity, zero initial curvatures $b_x(0)=b_y(0)=c(0)=0$ are assumed.

For the pulse energy exceeding the critical one ($N_c=35.3$), two absolute extrema, a minimum and a maximum, appear, i.e., a three-dimensional potential well is created. For illustration, for the energy $N=36$, we plot in Fig. 1 the corresponding two-dimensional potential as a function of dura-

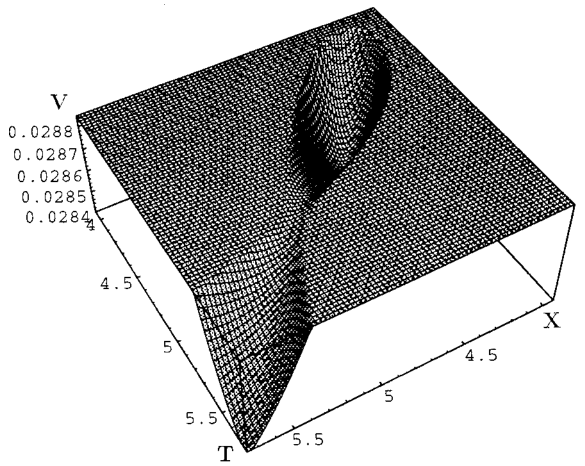

FIG. 1. Two-dimensional potential well as a function of duration T and width X for the energy $N=36$ slightly exceeding the critical one.

tion T and width X assuming that the transverse spatial symmetry of the pulse holds during its evolution $(X=Y)$. If the initial width and duration are inside the well, the self-trapping occurs preventing the spreading of the pulse. With increasing energy the potential well deepens and the trapping area becomes wider, as can be seen in Fig. 2, where the energy is $N=80$. The minimum at the bottom of the potential well corresponds to the stable equilibrium, i.e., to the light bullet. The maximum corresponds to the unstable equilibrium. For convenience, only the potential between its extrema is plotted in Figs. 1 and 2.

The effective particle trapped in the potential well in general follows a complex trajectory around the stable equilibrium. The variational approach does not account for any attenuation that obviously leads to the relaxation on the bottom of potential well, as can be seen by numerical simulations of the NSE.

In order to check the predictions of our analytical approach concerning the appearance of a potential well that

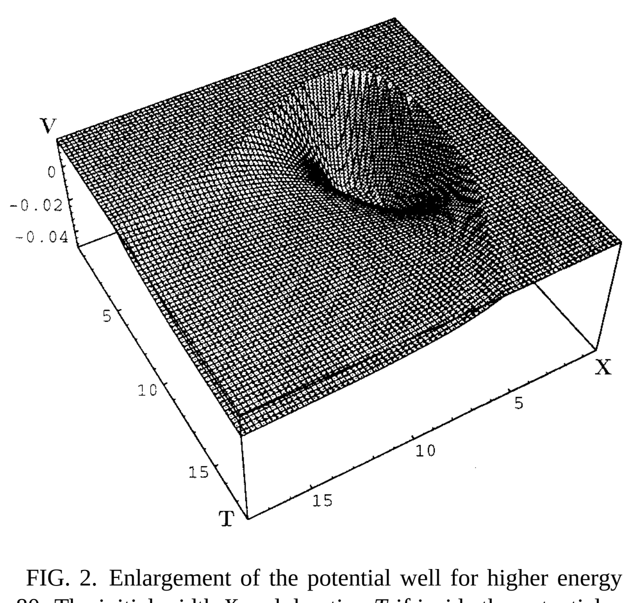

FIG. 2. Enlargement of the potential well for higher energy $N=80$. The initial width X and duration T if inside the potential well lead to the light bullet generation.

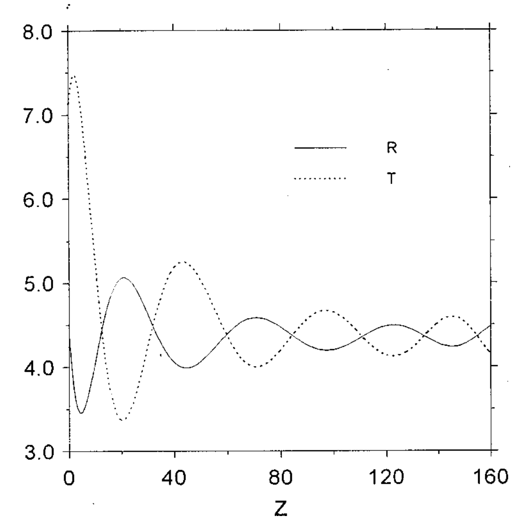

FIG. 3. Numerical simulations of the Schrödinger equation (2). The width R and duration T approach their common equilibrium value.

delineate the trapping region, the numerical simulations of Eq. (2) for cylindrically symmetric pulses are carried out $[\Delta_\perp=\partial^2/\partial r_\perp^2+r_\perp^{-1}\partial/\partial r_\perp$, where $r_\perp=(x^2+y^2)^{1/2}]$. We use a finite-difference method in both the radial and temporal directions for an initially Gaussian-shaped pulse $|E(0,r_\perp,\tau)|=A_0\exp(-r_\perp^2/2R_0^2-\tau^2/2T_0^2)$. The simulations show that the pulse above a critical energy is trapped provided its initial parameters belong to the range reasonably close to the one predicted by the analytical approach (see Fig. 3 in Ref. [5]). The numerical simulations for different values of the pulse energy (up to $N=150$) and different levels of asymmetry $(R_0/T_0\approx0.1-10)$ are carried out. The dynamics of the pulse sizes in the transverse and temporal directions with the asymmetric initial conditions $R_0=4$ and $T_0=7$ for the energy $N=80$ is given in Fig. 3. Both size parameters undergo a damped oscillation around the same equilibrium state with the period closed to the one that follows from the variational approach. Thus, in spite of the initial asymmetry in the final stage of evolution, the symmetric stable pulse, i.e., light bullet, will be formed. For other initial conditions studied the pulse dynamics exhibits essentially the same behavior as in Fig. 3. In these simulations the modulation instability that may lead to the spontaneous decay of the pulse into a number of fragments did not take place. A longitudinally modulated cylindrical beam can indeed spontaneously break into a train of light bullets, as it has been shown in Ref. [10]. Such a case corresponds to the conditions $N\rightarrow\infty$ and $R_0/T_0\rightarrow0$ and it is beyond the scope of our investigation in this Brief Report.

In conclusion, we considered the light bullet generation from initially asymmetric input pulses propagating in saturating nonlinear media. The numerical simulations confirm the analytical prediction based on variation method that an asymmetric pulse with the energy above a critical one evolves towards the light bullet for a large range of parameters. Light bullets are exceptionally robust objects that can be generated even far from stable equilibrium. It is much less complicated to realize the initial pulse parameters without restrictions on their symmetry in an experiment.

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