Stochastic theory of freeway traffic

R. Mahnke*
Fachbereich Physik, Universität Rostock, D-18051 Rostock, Germany

J. Kaupužs
Institute of Mathematics and Computer Science, University of Latvia, 29 Rainja Boulevard, LV-1459 Riga, Latvia

(Received 30 March 1998; revised manuscript received 16 September 1998)

Extending a new stochastic approach to congestion in traffic flow [R. Mahnke and N. Pieret, Phys. Rev. E **56**, 2666 (1997)], the nucleation, growth, and condensation of car clusters in a circular one-lane freeway traffic model is investigated. In analogy to usual aggregation phenomena such as the formation of liquid droplets in supersaturated vapor, the clustering behavior in traffic flow is described by the Master equation. At overcritical densities the transition from the initial free-particle situation (free flow of vehicles) to the final congested cluster state, where one big aggregate of cars has been formed, is shown. In dependence on the concentration of cars on the road, the stationary solution of the Master equation is derived analytically. The obtained fundamental diagram as a flow-density relation indicates clearly the different regimes of traffic flow (free jet of cars, coexisting phase of jams and isolated cars, and highly viscous heavy traffic). In the (thermodynamic) limit of an infinite number of vehicles on an infinitely long road, the analytical solution for the fundamental diagram is in agreement with experimental traffic flow data. As a particular example, we take into account measurements from German highways presented by Kerner and Rehborn. [S1063-651X(99)00301-3]

PACS number(s): 02.50.Ey, 05.70.Fh, 89.40.+k

I. INTRODUCTION

The aggregation of particles out of an initially homogeneous situation is well known in physics. Depending on the system under consideration and its control parameters, the cluster formation in a supersaturated (unstable) situation has been observed in nuclear physics as well as in other branches. We mention the well-known example of condensation (formation of liquid droplets) in undercooled water vapor. The formation of binding states as an aggregation process is related to self-organization phenomena [1]. The emergence of car clusters (jams) in traffic flow has been studied by different authors (see, e.g., [2–7]) and could be summarized in the so-called fundamental diagram of traffic flow. Up to now there have been numerous discussions about measured and calculated flow-density curves on freeways. A complete understanding of traffic flow should be based on empirical investigations as well as on theoretical descriptions. Well-defined mathematical equations in analogy to established physical models are very helpful to compare theory and experiment. The support given by relatively simple traffic systems such as particle hopping simulations (cellular automata [8–11]) and deterministic car following theory [12–14] becomes more and more important for practical traffic engineering.

Since the pioneering work by Prigogine and Herman [15] on the kinetic theory of vehicular traffic, cars have been considered as interacting particles. Based on a new stochastic description by Mahnke and Pieret [16], the clustering behavior in an initially homogeneous traffic flow is understood as nucleation, growth, and condensation of car clusters (jams) on a freeway. The purpose of this paper is to improve the stochastic description by taking into account the experimentally known vehicle density in the jam or the real distance between cars in the congested phase, respectively. We also would like to improve the current understanding of the fundamental diagram of traffic flow in analogy to aggregation phenomena in metastable and unstable systems, e.g., van der Waals gases, in comparison to experimental data from German highways presented by Kerner and Rehborn [17–19]. These data are a particular example of a large amount of high-resolution traffic values recorded on freeways in different countries by induction loop detectors [20,21,7].

II. EXPERIMENTAL SITUATION ON HIGHWAYS

Vehicular traffic flow on freeways (and in cities too) tends to suffer from a jamming transition when the total traffic density exceeds a critical threshold value. The phenomenon of car cluster formation (traffic jams) has been studied, as already mentioned, by different approaches and has to be related to the large variety of experimental observations [14,22,2,20,17–19,8,6,21,7]. The accumulation of a great number of measurements of traffic flow on Dutch [20], German [17–19,21], Japanese [14], U.S. [22], and other highways shows common macroscopic properties. In general, German data do not differ from data from elsewhere. The interpretation of experimental measurements is rather complex and the data are highly nonstationary. However, there are basically three different regimes of traffic (free flow, highly correlated or synchronized flow, and heavy congested flow) and phase transitions between them. A short-time localized perturbation is able to generate a phase transition and later on a flow of slowly moving bounded vehicles can be observed on a highway for several hours.

The description of traffic flow has to be based on two fundamental essentials: First, the variability of vehicle per-

*Electronic address: mahnke@darss.mpg.uni-rostock.de

formance and driver behavior demands for a stochastic approach of description. Second, the reality can only be described based on empirical investigations. Moreover, since external parameters have a remarkable influence on driver behavior, a specific description of traffic flow can only be valid for one specific point along the highway network, and for one specific time period. This is the reason why we have used empirical data of Kerner and Rehborn [18] in detail to find values of control parameters for our model. In general, we allow that these values are slightly dependent on the specific experimental situation.

Below we have discussed the experimental situation of Ref. [18] to illustrate the complexity of real traffic. In Ref. [18] nonstationary traffic flow related to the appearance of complex time-space structures has been discussed on the basis of flux-density measurements made on a certain section of German highway A5. The so-called ''synchronized'' traffic was discussed. A characteristic feature of this regime of traffic is that the traffic flow is synchronized in all (three) lanes of the road. The density in this case is larger than in free traffic at the same value of flux, and the time evolution of flux and density measured at a given coordinate by averaging over relatively short time intervals (1 min) exhibits a very complex nonstationary behavior. Based on measurements on Friday, August 25, 1995, when 10 different traffic jams occurred on highway A5, a behavior of nonstationary jams has been considered. In particular, a random emergence of a nonstationary jam, nonstationary growth of the amplitude and/or of the width of jams, an extinction of a jam, a merger of a few jams into one jam, as well as an appearance of nonstationary moving blanks inside wide jams have been discussed. It was found that for wide jams the average velocity of downstream front (the velocity of backwards motion of a jam) is nearly the same, about -15 km/h. The fronts of jams, where the flux and the average speed abruptly change, have been interpreted as nonstationary and nonhomogeneous states of synchronized traffic flow. The results of measurements have been reflected in the fundamental diagram of traffic flow. The measured average flux and density in the outflow from the jam was 1800 vehicles/h and 20 vehicles/km, respectively. The average density inside the jam was found to be 140 vehicles/km.

In conclusion, the experimental situation on highways is rather complex (see also Refs. [17,19]). Nevertheless, we believe that some general features of traffic flow exist. Considerable attempts were made to investigate scaling properties to find power-law behavior in which the power spectral density is proportional to $1/f^{\alpha}$ (f is the frequency). The exponent α is obtained by fitting traffic flow data [9,21].

Our purpose is to give a relatively simple description of traffic flow. Our model does not include all the complexity of the experimental situation, but it includes the most important general features. It allows us to describe spontaneous formation of traffic jams and to interpret different regimes of traffic flow as well as to calculate the fundamental diagram (flux-density plane) in comparison with experimental data.

III. THE CAR CLUSTER MODEL AND ITS STOCHASTIC DESCRIPTION

Here we consider a model of traffic flow on a one-lane road according to which N cars are moving along a circle of length L. The motivation of the model and the details of its stochastic description have already been discussed in [16]. If a road is crowded by cars, each car requires some minimal space or length which, obviously, is larger than the real length of a car. We call this the effective length l of a car. The distance between the front bumpers of two neighboring cars, in general, is $l+\Delta x$. The distance Δx can be understood as the headway between two ''effectice'' cars which, according to our definition, is always smaller than the real bumper-to-bumper distance. The maximal velocity of each car is v_{max}. The desired (optimal) velocity v_{opt}, depending on the distance between two cars Δx, is given in dimensionless variables $w_{opt}=v_{opt}/v_{max}$ and $\Delta y=\Delta x/l$ by the formula

$$w_{opt}(\Delta y)=\frac{(\Delta y)^2}{d^2+(\Delta y)^2}, \tag{1}$$

where the parameter $d=D/l$ is the interaction distance. D is the distance between two cars corresponding to the velocity value $v_{max}/2$. According to Eq. (1) the optimal velocity is represented by a sigmoidal function with values ranging from 0, corresponding to zero distance between cars, to 1, corresponding to an infinitely large distance or absence of interaction between cars. We have chosen one of the simplest functions having these properties, since this allows analytical solutions. Our choice is also justified physically, since it provides a good agreement with the experimental data (cf. Sec. VIII). In nonlinear dynamics based on car following models [12–14] another functional representation was applied. In particular, the optimal velocity function used in Ref. [14] reads in our notation

$$w_{opt}(\Delta y)=\frac{\tanh(d_1\Delta y-d_2)+\tanh(d_2)}{1+\tanh(d_2)}. \tag{2}$$

The most important difference between the optimal velocity functions (1) and (2) is that in our case [Eq. (1)] the first derivative of $w_{opt}(\Delta y)$ with respect to spacing Δy vanishes at $\Delta y=0$. The property that the first derivative should be small at vanishing distances ($\Delta y\to 0$) is important to understand the qualitative behavior of traffic flow at large densities. Details are discussed in Sec. VII. Following our intention to minimize the number of control parameters, we use in Eq. (1) only one free parameter d with a precise understanding. Two parameters d_1 and d_2 in Eq. (2) obviously allow a better fit with data, but this is not so important in our investigation.

Measurements on highways have shown that the density of cars in congested traffic ϱ_{clust} is independent of the size of the dense phase (jam). The quantity ϱ_{clust} is known experimentally (cf. Sec. II). As a consequence, the distance between jammed cars, the spacing Δx_{clust}, is well known and has to be treated as a given measured quantity. Therefore, it is assumed that the distance between cars inside the cluster is $\Delta x_{clust}=l\Delta y_{clust}=\mathrm{const}\geqslant 0$. In earlier work (see [23,16]) the case $\Delta x_{clust}=0$ was considered. This corresponds to the special situation where the congested cars are close together without spacing, and according to Eq. (1) the velocity of all cars bounded in the jam is zero. This simple approximation dealing with a nonmoving jam of highest car density is not very often valid on highways. From the physical point of view, the experimentally known jam density ϱ_{clust} or the spacing in jams Δx_{clust}, respectively, connected by the relationship $\varrho_{clust}=1/(l+\Delta x_{clust})$, is a given measured parameter.

The length of the cluster (jam) depending on the number of congested cars n is defined by

$$L_{clust}=ln+(n-1)\Delta x_{clust}. \tag{3}$$

According to this, the average distance $\Delta x_{free}=l\Delta y_{free}$ between two cars outside the jam (or free cars) distributed over the free part of the road with length $L_{free}=L-L_{clust}$ is given by

$$\Delta y_{free}(n)=\frac{L/l-N-(n-1)\Delta y_{clust}}{N-n+1}, \tag{4}$$

where N is the total number of cars on the road (circle of length L).

The traffic flow is described as a stochastic process where adding a vehicle to a car cluster of size n is characterized by a transition frequency (attachment probability per time unit) $w_+(n)$ and the opposite process by a frequency $w_-(n)$. The number n of cars in the cluster is the stochastic variable which may have values from 1 to N. The situation of a single ''congested'' car $n=1$ is treated as a cluster of size 1, and therefore we do not consider transitions between states with $n=1$ and $n=0$. According to our model, only one car cluster exists at any given time. The basic equation for the evolution of the probability distribution $P(n,t)$ to find a cluster of size n at time t with probability P is known as the Master equation. The one-dimensional stochastic equation (see, e.g., [24,25]) reads

$$\begin{aligned}\frac{1}{\tau}\frac{dP(n,T)}{dT}=&\,w_+(n-1)P(n-1,T)\\&-[w_+(n)+w_-(n)]P(n,T)\\&+w_-(n+1)P(n+1,T),\end{aligned} \tag{5}$$

where $T=t/\tau$ is the dimensionless time. The time constant τ will be specified below.

The main task is to formulate expressions for both transition probabilities w_+ and w_-. As already explained in our previous work [23,16], we have assumed that the detachment frequency $w_-(n)$ or the average number of cars leaving the cluster per time unit is a constant independent of cluster size n. The ansatz for $w_+(n)$ is now corrected allowing for Δx_{clust} to be nonzero. Our general assumption is that a vehicle changes the velocity from $v_{opt}(\Delta x_{free})$ in free flow to $v_{opt}(\Delta x_{clust})$ in jam and approaches the cluster as soon as the distance to the next car (the last car in the cluster) reduces from Δx_{free} to Δx_{clust}. This assumption allows one to calculate the average number of cars joining the cluster per time unit or the attachment frequency $w_+(n)$. Thus, we have the ansatz which in dimensionless quantities reads

$$w_+(n)=\frac{b}{\tau}\,\frac{w_{opt}(\Delta y_{free}(n))-w_{opt}(\Delta y_{clust})}{\Delta y_{free}(n)-\Delta y_{clust}}, \tag{6}$$

$$w_-(n)=1/\tau=\text{const}, \tag{7}$$

where $b=v_{max}\tau/l$ denotes a dimensionsless parameter. The parameter τ is a time constant, which can be understood as the waiting time for the escape (detachment) of the first car out of the jam into free flow. This parameter characterizes the fastness of the driver's reaction or adaptation to a new situation (the road is free and one can start to move). According to this interpretation, our assumption that τ and w_- are constants is well justified. On the other hand, in a real situation the detachment of cars from the jam occurs more or less continuously, therefore the above interpretation of τ as a waiting time interval to escape is appropriate.

In any case, w_- is the average number of cars leaving the jam per time unit. This allows us to relate τ to the velocity of backwards motion of the jam v_{back}, i.e.,

$$v_{back}=\frac{l+\Delta x_{clust}}{\tau}-v_{opt}(\Delta x_{clust})=\frac{1}{\tau\varrho_{clust}}-v_{opt}(\Delta x_{clust}), \tag{8}$$

and to determine the value of parameter τ. According to the data of Ref. [18], we have $\varrho_{clust}=140$ vehicles/km and $v_{back}=15$ km/h, which yields the value of τ about 1.7 s or smaller.

The Master equation (5) can be rewritten as follows:

$$\begin{aligned}\frac{dP(n,T)}{dT}=&\,j(n-1\to n)-j(n\to n+1)-j(n\to n-1)\\&+j(n+1\to n),\end{aligned} \tag{9}$$

where $j(n-1\to n)$, $j(n\to n+1)$, $j(n\to n-1)$, and $j(n+1\to n)$ are probability fluxes which are equal to the corresponding terms in Eq. (5). The boundary conditions for Eq. (9) are

$$j(1\to 0)=j(0\to 1)=j(N\to N+1)=j(N+1\to N)=0. \tag{10}$$

Our purpose is to solve this equation and to extract from this solution information about the formation of traffic jams and about the various possible regimes of traffic flow depending on the parameters of the system, as well as to calculate the flux-density or the fundamental diagram of traffic flow. Finally, our aim is to compare the results of calculation with experimental data.

IV. STATIONARY PROBABILITY DISTRIBUTION OF TRAFFIC JAMS

Now let us consider the stationary solution $P(n)=\lim_{T\to\infty}P(n,T)$ of the Master equation (5) corresponding to the condition

$$\frac{dP(n,T)}{dT}=0. \tag{11}$$

The general solution $P(n)$ of Eq. (11) is well known and reads (see [24])

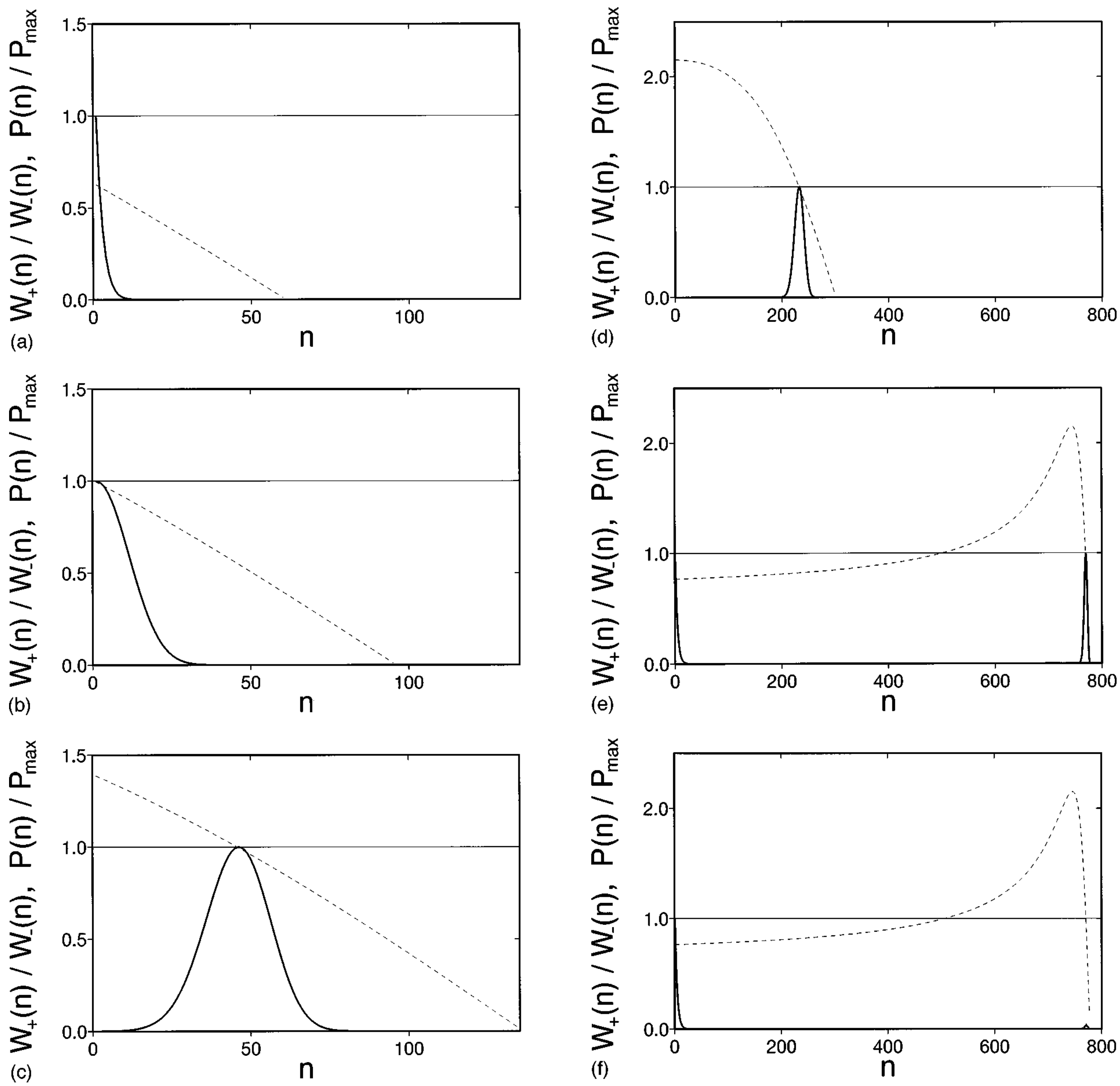

FIG. 1. Series of different stationary probability distributions $P(n)$ (solid lines) and ratios of transition rates $w_+(n)/w_-$ (dashed lines) showing the formation of a jam of size n depending on the total number of cars N on the road. The values of N and P_{max} are (a) $N=55$, $P_{max}=0.439$; (b) $N=96$, $P_{max}=0.070$; (c) $N=135$, $P_{max}=0.039$; (d) $N=300$, $P_{max}=0.045$; (e) $N=776$, $P_{max}=0.088$; (f) $N=777$, $P_{max}=0.227$. The parameters of the system are $L/l=1000$ ($L=5000$ m, $l=5$ m), $b=10$, $d=2.5$, and $\Delta y_{clust}=0.2$. The maximum of the probability distribution corresponds to the stable cluster size of congested cars.

$$P(n+1)=P(n)Q(n) \quad \text{with} \quad Q(n)=\frac{w_+(n)}{w_-(n+1)} \tag{12}$$

or

$$P(n)=\frac{\prod_{m=0}^{n-1} Q(m)}{\sum_{n=1}^{N}\prod_{m=0}^{n-1} Q(m)} \quad \text{with} \quad Q(0)=1. \tag{13}$$

Here we propose a short way of getting this established result and proving that it is the only stationary solution and the detailed balance is true. From Eqs. (9) and (11) we conclude that the relation

$$j(m\rightarrow m+1)=j(m+1\rightarrow m) \tag{14}$$

holds for $m=n$ if it is satisfied at $m=n-1$. According to Eqs. (9)–(11), Eq. (14) holds for $m=1$, therefore it holds for any $1\leqslant m\leqslant N-1$. This means that the detailed balance (14) is true and leads unambiguously to the final solution given by Eq. (13).

V. NUMERICAL RESULTS FOR FINITE SYSTEMS

A series of probability distributions $P(n)$ for increasing dimensionless densities $c=lN/L$, calculated on the basis of Eq. (13) together with Eqs. (6) and (7) indicating the formation of a cluster on the road, is depicted in Fig. 1. The ratio of transition rates $w_+(n)/w_-(n)$ is shown too. Calculations are made for a finite system with $L=5000$ m, $l=5$ m, or

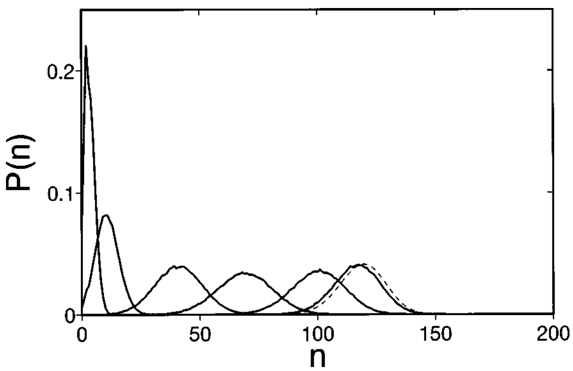

FIG. 2. Evolution of probability distribution $P(n,T)$ as a function of time for a system of $N=200$ cars simulated by Monte Carlo method starting with $P(n,0)=\delta_{n,1}$. Solid curves from left to right correspond to increasing times $T=2$, 10, 50, 100, 200, and 400. The dashed line depicts the stationary solution for $T\rightarrow\infty$. The parameters of the system are $L/l=1000$, $b=10$, $d=2.5$, and $\Delta y_{clust}=0.2$.

$L/l=1000.0$, $d=2.5$, $b=10.0$, and $\Delta y_{clust}=0.2$. One can conclude from Eq. (12) that the condition $w_+(n)/w_-(n)=1$ corresponds to a probability maximum (or minimum) located at $n/N\neq 0$ if $N\rightarrow\infty$ and $L\rightarrow\infty$. This holds approximately for a finite system, as it can be seen from Fig. 1. At small densities ($N=55$), $w_+(n)/w_-(n)<1$ holds for any n, therefore the absolute maximum of $P(n)$ is located at $n=1$, indicating the absence of a macroscopic cluster. This corresponds to the regime of free traffic without congestions. If the density increases above some critical value (the first critical density c_1 with the value $N\approx 96$ in this case) corresponding to $w_+(1)/w_-(1)=1$, then a macroscopic cluster appears ($n/N\neq 0$ at $N\rightarrow\infty$). The series of snapshots with $N=135$, 300, and 776 illustrates the growth of the cluster with increasing density. This corresponds to a partly congested road where one cluster of cars coexists with a region of free traffic. The probability maximum corresponds to the average cluster size. This holds exactly at $N\rightarrow\infty$ and represents a reasonable approximation for the considered finite system. At large densities ($N=776{,}777$), the equation $w_+(n)/w_-(n)=1$ has two solutions with respect to n. The first (the smallest) root corresponds to the minimum, whereas the second corresponds to the maximum of $P(n)$. In this case, the stationary probability distribution $P(n)$ has another maximum at $n=1$. At some critical density (the second critical density c_2 corresponding to $N\approx 777$), the maximum at $n=1$ becomes the absolute maximum and the macroscopic cluster disappears. The latter means that the density of cars is large over the whole road (highly viscous overcrowded situation). Generally Fig. 1 shows the phase transition from dilute gas (free flow, one phase), cluster phase (congestion and free flow, two coexisting phases), to liquid state (heavy traffic, one dense phase).

The time evolution of probability distribution $P(n,T)$ represented by Master equation (5) has been simulated for a finite system of 200 cars moving along the road of length $L=1000\ l$ with the same values of control parameters as before. Results are shown in Fig. 2. The probability distribution $P(n,T)$ at various time moments $T=2$, 10, 50, 100, 200, and 400 starting with $P(n,0)=\delta_{n,1}$ as initial condition are obtained by averaging over 500 000 stochastic trajectories simulated by the Monte Carlo method [25]. The dashed line in Fig. 2 represents the stationary solution given by Eq. (13). The same simulations have been made starting with $P(n,0)=\delta_{n,N}$. In both cases the probability distribution tends asymptotically to the stationary solution for $T\rightarrow\infty$.

VI. THE FUNDAMENTAL DIAGRAM

One of the most important characteristics of traffic flow is the fundamental diagram showing the flux J of cars as a function of the total density $\varrho=N/L$ (or dimensionless total density $c=l\varrho$) on the road. We define J as a local flux $\varrho(x,t)v(x,t)$ averaged over an infinite time interval, where $\varrho(x,t)$ is the local density and $v(x,t)$ is the local velocity of cars at a time moment t and space coordinate x, i.e.,

$$J=\lim_{t\rightarrow\infty}\frac{1}{t}\int_0^t \varrho(x,t')v(x,t')dt'. \tag{15}$$

In our model the local velocity as well as the density of cars are defined by the cluster size n and the distance $x-x'$ between the considered local coordinate x and the coordinate x' of the first car in the jam. Thus, we have $\varrho(x,t)=\varrho(x-x'(t),n(t))$ and $v(x,t)=v(x-x'(t),n(t))$, and after averaging over time we get

$$J=\sum_n \int dx' P(n,x')\varrho(x-x',n)v(x-x',n), \tag{16}$$

where $P(n,x')dx'$ denotes the part of the total time during which the size of the cluster is n and the coordinate of the first car of the jam is between x' and $x'+dx'$. The cluster can be found with equal probability at any coordinate x' along the circle if an averaging over an infinite time interval t is considered. Thus we have $P(n,x')=P(n)/L$. According to our assumptions, the velocity of congested cars is $v_{opt}(\Delta x_{clust})$ and their density is $\varrho_{clust}=n/L_{clust}$ inside the jam of length L_{clust}. Outside the jam we have $v=v_{opt}(\Delta x_{free}(n))$ and $\varrho_{free}=(N-n)/L_{free}$. By these assumptions the integration (16) can be performed easily, and this yields

$$j=b\sum_n P(n)\left[w_{opt}(\Delta y_{clust})\frac{nl}{L}+w_{opt}(\Delta y_{free}(n))\left(c-\frac{nl}{L}\right)\right], \tag{17}$$

where $j=J\tau$ is the dimensionless flux.

Now we consider the behavior of the system in the (thermodynamic) limit $N\rightarrow\infty$ under the condition

$$\sigma=(Rd)^2+4R\Delta y_{clust}-4>0, \tag{18}$$

where $R=b/[d^2+(\Delta y_{clust})^2]$. This is the condition at which the equation $w_+(n)/w_-(n)=1$ has real physical solution(s) and a cluster with $n/N\neq 0$ emerges, i.e., a phase transition takes place at some value of car density c. In the opposite case there is no phase transition (cluster formation) at all. In the special situation $\Delta y_{clust}=0$, condition (18) reduces to $b>2d$. The analysis of solution (13) shows that $P(z)=N^{-1}\delta(z-z_0)$ holds in the thermodynamic limit $N\rightarrow\infty$, where z is defined as $z=n/N$ with the value z_0 corresponding to the absolute maximum of $P(z)$. $z_0=0$ holds if $c\leqslant c_1$ or

$c > c_2$. If $c_1 \leqslant c < c_2$, then $z_0 = z'_0$, where z'_0 is defined by $z'_0 = (1 + \Delta y_{free} - 1/c)/(\Delta y_{free} - \Delta y_{clust})$ and Δy_{free} has the constant value $(d/2)(Rd + \sqrt{\sigma})$, as it follows from the equation $w_+(n)/w_-(n) = 1$. The critical densities c_1 and c_2 have the same meaning as discussed in Sec. V. They are defined by $z'_0 = 0$ or $c_1 = 1/[1 + (d/2)(Rd + \sqrt{\sigma})]$, and $\ln(P(z=0)) = \ln(P(z = z'_0))$, respectively.

In the following, an equation will be derived (under the assumption $\Delta y_{clust} = 0$) from which both critical densities c_1 and c_2 can be determined. Taking account of Eq. (12), the condition $\ln(P(z=0)) = \ln(P(z = z'_0))$ in the thermodynamic limit reduces to

$$\int_0^{z'_0} \ln[Q(z)]\, dz = 0. \tag{19}$$

This equation is satisfied both at $c = c_2$ and $c = c_1$ because in the latter case we have $z'_0 = 0$. Using partial integration [with account for $Q(z'_0) = 1$] and change of integration variable to $h = \Delta y_{free}$ [defined by Eq. (4)], we get

$$\int_{sd}^{Bd} \left(\frac{1}{h} - \frac{2h}{d^2 + h^2} \right) \left(1 - \frac{sd}{h} \right) dh = 0, \tag{20}$$

where $s = (1-c)/(cd)$ and $B = b/(2d) + \sqrt{b^2/(4d^2) - 1}$. This integral can be calculated analytically, and this yields

$$\ln \left[\frac{B(1+s^2)}{s(1+B^2)} \right] + \frac{s}{B} - 1 + 2s(\arctan B - \arctan s) = 0. \tag{21}$$

One of the solutions is, obviously, $s_1 = B$ corresponding to the first critical value $c_1 = 1/(1 + Bd)$. A complete analytical solution is possible in some asymptotic cases. At $B = 1 + \epsilon$, where $\epsilon \to 0$, the solution can be found in the form $s = 1 + \delta$, where $\delta \to 0$. Neglecting terms of fourth and higher orders we get $(\delta - \epsilon)^2 (\delta + 2\epsilon) = 0$. Thus, we have $s_1 = 1 + \epsilon = B$ and $s_2 \simeq 1 - 2\epsilon$. At the critical point $\epsilon = 0$ or $b = 2d$ (at $b > 2d$ the cluster emerges) we get $s_1 = s_2 = 1$ or $c_1 = c_2 = c_{crit}$, where $c_{crit} = 1/(d+1)$ is the critical value of c. Another asymptotical case is $B \to \infty$, where we have a solution with $s_2 \to 0$. Retaining only the main terms in the equation, we get $\ln(s_2 B) + 1 = 0$ or $s_2 = 1/(eB)$.

It should be noted that cases with $c \leqslant c_{clust}$ have a physical meaning only, because the total density c cannot exceed the density of cars in the cluster $c_{clust} = 1/(1 + \Delta y_{clust})$. In general (with $\Delta y_{clust} > 0$), a situation is possible where the equation for c_2 has no solution at $c_2 < c_{clust}$. In this case the following flux equations for an infinite system are correct, formally setting $c_2 = c_{clust}$. Thus, taking into account the above discussed solution for $P(n)$, we get the following flux-density relation:

$$j(c) = \begin{cases} \dfrac{bc(1-c)^2}{(cd)^2 + (1-c)^2} : c \in [0; c_1] \cup]c_2; c_{clust}] \\ 1 - c + c[b w_{opt}(\Delta y_{clust}) - \Delta y_{clust}] : c \in [c_1; c_2[. \end{cases} \tag{22}$$

These equations represent an exact analytical solution for the fundamental diagram of traffic flow in the framework of our relatively simple model, calculated in the thermodynamic limit. Since the fundamental diagram represents one of the most important characteristics of traffic flow, this result has a fundamental significance and has to be compared with vehicular experiments. As can be seen from these equations, the fundamental diagram consists of fragments of a nonlinear curve and of a straight line. The nonlinear curve represented by the first formula of Eq. (22) corresponds to homogeneous flow, whereas the straight line corresponds to nonhomogeneous (or congested) flow.

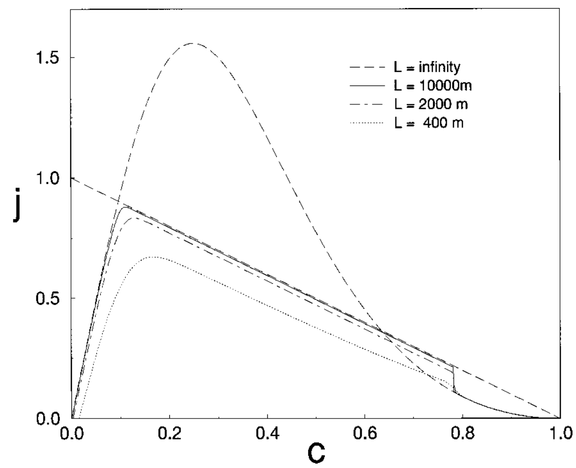

FIG. 3. Based on the stationary solution of the stochastic Master equation, the fundamental diagram [dimensionless flow rate (flux) j vs dimensionless car density c] is calculated. The dimensionless control parameters are $b = 10$, $d = 7/3$, and $\Delta y_{clust} = 0$. The length of road L varies, the effective length of a car being fixed $l = 6$ m. For finite roads ($L < \infty$) as well as for infinitely long roads ($L \to \infty$), the flow j can be divided in two homogeneous regimes (left: free flow as gaseous phase; right: heavy traffic as liquid phase) and a transition regime with free and congested vehicles (formation of a car cluster).

VII. PHASE TRANSITIONS AND DIFFERENT REGIMES OF TRAFFIC FLOW

Based on the analysis of the fundamental diagram (22), different regimes of traffic flow and phase transitions between these regimes have been discussed. The fundamental diagram is shown in Fig. 3, calculated at $b = 10$, $d = 7/3 \simeq 2.333$, and $\Delta y_{clust} = 0$ for different values of the length of the road L. The effective length of a car is fixed to $l = 6$ m, which corresponds to the experimental data discussed in Sec. II. The set of control parameters in this case is chosen slightly different from that in Sec. V. Our intention is to show two phase transitions at critical densities c_1 and c_2, respectively, which are more distinct at $\Delta y_{clust} = 0$ than at $\Delta y_{clust} = 0.2$. The regions $c \in [0; c_1]$, $c \in [c_1; c_2[$, and $c \in]c_2; 1]$ correspond to three different regimes of traffic flow. Besides, there is a breakpoint in flux j at $c = c_1$ and a jump at $c = c_2$, as may be seen from Fig. 3, where the analytical solution (22) is shown by dashed lines [corresponding to the first and the second formulas in Eq. (22)]. The jump Δj tends to zero in the case $B = 1 + \epsilon$ at $\epsilon \to 0$, i.e., $\Delta j \simeq 3 d c_{crit} \epsilon$, and j in the vicinity of the jump tends to the value $j_{crit} = d/(1+d)$. In general, the value $Bd/(1 + Bd)$ corresponds to j at $c = c_1$ and $\Delta y_{clust} = 0$. The results of the calculation for finite systems, shown in Fig. 3, coincide with the above analytical solution if $L\to\infty$. The region $c\in[0;c_1]$ corresponds to the free traffic flow where the relative part of the cars involved in the jam tends to zero if $N\to\infty$. The region $c\in[c_1;c_2[$ corresponds to the traffic flow on a partly congested road where one cluster of cars coexists with a region of free traffic. The region $c\in]c_2;1]$ corresponds to a highly viscous overcrowded situation, where the density of cars is high and their velocity is small over the whole road.

In our model, two phase transitions are always present at $\Delta y_{clust}=0$ and $b>2d$, since Eq. (21) has two real positive solutions with respect to s. The physical reason for the first phase transition is that the homogeneous flow, which is stable at low densities, becomes unstable at some critical density c_1. This phenomenon has been shown experimentally and theoretically by many authors (see, e.g., [12–14,3,4,17–19,5,23,16,8–10]), and it is not a unique feature of our model. The second phase transition occurs because the homogeneous flow becomes stable again at large densities of cars, i.e., at $c>c_2$. This phenomenon can be explained in terms of transition probabilities between two different macroscopic states in a bistable system, i.e., between the states without and with a macroscopic cluster. At sufficiently large densities, $P(n)$ has a minimum $[w_+(n)/w_-(n)=1$ has two solutions] at $n=n_{unst}$, where n_{unst} is the unstable cluster size. It means that there exists a nucleation barrier for the formation of the macroscopic cluster from an initially homogeneous state. A stable growth of the cluster can occur merely at $n>n_{unst}$, whereas a stable dissolution of the cluster can take place at $n<n_{unst}$ only. A switching between the two macroscopic states of the system is possible due to stochastic fluctuations. In our model, n_{unst} increases with c and approaches N if $c\to 1$. The velocity of cars is strongly decreased if the distance between them becomes small. As a result, the average flow of cars, joining the cluster, can exceed the opposite flow merely if the density of cars in the free phase is not too large. From a purely geometrical aspect, the necessary condition of this at $c\to 1$ is $n\to N$. In such a way, because of a large nucleation barrier and a small dissolution barrier, a spontaneous dissolution of the cluster becomes much more probable than its formation at $c\to 1$. Thus, in this case the highly dense homogeneous state with the probability maximum at $n=1$ is the stable state of the system. It is noteworthy that two phase transitions take place in our model at not too large values of Δy_{clust} only. If Δy_{clust} is large, then the average distance between cars (at $c<c_{clust}$) is never small enough to ensure the above discussed effects. A question arises whether the two phase transitions are present at an optimal velocity function different from Eq. (1). The necessary condition for the second phase transition at $\Delta y_{clust}=0$ and $N\to\infty$ is

$$\left.\frac{dw_{opt}(\Delta y)}{d(\Delta y)}\right|_{\Delta y=0}<\frac{1}{b}. \tag{23}$$

If this condition is not satisfied, then the homogeneous flow is unstable at $c\to 1$, since in this case $w_+(1)>w_-(1)$ holds. As a consequence, the existence of the second phase transition depends on the specific choice of the optimal velocity function.

TABLE I. Values of parameters for the stochastic traffic model determined from experimental observations on German highways.

Quantity	Symbol	Value
Effective length of a car	l	6 m
Interaction distance	D	13 m
Distance in jam	Δx_{clust}	1 m
Waiting time	τ	1.5 s
Maximal velocity	v_{max}	34 m/s

VIII. COMPARISON WITH EXPERIMENTAL DATA

The fundamental diagram discussed in the preceding section has been calculated for quite realistic values of the parameters. These parameter values are suitable to show some general features of behavior of a vehicular system described by our model. Nevertheless, these parameters have up to now not been adjusted to real experimental data. In this section we present a comparison with the experimental data reported partly in Sec. II (see [17–19]). The parameter values corresponding to the best (or near to the best) fit with these experimental data are given in Table I. The corresponding dimensionless parameters are $b=v_{max}\tau/l=8.5$, $d=D/l=13/6\simeq 2.167$, and $\Delta y_{clust}=\Delta x_{clust}/l=1/6\simeq 0.167$. We have considered v_{max} and $l+\Delta x_{clust}=1/\rho_{clust}$ as experimentally measured quantities consistent with data given in Ref. [18]. In this case there are only three independent free parameters, i.e., D, Δx_{clust}, and τ, which can be used for matching the theoretically calculated fundamental diagram with the experimental one in the case if N and L are large. The parameter L/l is responsible for finite-size effects.

The velocity of backwards motion of a jam $v_{back}\approx 16$ km/h calculated with these parameters from Eq. (8) is close to the experimental value 15 km/h [18]. The latter fact provides good evidence of validity of the used optimal velocity function (1). To ensure a good fit with the experimental value of v_{back}, it is important that $v_{opt}(\Delta x_{clust})$ in Eq. (8) is small compared to v_{back}. Thus, not any function of the optimal velocity is valid. In our case, a sufficiently small value of $v_{opt}(\Delta x_{clust})\approx 1$ km/h is ensured by the fact that the first derivative of the optimal velocity function (1) vanishes at $\Delta y=0$. In any case, this derivative should be remarkably smaller than 1. Although our choice is not the only possible one, Eq. (1) represents one of the simplest functions having this property. The optimal velocity function (2) used in Ref. [14] with the parameter values $d_1=0.602$ and $d_2=1.548$, determined on the basis of the experimental data measured on the Chuo Motorway, has also a sufficiently small value 0.052 of the first derivative at $\Delta y=0$. However, this function is more complicated as compared to Eq. (1).

Under the condition $v_{opt}(\Delta x_{clust})\ll v_{back}$, the waiting time τ may be considered as a directly measured quantity $[\tau\approx 1/(v_{back}\varrho_{clust})]$. As already mentioned in Sec. III, this yields the value of τ about 1.7 s, which is compatible with the value 1.5 s obtained in this section by matching the experimental and the theoretical fundamental diagrams. It is also interesting to compare the interaction distances d [d in any case is defined by $w_{opt}(d)=1/2$], $d\simeq 2.167$ and

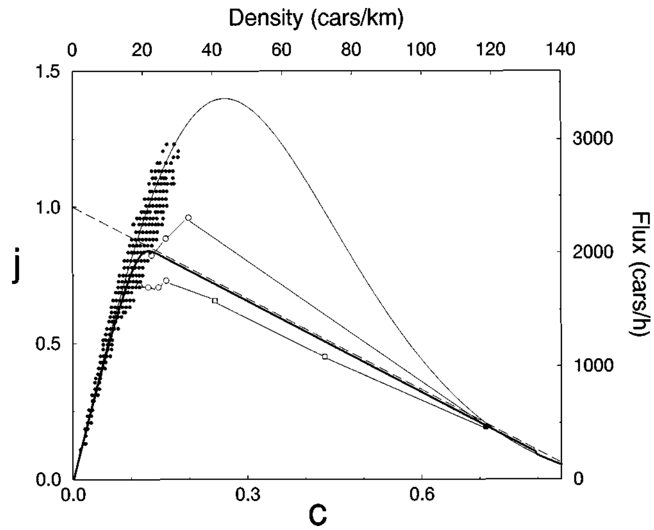

FIG. 4. Comparison of the fundamental diagram of traffic flow calculated at fitted parameter values $l=6$ m, $v_{max}=34$ m/s, $\tau=1.5$ s, $D=13$ m, and $\Delta x_{clust}=1$ m ($b=8.5$, $d=13/6$, and $\Delta y_{clust}=1/6$) with experimental data of Ref. [18] (denoted by separate points and a thin solid line connecting measured points). The thick solid line shows the solution for the finite road of length $L=5000$ m; the theoretical curves for an infinite system represented by the first and the second formulas in Eq. (22) are shown by a smooth thin solid line and a dashed line, respectively.

$d\simeq 2.643$, following from Eqs. (1) and (2), respectively. These values are compatible. The agreement is satisfactory, taking into account that different models and different experimental data have been used to determine the values of the control parameters in these two equations.

The values of the dimensionless parameters are rather close to those used in Sec. VII ($b=10$, $d=7/3\simeq 2.333$, $\Delta y_{clust}=0$). Therefore qualitatively similar behavior of the system is expected. The fundamental diagram calculated with these parameters for a circular road of length $L=5000$ m is shown by the thick solid line in Fig. 4, where the experimental data of Ref. [18] corresponding to free traffic flow are shown by separate points. The experimental points measured inside the region of wide traffic jams are connected by a thin solid line. Since data, taken from Fig. 3 in [18] (with the same notation), are measured in a nonstationary situation, a hysteresis appears (points starting from upper ones are connected in a sequence of measurement time). It is necessary to make an averaging over a sufficiently large time interval to get the value of flux defined by Eq. (15) corresponding to an average (total) density of cars. According to this, there are some difficulties in comparing the experimentally measured nonstationary values of the flux with our theoretical results. In the first approximation we have assumed that the result of averaging of nonstationary measurements at a given density corresponds to the value of flux calculated at this density from our theoretical model. This is a criterion we have used to find the values of parameters corresponding to the experimental data optimally. The theoretical line in Fig. 4 is in good agreement with the averaged experimental values of flux at both small densities and at high densities. The most significant difference is at densities slightly above the first critical density c_1 corresponding to the maximum of theoretical curve for an infinite system. The theoretical curves for an infinite system represented by the first and the second formulas in Eq. (22) are shown in Fig. 4 by a (smooth) thin solid line and a dashed line, respectively. The thin solid line corresponds to homogeneous flow without congestions, whereas the dashed line reflects the congested traffic flow. The experimental data shown in Fig. 4 by separate points correspond to homogeneous traffic flow. The homogeneous situation in the considered experiments is observed at densities slightly larger than c_1, as can be seen from the figure. In such a way, the separated experimental points above the maximum of the theoretical curve can be interpreted as metastable homogeneous flow.

IX. CONCLUSIONS

The stochastic Master equation approach to describe traffic flow developed in Refs. [16,23] is extended allowing that cars in a jam are not close to each other and that they are moving. The stationary probability distribution over jam sizes and the fundamental (flux-density) diagram is calculated on the basis of the developed model depending on total density of cars on a circular one-lane road. The evolution of probability distribution, showing formation of a jam on a road, is investigated on the basis of the time-dependent Master equation solved by the Monte Carlo method.

The obtained results indicates the existence of three different regimes of traffic flow, i.e., free flow at small densities of cars, congested traffic or coexisting phase, where a car cluster coexists with a region of free traffic, at intermediate densities, and a highly viscous overcrowded situation, where the density of the cars is large and their velocity is small over the whole road, at high densities.

The obtained results represented in the flux-density plane allow us to interpret the main features of traffic flow observed experimentally. The analytical solution of the fundamental diagram for an infinite road, as well as solutions for finite roads of different lengths, are in qualitative agreement with experimental data. A good quantitative agreement with experimental data [17–19], especially with Fig. 3 from Ref. [18], is obtained at appropriate realistic values of control parameters (Table I). It is shown that the experimental points corresponding to free traffic are in good agreement with the analytical curve for homogeneous flow. Besides, the largest experimental values of flux correspond to a metastable homogeneous state. The experimental values of flux measured in a region of wide jams on average correspond to the theoretical (straight) line of the coexisting phase.

In conclusion, we summarize that the interpretation of traffic measurements by physically motivated car models based on particle interaction gives support to the current understanding of traffic behavior of real road networks.

ACKNOWLEDGMENTS

One of us (J.K.) gratefully acknowledges support under Project No. 0023 0040/97 (Kultusministerium des Landes Mecklenburg–Vorpommern) during the stay at the Physics Department of Rostock University. The authors thank Nicola Pieret, Marcel Ausloos (Liège), and Boris Kerner (Stuttgart) for fruitful discussions.

[1] *Self-Organization of Complex Structures*, edited by F. Schweitzer (Gordon and Breach Science Publ., Amsterdam, 1997).
[2] D. Helbing, *Verkehrsdynamik: Neue Physikalische Modellierungskonzepte* (Springer, Berlin, 1997).
[3] B. S. Kerner and P. Konhäuser, Phys. Rev. E **48**, R2335 (1993).
[4] B. S. Kerner and P. Konhäuser, Phys. Rev. E **50**, 54 (1994).
[5] W. Leutzbach, *Einführung in die Theorie des Verkehrsflusses* (Springer, Berlin, 1972); *Introduction to the Theory of Traffic Flow* (Springer, Berlin, 1988).
[6] *Traffic and Granular Flow '97*, edited by M. Schreckenberg and D. E. Wolf (Springer, Singapore, 1998).
[7] *Traffic and Granular Flow*, edited by D. E. Wolf, M. Schreckenberg, and A. Bachem (World Scientific Publ., Singapore, 1996).
[8] K. Nagel, Phys. Rev. E **53**, 4655 (1996).
[9] K. Nagel and M. Paczuski, Phys. Rev. E **51**, 2909 (1995).
[10] K. Nagel and M. Schreckenberg, J. Phys. I **2**, 2221 (1992).
[11] M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito, Phys. Rev. E **51**, 2939 (1995).
[12] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama, Jpn. J. Indust. Appl. Math **11**, 203 (1994).
[13] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama, Phys. Rev. E **51**, 1035 (1995).
[14] M. Bando, K. Hasebe, K. Nakanishi, A. Nakayama, A. Shibata, and Y. Sugiyama, J. Phys. I **5**, 1389 (1995).
[15] I. Prigogine and R. Herman, *Kinematic Theory of Vehicular Traffic* (Elsevier, New York, 1971).
[16] R. Mahnke and N. Pieret, Phys. Rev. E **56**, 2666 (1997).
[17] B. S. Kerner and H. Rehborn, Phys. Rev. E **53**, R1297 (1996).
[18] B. S. Kerner and H. Rehborn, Phys. Rev. E **53**, R4275 (1996).
[19] B. S. Kerner and H. Rehborn, Phys. Rev. Lett. **79**, 4030 (1997).
[20] D. Helbing, Phys. Rev. E **55**, R25 (1997).
[21] P. Wagner and J. Peinke, Z. Naturforsch., A: Phys. Sci. **52a**, 600 (1997).
[22] W. Brilon and M. Ponzlet (Ref. [7]), p. 23.
[23] R. Mahnke and J. Kaupužs, *One More Fundamental Diagram of Traffic Flow, Presentation at the Second Workshop on Traffic and Granular Flow, Duisburg, 1997*, edited by M. Schreckenberg and D. E. Wolf (Springer, Singapore, 1998), see [6].
[24] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, New York, 1983).
[25] J. Honerkamp, *Stochastische Dynamische Systeme* (VCH, Weinheim, 1990); *Stochastic Dynamical Systems* (VCH, New York, 1994).