Ultrahigh-intensity inverse bremsstrahlung

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We study inverse bremsstrahlung in the ultrahigh intensity relativistic regime. The fully relativistic ultrahigh intensity absorption (emission) coefficient is derived for an arbitrary scattering potential and small-angle scattering. We find that in the Coulomb field case this absorption (emission) coefficient can be calculated as a function of the quiver energy, drift momentum, and impact parameter in two complementary regimes: (i) for remote collisions when the impact parameter is larger than the amplitude of the quiver motion, and (ii) for instantaneous collisions when the scattering time is shorter than the period of the wave. Both circular and linear polarizations are considered, and this study reveals that in this relativistic regime inverse bremsstrahlung absorption can be viewed as a harmonic Compton resonance heating of the laser-driven electron by the virtual photon of the ion Coulomb field. The relativistic modification of Marcuse's effect [Bell Syst. Tech. J. **41**, 1557 (1962)] are also discussed, and relations with previous nonrelativistic results are elucidated.
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I. INTRODUCTION

The basic physical processes involved in laser-plasma interaction, up to 10^{17} W/cm^2, are now well understood; on the other hand, a large number of issues remain open in the study of the relativistic interaction regime above 10^{18} W/cm^2. Recent advances in pulse compression now make possible an exploration of laser-plasma interactions with such fluxes above 10^{18} W/cm^2, thus there is a clear need to identify and analyze the issues relevant to this ultrahigh intensity (UHI) regime [1]. Among these various issues, UHI inverse bremsstrahlung (IB) is particularly important to understand energy momentum transfer from an electromagnetic (EM) laser field to a plasma during solid target experiments.

Terawatt to petawatt UHI laser pulses are now considered as potential candidates to provide highly localized, nonlinear energy deposition (through fast electrons production) in an inertial fusion target in order to obtain spark ignition of the compressed fuel. Laser intensities in a range from 10^{18} to 10^{21} W/cm^2 are thus considered, and the impact of UHI-IB on these scenarios remains to be evaluated. Besides solid target experiments, UHI-IB is also relevant to understand the early stages of some recently reported efficient heating and emission of energetic electrons from atomic clusters submitted to UHI laser pulses [2].

Up to now, two types of approximations have been used to study IB in the framework of quantum mechanics: the Born approximation (when the scattering potential is assumed to be weak) [3], and the low-frequency approximation (when the frequency of the EM wave is smaller than the interaction time) [4]. Krall and Watson [5] combined both results, and showed that the differential cross section for laser potential scattering has the same form for both limiting cases and can be expressed in terms of the differential cross section for electron scattering without the lase EM field, the so-called Krall-Watson formula.

Relativistic quantum IB has been studied within the framework of the Born approximation [6,7] and in the low-frequency approximation [8]; however this quantum formalism is limited to classical perturbation expansions and is not relevant to UHI-IB during laser-plasma interaction. In order to study UHI-IB during laser-plasma interaction, we will use a classical relativistic formalism. Classical mechanics provides the right framework to describe IB for fluxes in the range from 10^{18} to 10^{21} W/cm^2, where the energy exchange between electrons and EM wave exceeds the energy of one EM quantum $\hbar\omega$ (i.e., the interaction is essentially a multiphoton process) [9].

Nonrelativistic high-frequency resistivity was evaluated in the 1960s by Dawson and Oberman [10] and Silin [11]. In addition to this Maxwellian averaged quantity, the nonrelativistic rate of IB for classical collisions between one electron and an ion population has been calculated in the small-angle approximation (or ''straight-line path'' approximation) [12–14], but the final result remains an infinite sum of integrals and does not reveal the scaling of the process with respect to the various parameters. The classical description of the instantaneous collisions has also been considered in the impact approximation, where it is assumed that the collisions take place on a time scale far shorter than the interaction time between the colliding species [15,16].

The Coulomb field can be treated as a perturbation during the scattering process for relativistic laser fluxes ($eA/mc \approx 1$, where A is vector potential of the EM wave) provided the minimum distance between the electron and the ion is such that the Coulomb field remains smaller than the laser field. Given the fact that the velocity of the quiver motion and drift motion is of the order of the velocity of light, this minimum distance is $r_{\min} \sim \sqrt{r_e \lambda}$ ($r_e = e^2/mc^2$ is the classical electron radius, λ is the wavelength of the radiation, c is the velocity of light, and e and m are the charge and mass of the electron). A classical relativistic description can be used if the de Broglie wavelength of a relativistic electron $\lambda_e = \hbar/p \sim 10^{-9}$ cm (p is the electron momentum) is smaller than the distance between two particles. For typical lasers, $r_{\min} = 10^{-9}$ cm; thus classical mechanics is valid and the ion field remain smaller than the EM field and the Coulomb field can be treated as a perturbation [17].

In this study we shall calculate the energy exchange between an EM laser field and an electron population within the framework of the random phase approximation; that is to say, we average the final results over the initial phase of electrons entering the scattering region. This averaging can also be presented as the interaction between an infinitely narrow electron beam and an EM field in the presence of a scattering potential. Besides the fact that the experimental measurement is naturally phase averaged, this averaging over scattering phases has another advantage: it essentially simplifies the calculation of the energy exchange. This latter point was noted some time ago in the theory of the free-electron laser, and formulated as Madey's theorem [18]. This theorem simplifies the evaluation of the gain for fast wave devices [19–21].

This theorem has been demonstrated in general Hamiltonian form for one-dimensional cases, and extended to near-integrable Hamiltonian systems with arbitrary degrees of freedom [20–22]. For a perturbed classical Hamiltonian system expressed in terms of actions and angles, the statement of this theorem can be summarized as follow: the second order change (due to the perturbation) in action variables averaged over the initial phase can be expressed in terms of the first order perturbation, this substantially simplifies calculations of second order quantities. It can be shown that this theorem has relations with other results, such as the classical limit of Einstein relations between spontaneous and stimulated emission, [23] and the fluctuation dissipation theorem [24].

In addition to this methodological issue concerning Madey's theorem, from a phenomenological point of view IB can be considered as a conversion of the regular quiver motion energy into drift thermal energy as a result of the reorientation of the quiver motion into translational motion during the collision. This efficient conversion process is due to the occurrence of a set of resonances, identified as Compton resonances, induced by the beating between the laser wave and the virtual photon of the ion Coulomb field [Eq. (27)].

The paper is organized as follows. In Sec. I the problem is formulated within the framework of Hamiltonian dynamics. In Sec. II the expression of the energy exchange between a relativistic electron and an UHI EM laser field is derived with the help of Madey's theorem. In Sec. III this energy exchange is calculated and analyzed as a function of the electron momentum, impact parameter, and azimuthal angle for remote collisions, when the impact parameter is larger than the amplitude of the quiver oscillations. In Sec. IV we consider the complementary situation of instantaneous collisions when the scattering time is shorter than the period of the UHI laser wave. In Sec. V the relation with previous results obtained with different approximations is elucidated and discussed, and we summarize the main original results of our study and give our conclusions.

II. ACTION-ANGLE VARIABLES FOR RELATIVISTIC SCATTERING

The electron motion in a scattering field $U(\mathbf{r}-\mathbf{r}_0)$ and a linearly polarized (LP) EM wave with vector potential $\mathbf{A}(\mathbf{r},t) = A\cos(\omega x/c - \omega t)\mathbf{e}_y$, propagating along the vector $\boldsymbol{\nu} = \mathbf{e}_x$, is described by Hamilton's equations derived from the Hamiltonian:

$$H(\mathbf{p},\mathbf{r},t) = \sqrt{1+p_x^2+[A\cos(x-t)+p_y]^2+p_z^2} + U(\mathbf{r}-\mathbf{r}_0), \tag{1}$$

where we have used the set of natural units $m=c=e=\omega=1$, $\mathbf{r_0}$ is the vector specifying the position of the Coulomb center.

We extend the phase space of this dynamical system through the introduction of a set of new canonical variables $t, E=-H$. Then the Hamiltonian becomes

$$H(\mathbf{p},\mathbf{r},E,t) = \sqrt{1+p_x^2+[A\cos(x-t)+p_y]^2+p_z^2} + U(\mathbf{r}-\mathbf{r}_0) + E. \tag{2}$$

It is known [25] that the problem of electron motion in a plane wave is integrable, and it is possible to introduce a set of action-angles variables $\gamma, \mathbf{P}$ and $\Phi, \boldsymbol{\Theta}$ with the help of the generating function [26]

$$S(\gamma,\mathbf{P},t,\mathbf{r}) = (\mathbf{P}\cdot\mathbf{r}) - \gamma t + \frac{P_y A}{\gamma - P_x}\sin(x-t) + \frac{A^2}{8(\gamma-P_x)}\sin(2x-2t). \tag{3}$$

Thus the transformation from the old to the new canonical variables is defined by the relations

$$\begin{aligned}
p_x &= P_x + \frac{P_y A}{\gamma-P_x}\cos(\Theta_1+\Phi) + \frac{A^2}{4(\gamma-P_x)}\cos(2\Theta_1+2\Phi),\\
p_y &= P_y,\\
p_z &= P_z,\\
E &= -\gamma - \frac{P_y A}{\gamma-P_x}\cos(\Theta_1+\Phi) - \frac{A^2}{4(\gamma-P_x)}\cos(2\Theta_1+2\Phi),\\
x &= \Theta_1 - \frac{P_y A}{(\gamma-P_x)^2}\sin(\Theta_1+\Phi) - \frac{A^2}{8(\gamma-P_x)^2}\sin(2\Theta_1+2\Phi), \qquad (4)\\
y &= \Theta_2 - \frac{A}{(\gamma-P_x)}\sin(\Theta_1+\Phi),\\
z &= \Theta_3,
\end{aligned}$$

$$t=-\Phi-\frac{P_yA}{(\gamma-P_x)^2}\sin(\Theta_1+\Phi)-\frac{A^2}{8(\gamma-P_x)^2}\sin(2\Theta_1+2\Phi).$$

The new Hamiltonian as a function of the new variables becomes

$$H(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})=\sqrt{\mathbf{P}^2+M^2+2\gamma B(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})+B^2(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})}-\gamma-B(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})+U[\mathbf{r}(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})-\mathbf{r}_0], \tag{5}$$

$$B(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})=\frac{P_yA}{\gamma-P_x}\cos(\Theta_1+\Phi)+\frac{A^2}{4(\gamma-P_x)}\cos(2\Theta_1+2\Phi),$$

where $M^2=1+A^2/2$ is the averaged electron energy in the LP EM wave without drift motion. The parameter M plays the role of an effective mass of an electron inside the EM field. This effective mass is an important concept of both classical and quantum electrodynamics in a strong field. Among other processes, M is responsible for a nonlinear frequency shift in strong field Compton scattering, and for the enhanced penetration of intense waves in dense plasmas. The corresponding Hamilton's equations describe the evolution with respect to the time t, so in order to perform further calculations with the previous Hamiltonian we introduce a new time $\tau\equiv\Phi$ instead of t, the action conjugated to Φ can be taken as a new Hamiltonian [27]. Then expressing this action γ in terms of $H,\mathbf{P},\Phi$, and $\mathbf{\Theta}$ by Eq. (5), the final Hamiltonian describing the evolution becomes

$$H(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})=\gamma(H=0,\mathbf{P},\Phi,\mathbf{\Theta})+\gamma. \tag{6}$$

Comparing the dynamical equations of Hamiltonians (5) and (6) it is easy to check that both Hamiltonians (6) describe the same dynamical system. Considering the scattering field as a perturbation, $U(\mathbf{r}-\mathbf{r}_0)=\mu U(\mathbf{r}-\mathbf{r}_0)$, $\mu\ll 1$, we obtain the final result to first order in μ:

$$H(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})=\gamma+H_0(\mathbf{P})+\mu H_\sim(\mathbf{P},\Phi,\mathbf{\Theta})+O(\mu^2),$$

$$H_0(\mathbf{P})=\sqrt{\mathbf{P}^2+M^2},$$

$$H_\sim(\mathbf{P},\Phi,\mathbf{\Theta})=\left[1+\frac{B(\mathbf{P},\Phi,\mathbf{\Theta})}{\gamma(\mathbf{P})}\right]U[\mathbf{r}(\mathbf{P},\Phi,\mathbf{\Theta})-\mathbf{r}_0], \tag{7}$$

$$B(\mathbf{P},\Phi,\mathbf{\Theta})\equiv B[\gamma(\mathbf{P}),\mathbf{P},\Phi,\mathbf{\Theta}],$$

$$\mathbf{r}(\mathbf{P},\Phi,\mathbf{\Theta})\equiv\mathbf{r}[\gamma(\mathbf{P}),\mathbf{P},\Phi,\mathbf{\Theta}],$$

where $\gamma(\mathbf{P})=H_0(\mathbf{P})$ [the relations between $\mathbf{r}$ and $\gamma,\mathbf{P},\Phi,\mathbf{\Theta}$ are given by Eq. (4)]. Similarly, for a circularly polarized (CP) EM wave $\mathbf{A}(\mathbf{r},t)=\mathbf{e}_1A\cos(\boldsymbol{\nu}\mathbf{r}-t)+\mathbf{e}_2A\sin(\boldsymbol{\nu}\mathbf{r}-t)=\mathbf{e}_yA\cos(x-t)+\mathbf{e}_zA\sin(x-t)$, Hamiltonian (7) can be derived, and we obtain

$$x=\Theta_1-\frac{P_yA}{[\gamma(\mathbf{P})-P_x]^2}\sin(\Theta_1+\Phi)+\frac{P_zA}{[\gamma(\mathbf{P})-P_x]^2}\cos(\Theta_1+\Phi),$$

$$y=\Theta_2-\frac{A}{\gamma(\mathbf{P})-P_x}\sin(\Theta_1+\Phi),$$

$$z=\Theta_3+\frac{A}{\gamma(\mathbf{P})-P_x}\cos(\Theta_1+\Phi), \tag{8}$$

$$B(\mathbf{P},\Phi,\mathbf{\Theta})=\frac{P_yA}{\gamma(\mathbf{P})-P_x}\cos(\Theta_1+\Phi)-\frac{P_zA}{\gamma(\mathbf{P})-P_x}\sin(\Theta_1+\Phi),$$

$$M^2=1+A^2.$$

Let us consider the unperturbed motion ($\mu=0$) described by the Hamiltonian $H_0(\mathbf{P})$, and integrate Hamilton's equation to obtain

$$\Phi=\int_0^\tau\frac{\partial H(\gamma,\mathbf{P},\Phi,\mathbf{\Theta})}{\partial\gamma}d\tau=\tau,$$

$$\Theta_1^{(0)}(\mathbf{P},\tau)=\int_0^\tau\frac{\partial H_0(\mathbf{P})}{\partial P_x}d\tau=\frac{\tau P_x}{\gamma(\mathbf{P})},$$

$$\Theta_2^{(0)}(\mathbf{P},\tau)=\int_0^\tau\frac{\partial H_0(\mathbf{P})}{\partial P_y}d\tau=\frac{\tau P_y}{\gamma(\mathbf{P})}, \tag{9}$$

$$\Theta_3^{(0)}(\mathbf{P},\tau)=\int_0^\tau\frac{\partial H_0(\mathbf{P})}{\partial P_z}d\tau=\frac{\tau P_z}{\gamma(\mathbf{P})}.$$

Equations (4) and (9) define the well-known revivalistic orbit of an electron interacting with the LP EM wave; the so-called drifting ''figure-eight'' motion [26]. Using Eq. (4) and averaging Eqs. (9) over the electron oscillations in the EM wave, we obtain

$$\langle x\rangle=\tau V_x,\quad \langle y\rangle=\tau V_y,\quad \langle z\rangle=\tau V_z. \tag{10}$$

Therefore Eqs. (10) describe the drift motion of the ''figure eight'' with a drift velocity $\mathbf{V}=\mathbf{P}/\gamma(\mathbf{P})$ and an average kinetic energy $\gamma(\mathbf{P})=\sqrt{\mathbf{P}^2+M^2}$. Note that, in our formulation, the drift trajectory always passes through the center of the coordinate system, and the position of the scattering center is determined by $\mathbf{r}_0$. For $M^2=1+A^2$, Eqs. (8) and (9) define the relativistic orbit of an electron interacting with the CP EM wave corresponding to drifting circular oscillations [26].

III. ULTRAHIGH INTENSITY INVERSE BREMSSTRAHLUNG

The calculation of the action second order change averaged over initial phases is simplified by the generalized Madey's theorem [22]. Usually this averaging over all initial angle variables is assumed for multidimensional Hamiltonian systems [21,23]; however, this theorem can be employed in the case of averaging over only one of all initial angle variables [20,22]. If we have a nonautonomous dynamic system with N degrees of freedom described by the Hamiltonian

$$H(\mathbf{I},\boldsymbol{\theta},t)=H_0(\mathbf{I},t)+\mu H_{\sim}(\mathbf{I},\boldsymbol{\theta},t), \tag{11}$$

where $H_0(\mathbf{I},t)$ is the Hamiltonian of unperturbed problem, $H_{\sim}(\mathbf{I},\boldsymbol{\theta},t)$ is the perturbation, a 2π periodical function of the variable θ_1. Then the change in I_1 to first order in μ vanishes after averaging over $\theta_1^0=\theta_1(t=-\infty)$ and the change in I_1 to second order in μ averaged over θ_1^0 can be expressed in terms of the change in I_1 found to first order in μ:

$$\langle \Delta I_1^{(2)}\rangle=\sum_{i=1}^{N}\left\langle \Delta I_i^{(1)}\frac{\partial \Delta I_1^{(1)}}{\partial I_i^{(0)}}\right\rangle+O[\mu^3]. \tag{12}$$

Here $\langle\cdots\rangle$ means the averaging over θ_1^0 and $\mathbf{I}^{(0)}=\mathbf{I}(t=-\infty)$, and $\Delta I_1^{(2)}$ and $\Delta I_1^{(1)}$ are the second and first order changes of I_1. If Hamiltonian (11) is the periodical function of all angle variables, then, after averaging expression (12) over all initial angle variables, we obtain the usual form of Madey's theorem:

$$\langle \Delta I_1^{(2)}\rangle=\frac{1}{2}\sum_{i=1}^{N}\frac{\partial}{\partial I_i^0}\langle \Delta I_1^{(1)}\Delta I_i^{(1)}\rangle+O[\mu^3]. \tag{13}$$

Since Hamiltonian (7) is a periodic function of Φ, and assuming that the amplitude of the perturbation is weak ($\mu \ll 1$), the theorem in form (12) can be employed to calculate the energy exchange between the electrons and the EM field averaged over Φ_0 to second order in μ. Here $\mathbf{I}^{(0)}=[\gamma(\tau=-\infty),\ \mathbf{P}(\tau=-\infty)]$, $\theta_1^0=\Phi_0=\Phi(\tau=-\infty)$, and $\Delta I_1=\Delta\gamma=\langle\gamma(\tau=+\infty)-\gamma(\tau=-\infty)\rangle_{\Phi_0}$. Averaging over Φ_0 means averaging over an electron population that is evenly distributed along the unperturbed trajectory. Thus $n_b V\Delta\gamma$ is the power transferred from the EM laser field to an infinitely narrow electron beam with velocity $\mathbf{V}\equiv\mathbf{P}c^2/\gamma$ and linear density n_b. Integrating Hamilton's equations along the unperturbed trajectories, we can calculate the variations of the actions to first order in μ:

$$\Delta P_i^{(1)}(\mathbf{P},\mathbf{r_0},\Phi_0)=-\mu\int_0^t\frac{\partial H_{\sim}[\mathbf{P},\mathbf{r_0},\tau+\Phi_0,\boldsymbol{\Theta}^{(0)}(\mathbf{P},\tau)]}{\partial\Theta_i}d\tau, \tag{14}$$

$$\Delta\gamma^{(1)}(\mathbf{P},\mathbf{r_0},\Phi_0)=-\mu\int_0^t\frac{\partial H_{\sim}[\mathbf{P},\mathbf{r_0},\tau+\Phi_0,\boldsymbol{\Theta}^{(0)}(\mathbf{P},\tau)]}{\partial\Phi_0}d\tau,$$

where unperturbed motions $\Theta_1^{(0)}(\mathbf{P}^0,\tau)$, $\Theta_2^{(0)}(\mathbf{P}^0,\tau)$, and $\Theta_3^{(0)}(\mathbf{P}^0,\tau)$ are determined by Eqs. (9), and $\mathbf{P}^0=\mathbf{P}(\tau=-\infty)$. In the following discussion we will use the notation $\mathbf{P}$ instead of $\mathbf{P}^0$ unless otherwise specified.

Using the relations

$$\frac{\partial H_{\sim}}{\partial\Phi}=\frac{\partial H_{\sim}}{\partial\Phi_0},\quad \frac{\partial H_{\sim}}{\partial\Theta_1}=\frac{\partial H_{\sim}}{\partial\Phi_0}-\frac{\partial H_{\sim}}{\partial x_0},$$

$$\frac{\partial H_{\sim}}{\partial\Theta_2}=-\frac{\partial H_{\sim}}{\partial y_0},\ \frac{\partial H_{\sim}}{\partial\Theta_3}=-\frac{\partial H_{\sim}}{\partial z_0}, \tag{15}$$

and Eq. (12), the energy exchange averaged over Φ_0 can be rewritten to the second order in μ as follows:

$$\Delta\gamma=\left\langle-\frac{\partial^2 T}{\partial P_x\partial\Phi_0}\left(\frac{\partial T}{\partial x_0}-\frac{\partial T}{\partial\Phi_0}\right)-\frac{\partial T}{\partial y_0}\frac{\partial^2 T}{\partial P_y\partial\Phi_0}-\frac{\partial T}{\partial z_0}\frac{\partial^2 T}{\partial P_z\partial\Phi_0}\right\rangle_{\Phi_0}, \tag{16}$$

where

$$T(\mathbf{P},\mathbf{r_0},\Phi_0)=\int_{-\infty}^{+\infty}H_{\sim}[\mathbf{P},\mathbf{r_0},\tau+\Phi_0,\Theta_1^{(0)}(\mathbf{P},\tau),\Theta_2^{(0)}(\mathbf{P},\tau),\Theta_3^{(0)}(\mathbf{P},\tau)]d\tau. \tag{17}$$

Equation (16) can also be presented in the form

$$\Delta\gamma=\left\langle\frac{\partial\Delta\gamma^{(1)}}{\partial P_x}\Delta P_x^{(1)}+\frac{\partial\Delta\gamma^{(1)}}{\partial P_y}\Delta P_y^{(1)}+\frac{\partial\Delta\gamma^{(1)}}{\partial P_z}\Delta P_z^{(1)}\right\rangle_{\Phi_0}, \tag{18}$$

where $\Delta\gamma^{(1)}=\partial T/\partial\Phi_0$, and $\Delta\mathbf{P}^{(1)}=-\partial T/\partial\mathbf{r}_0+\boldsymbol{\nu}\partial T/\partial\Phi$ are the first order change in energy and momentum of the electron. Using Eqs. (4), (7), (8), and (9) the integral T [Eq. (17)] can be rewritten

$$T(\mathbf{P},\mathbf{r}_0,\Phi_0)=-\int_{-\infty}^{+\infty}d\tau\left[1+\frac{B(\mathbf{P},\Phi_0,\tau)}{\gamma(\mathbf{P})}\right]\times U[\mathbf{P}\tau/\gamma(\mathbf{P})-\mathbf{r}_{\sim}(\mathbf{P},\mathbf{r_0},\Phi_0,\tau)-\mathbf{r}_0], \tag{19}$$

where, for the LP EM wave,

$$B(\mathbf{P},\Phi_0,\tau)=\frac{P_yA}{\eta(\mathbf{P})}\cos[f(\mathbf{P})\tau+\Phi_0]+\frac{A^2}{4\eta(\mathbf{P})}\cos[2f(\mathbf{P})\tau+2\Phi_0],$$

$$x_{\sim}(\mathbf{P},\Phi_0,\tau)=-\frac{AP_y}{\eta(\mathbf{P})^2}\sin[f(\mathbf{P})\tau+\Phi_0]-\frac{A^2}{8\eta(\mathbf{P})^2}\sin[2f(\mathbf{P})\tau+2\Phi_0], \tag{20}$$

$$y_{\sim}(\mathbf{P},\Phi_0,\tau)=-\frac{A}{\eta(\mathbf{P})}\sin[f(\mathbf{P})\tau+\Phi_0],$$

$$z_{\sim}(\mathbf{P},\Phi_0,\tau)=0,\quad \eta(\mathbf{P})=\gamma(\mathbf{P})-P_x,\quad f(\mathbf{P})=\eta(\mathbf{P})/P$$

and, for the CP EM wave,

$$B(\mathbf{P},\Phi_0,\tau)=\frac{P_yA}{\eta(\mathbf{P})}\cos[f(\mathbf{P})\tau+\Phi_0]-\frac{P_zA}{\eta(\mathbf{P})}\sin[f(\mathbf{P})\tau+\Phi_0],$$

$$x_{\sim}(\mathbf{P},\Phi_0,\tau)=-\frac{P_yA}{\eta(\mathbf{P})^2}\sin[f(\mathbf{P})\tau+\Phi_0]+\frac{P_zA}{\eta(\mathbf{P})^2}\cos[f(\mathbf{P})\tau+\Phi_0],\tag{21}$$

$$y_{\sim}(\mathbf{P},\Phi_0,\tau)=-\frac{A}{\eta(\mathbf{P})}\sin[f(\mathbf{P})\tau+\Phi_0],$$

$$z_{\sim}(\mathbf{P},\Phi_0,\tau)=\frac{A}{\eta(\mathbf{P})}\cos[f(\mathbf{P})\tau+\Phi_0].$$

We can introduce the Fourier transform $U(r)=\int U(k)\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k}$ in the integral T to obtain

$$T(\mathbf{P},\mathbf{r}_0,\Phi_0)=\sum_{n=-\infty}^{+\infty}\int F_n(\mathbf{k},\mathbf{P})U(k)\times\delta[(\mathbf{k}\cdot\mathbf{P})-n(\mathbf{P}\cdot\boldsymbol{\nu})+n\gamma(\mathbf{P})]\times\frac{\exp[-i(\mathbf{k}\cdot\mathbf{r}_0)-in\Phi_0]}{(2\pi)^2}d\mathbf{k},\tag{22}$$

where, for the LP EM wave,

$$F_n(\mathbf{k},\mathbf{P})=\sum_{s=-\infty}^{+\infty}\gamma(\mathbf{P})J_{n-2s}\left[\frac{(\mathbf{A}\cdot\mathbf{Q})}{\eta(\mathbf{P})}\right]J_s\left[\frac{(\mathbf{k}\cdot\boldsymbol{\nu})A^2}{8\eta^2(\mathbf{P})}\right]\times\left[1+\frac{n(\mathbf{P}\cdot\mathbf{A})}{(\mathbf{A}\cdot\mathbf{Q})\gamma(\mathbf{P})}+\frac{2s\,\eta(\mathbf{P})}{(\mathbf{k}\cdot\boldsymbol{\nu})\gamma(\mathbf{P})}\right],\tag{23}$$

and, for the CP EM wave,

$$F_n(\mathbf{k},\mathbf{P})=\gamma(\mathbf{P})e^{in\beta}J_n\left[\frac{AQ}{\eta(\mathbf{P})}\right]+\frac{Ae^{i(n+1)\beta}}{\eta(\mathbf{P})}[\mathbf{P}\cdot(\mathbf{e}_1+i\mathbf{e}_2)]J_{n+1}\left[\frac{AQ}{\eta(\mathbf{P})}\right]+\frac{Ae^{i(n-1)\beta}}{\eta(\mathbf{P})}[\mathbf{P}\cdot(\mathbf{e}_1-i\mathbf{e}_2)]J_{n-1}\left[\frac{AQ}{\eta(\mathbf{P})}\right],\tag{24}$$

$$Qe^{i\beta}=(\mathbf{e}_1\cdot\mathbf{Q})+i(\mathbf{e}_2\cdot\mathbf{Q}),\quad \mathbf{Q}=\mathbf{k}-\frac{\mathbf{P}(\boldsymbol{\nu}\cdot\mathbf{k})}{\eta(\mathbf{P})},$$

$J_n(x)$ and $J_n'(x)$ are ordinary Bessel functions. The derivative of the integral $T(\mathbf{P},\mathbf{r}_0,\Phi_0)$ with respect of Φ_0 is the relativistic change in the energy to the first order in the scattering potential [28]. Using Eq. (18) the second order energy exchange is finally obtained as

$$\Delta\gamma(\mathbf{P},\mathbf{r}_0)=\sum_{n=-\infty}^{+\infty}\int U(k)U(k')nF_{-n}(\mathbf{k}'\cdot\mathbf{P})\delta\times[(\mathbf{k}'\cdot\mathbf{P})+n(\mathbf{P}\cdot\boldsymbol{\nu})-n\gamma(\mathbf{P})]\times\frac{\exp[-i(\mathbf{k}\cdot\mathbf{r}_0)-i(\mathbf{k}'\cdot\mathbf{r}_0)]}{(2\pi)^4}\tag{25}$$

$$\left(\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{P}}\right)\{F_n(\mathbf{k},\mathbf{P})\delta[(\mathbf{k}\cdot\mathbf{P})-n(\mathbf{P}\cdot\boldsymbol{\nu})+n\gamma(\mathbf{P})]\}d\mathbf{k}d\mathbf{k}'.$$

The problem of scattering in the plane wave is symmetrical with respect to the translation of the scattering center along direction $\mathbf{P}$, and we can choose $(\mathbf{r_0}\cdot\mathbf{P})=0$. In this case $\rho=\sqrt{\mathbf{r}_0^2}$ is the impact parameter that is the minimal distance of the drift (unperturbed) trajectory from the scattering center. Averaging this expression over the initial position of the electrons relative to the position of the scattering center, $\mathbf{r}_0$, we obtain the IB emission-absorption coefficient in the UHI regime:

$$\Delta\gamma(\mathbf{P})=\int d^2r_0\Delta\gamma(\mathbf{P},\mathbf{r}_0)$$

we obtain the IB emission-absorption coefficient in the UHI regime

$$\Delta\gamma(\mathbf{P})=\sum_{n=-\infty}^{+\infty}\int U(k)U(k')nF_{-n}(\mathbf{k}',\mathbf{P})\delta[(\mathbf{k}'\cdot\mathbf{P})+n(\mathbf{P}\cdot\boldsymbol{\nu})-n\gamma(\mathbf{P})]\frac{\delta(\mathbf{k}_\perp+\mathbf{k}'_\perp)}{(2\pi)^4}\left(\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{P}}\right)\times\{F_n(\mathbf{k},\mathbf{P})\delta[(\mathbf{k}\cdot\mathbf{P})-n(\mathbf{P}\cdot\boldsymbol{\nu})+n\gamma(\mathbf{P})]\}d\mathbf{k}\,d\mathbf{k}',\tag{26}$$

where $\mathbf{k}_\perp$ and $\mathbf{k}'_\perp$ are the components $\mathbf{k}$ and $\mathbf{k}'$, perpendicular to $\mathbf{P}$. Integrating this equation over $\mathbf{k}'$, the final expression of the UHI emission-absorption coefficient is thus

$$\Delta\gamma=\sum_{n=-\infty}^{+\infty}\int\frac{U(k)^2n}{2(2\pi)^4\mathbf{P}}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{P}}\right)\times\{|F_n(\mathbf{k},\mathbf{P})|^2\delta[(\mathbf{k}\cdot\mathbf{P})-n(\mathbf{P}\cdot\boldsymbol{\nu})+n\gamma(\mathbf{P})]\}d\mathbf{k}.\tag{27}$$

The facts that $F_{-n}(-\mathbf{k},\mathbf{P})=F_n^*(\mathbf{k},\mathbf{P})$ and $U(k)$ is isotropic were used to derive Eq. (27). This result is indeed interesting, and is relevant to an arbitrary scattering potential and an arbitrary intense laser field. In Sec. IV we will evaluate this integral in order to study the Coulomb scattering case relevant to UHI laser-plasma interaction.

The Dirac distribution involved in Eq. (27) clearly displays the origin of the energy exchange mechanism: this exchange mechanism is resonant, and these resonances are harmonic Compton resonances identified in Ref. [26]. In the nonrelativistic limits expression (22) for $T(\mathbf{P},\mathbf{r}_0,\Phi_0)$ becomes

$$T(\mathbf{P},\mathbf{r}_0,\Phi_0)=\sum_{n=-\infty}^{+\infty}\int J_n(\mathbf{k}\cdot\mathbf{A})U(k)\delta(\mathbf{k}\cdot\mathbf{P}-n)\times\frac{\exp[-i(\mathbf{k}\cdot\mathbf{r}_0)-in\Phi_0]}{(2\pi)^2}d\mathbf{k}, \tag{28}$$

so that coefficient $\Delta\gamma$ becomes

$$\Delta\gamma=\frac{1}{2(2\pi)^4}\sum_{n=-\infty}^{+\infty}\int nJ_n^2(\mathbf{k}\cdot\mathbf{A})k^2U^2(k)\left(\frac{\mathbf{P}}{P^2}\cdot\frac{\partial}{\partial\mathbf{k}}\right)\times\delta(\mathbf{k}\cdot\mathbf{P}-n)d\mathbf{k}. \tag{29}$$

This nonrelativistic expression has been obtained without Madey's theorem in previous studies [13,14] so the fact that the low energy limit agrees with these results gives confidence in this result, and formula (27) can be viewed as a generalization of the expression of the energy exchange for nonrelativistic small-angle classical collisions [Eq. (29)] to the UHI relativistic regime.

Let us now consider the Coulomb potential as the scattering potential. We assume that this Coulomb scattering potential is screened above distances $r>r_{\max}=1/k_{\min}$ on the order of the Debye length, and to avoid divergence near the scattering center we use the classical procedure of Coulomb potential "softening" below distances $r<r_{\min}=1/k_{\max}$ of the order of the Landau length which approaches the classical electron radius in the relativistic case. Then the classical screened and softened Coulomb potential can be written as [29,14]

$$U(\mathbf{r})=\frac{eZ\exp(-k_{\min}r)[1-\exp(-k_{\max}r)]}{r}. \tag{30}$$

Even for this classical Coulomb potential the integral $T(\mathbf{p},\mathbf{r}_0,\Phi_0)$ is rather complicated to evaluate, and the scaling of the process with respect to the various parameter difficult to display. Nevertheless $T(\mathbf{P},\mathbf{r}_0,\Phi)$ can be calculated in two complementary limiting cases which are physically relevant: for remote collisions when the amplitude of the electron oscillations in the EM field is smaller than the impact parameter,

$$\rho>r_{\sim}(\mathbf{P},\Phi_0,t), \tag{31}$$

and for instantaneous collisions when the duration of the scattering process is smaller than the wave period,

$$\rho<\frac{\omega(\gamma-\mathbf{P}\cdot\boldsymbol{\nu})}{Pc^2}. \tag{32}$$

In typical laser-plasma experiments the drift energy of the fast electrons is of the order of a few keV, so that both regimes can overlap between $r_{\min}$ and $r_{\max}$, and the relative importance of the two regimes is determined by the value of the Debye length.

IV. ENERGY EXCHANGE FOR REMOTE COLLISIONS

In this section we will perform an integration of the previous formulas for remote collision emission and absorption [Eq. (27)]. In the UHI regime $A>1$, the amplitude of the electron oscillations is of the order of the wavelength, and inequality (31) means that the impact parameter is larger than the wavelength. Expanding integral (19) to the first order in $\mathbf{r}_\sim$ we obtain, for the linear polarization,

$$\begin{aligned}T(\mathbf{P},\mathbf{r}_0,\Phi_0,k_{\min},k_{\max})&=g_{L,0}(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})+g_{L,1}(\mathbf{P})\kappa_1(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})\cos[\Phi_0+B(\mathbf{P},\mathbf{r}_0)]\\&\quad+g_{L,2}(\mathbf{P},\mathbf{r}_0)\kappa_2(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})\sin[\Phi_0+B(\mathbf{P},\mathbf{r}_0)]+g_{L,3}(\mathbf{P})\kappa_3(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})\\&\quad\times\cos[2\Phi_0+2B(\mathbf{P},\mathbf{r}_0)]+g_{L,4}(\mathbf{P},\mathbf{r}_0)\kappa_4(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})\sin[2\Phi_0+2B(\mathbf{P},\mathbf{r}_0)],\end{aligned} \tag{33}$$

where the coefficients g and κ are given by

$$\kappa_1(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})=K_0[h(\mathbf{P},k_{\min})\rho(\mathbf{P},\mathbf{r}_0)]-K_0[k_{\max}\rho(\mathbf{P},\mathbf{r}_0)],$$

$$\kappa_2(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})=h(\mathbf{P},k_{\min})K_1[h(\mathbf{P},k_{\min})\rho(\mathbf{P},\mathbf{r}_0)]-k_{\max}K_1[k_{\max}\rho(\mathbf{P},\mathbf{r}_0)],$$

$$\kappa_3(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})=K_0[2h(\mathbf{P},k_{\min})\rho(\mathbf{P},\mathbf{r}_0)]-K_0[k_{\max}\rho(\mathbf{P},\mathbf{r}_0)],$$

$$\kappa_4(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})=2h(\mathbf{P},k_{\min})K_1[2h(\mathbf{P},k_{\min})\rho(\mathbf{P},\mathbf{r}_0)]-k_{\max}K_1[k_{\max}\rho(\mathbf{P},\mathbf{r}_0)],$$

$$g_{L,1}(\mathbf{P})=\frac{ZAP_yM^2}{\eta(\mathbf{P})P^3},$$

$$g_{L,2}(\mathbf{P},\mathbf{r}_0)=\frac{ZA\gamma(\mathbf{P})[t_0(\mathbf{P},\mathbf{r}_0)\gamma(\mathbf{P})-y_0\eta(\mathbf{P})-P_yx_0]}{\eta(\mathbf{P})P^2\rho(\mathbf{P},\mathbf{r}_0)h(\mathbf{P},k_{\min})},$$

$$g_{L,3}(\mathbf{P})=\frac{ZA^2[\gamma(\mathbf{P})P_x-P^2]}{4\eta(\mathbf{P})P^3},$$

$$g_{L,4}(\mathbf{P},\mathbf{r}_0)=\frac{ZA^2\gamma(\mathbf{P})[t_0(\mathbf{P},\mathbf{r}_0)P_x-x_0]}{2\eta(\mathbf{P})P^2\rho(\mathbf{P},\mathbf{r}_0)h(\mathbf{P},k_{\min})},$$

$$h(\mathbf{P},k_{\min})=\sqrt{\eta(\mathbf{P})^2/P^2+k_{\min}^2};$$

the time $t_0(\mathbf{P},\mathbf{r}_0)=(\mathbf{P}\cdot\mathbf{r}_0)/P^2$ is the instant of time when the electron is at the minimal distance from the Coulomb center, and $\rho(\mathbf{P},\mathbf{r}_0)=\sqrt{\mathbf{r}_0^2-(\mathbf{P}\cdot\mathbf{r}_0)^2/P^2}$ is the impact parameter. $K_n(x)$ is a modified Bessel function of nth order, and $B(\mathbf{P},\mathbf{r}_0)=\eta(\mathbf{P})t_0(\mathbf{P},\mathbf{r}_0)$ is the value of the $x-t$ phase at the time $t_0(\mathbf{P},\mathbf{r}_0)$. We will not present the expression for $g_{L,0}(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})$, because the first term in Eq. (33) does not depend on Φ_0 and, therefore, does not give any contribution to $\Delta\gamma$. Similarly, for circular polarization, we obtain

$$\begin{aligned}T(\mathbf{P},\mathbf{r}_0,\Phi_0,k_{\min},k_{\max})=&g_{C,0}(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})+[g_{C,1}(\mathbf{P})\kappa_1(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})+g_{C,2}(\mathbf{P},\mathbf{r}_0)\kappa_2(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})]\\&\times\cos[\Phi_0+B(\mathbf{P},\mathbf{r}_0)]+[g_{C,3}(\mathbf{P})\kappa_1(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})+g_{C,4}(\mathbf{P},\mathbf{r}_0)\kappa_2(\mathbf{P},\mathbf{r}_0,k_{\min},k_{\max})]\\&\times\sin[\Phi_0+B(\mathbf{P},\mathbf{r}_0)],\end{aligned}\tag{34}$$

where the coefficients g are given by

$$g_{C,1}(\mathbf{P})=-\frac{ZAP_zM^2}{\eta(\mathbf{P})P^3},$$

$$g_{C,2}(\mathbf{P},\mathbf{r}_0)=\frac{ZA\gamma(\mathbf{P})[t_0(\mathbf{P},\mathbf{r}_0)\gamma(\mathbf{P})-y_0\eta(\mathbf{P})-P_yx_0]}{\eta(\mathbf{P})P^2\rho(\mathbf{P},\mathbf{r}_0)h(\mathbf{P},k_{\min})},$$

$$g_{C,3}(\mathbf{P})=\frac{ZAP_yM^2}{\eta(\mathbf{P})P^3},$$

$$g_{C,4}(\mathbf{P},\mathbf{r}_0)=\frac{ZA\gamma(\mathbf{P})[t_0(\mathbf{P},\mathbf{r}_0)\gamma(\mathbf{P})-z_0\eta(\mathbf{P})-P_zx_0]}{\eta(\mathbf{P})P^2\rho(\mathbf{P},\mathbf{r}_0)h(\mathbf{P},k_{\min})}.$$

Given these two results we can explicitly calculate [30] the energy transfer from a LP EM wave to one electron resulting from remote Coulomb collisions when $k_{\min}$ is set to zero, because, as we will see below, our approach provides a natural cutoff:

$$\begin{aligned}\Delta\gamma(\mathbf{P},\mathbf{r}_0)=&-\frac{K_0^2(h\rho)\gamma}{2P^2}\left(2g_{L,1}^2+\frac{g_{L,2}^2M^2}{\gamma^2}\right)+\frac{K_1^2(h\rho)\gamma}{2P^2}\left(\frac{g_{L,2}^2P^2c^2}{\gamma^2}-g_{L,1}^2\right)+\frac{K_0(h\rho)K_1(h\rho)\omega\gamma\eta\rho}{c^2P^3}\\&\times\left(g_{L,2}^2+g_{L,1}^2-\frac{M^2g_{L,2}^2c^4P^2}{(\omega\rho\gamma\eta)^2}-\frac{[M^2(\eta^2+c^2P_y^2)-\gamma^2\eta^2]c^4P^2g_{L,1}^2}{2(\omega\rho\eta)^2M^2P_y^2}\right)-\frac{2K_0^2(h\rho)\gamma}{P^2}\left(\frac{M^2g_{L,3}^2}{\gamma^2}+2g_{L,4}^2\right)\\&+\frac{2K_1^2(h\rho)\gamma}{P^2}\left(\frac{g_{L,4}^2P^2c^2}{\gamma^2}-g_{L,3}^2\right)+\frac{8K_0(h\rho)K_1(h\rho)\omega\gamma\eta\rho}{c^2P^3}\left(g_{L,3}^2+g_{L,4}^2+\frac{e^8Z^2A^4M^2[2P^2x_0^2+(P_x^2-P^2)\rho^2]}{2^7\omega P^4(\eta\rho)^4}\right),\end{aligned}\tag{35}$$

where

$$g_{L,1}=\frac{D_1M^2P_y}{\eta P^3},\quad g_{L,2}=-\frac{D_1\gamma(P_yx_0+y_0\eta)}{\eta P^2\rho},$$

$$g_{L,3}=\frac{D_2(M^2-\gamma\eta)}{4\eta P^3},\quad g_{L,4}=-\frac{D_2A^2x_0\gamma}{4\eta P^2\rho},$$

$$h=\sqrt{\frac{\omega^2\eta^2}{c^4P^2}+k_{\min}^2},\quad \gamma^2=M^2+c^2P^2,$$

$$M^2=m^2c^4+\frac{e^2A^2}{2},\quad \eta=\gamma-cP_x,$$

$$D_1=\frac{e^3ZA\omega}{c^4},\quad D_2=\frac{e^4ZA^2\omega}{c^5}.$$

From this point on, the results come in usual physical unities. The parameter $h=\omega\tau$ has a simple physical meaning in the nonrelativistic limit as the ratio of the period of the EM field to the scattering time: $(\tau=\rho/V)$. We see from Eq. (35) that the expression for $\Delta\gamma$ for IB is expressed in term of modified Bessel function as well of the spontaneous bremsstrahlung [31] and has the following asymptotic values: $\Delta\gamma\propto\exp(-h)$ at $h\to\infty$ and $\Delta\gamma\propto\ln(h)/h^2$ at $h\to 0$. Therefore, $\Delta\gamma$ has the natural cutoff $\rho=cP/\omega\eta$ (V/ω in the nonrelativistic limit) in the approach used even for $r_{\max}\to\infty$.

Note that in the nonrelativistic limit we recover the classical linear bremsstrahlung [14], so that we conclude that the nonrelativistic approximation always take place in the remote collision case. This is quite logical, as the quiver motion is supposed to be small with respect to all the other energy scales of the problem [$n=1$, $(\mathbf{k}\cdot\mathbf{A})\ll 1$, $J_1(\mathbf{k}\cdot\mathbf{A})$

$\approx(\mathbf{k}\cdot\mathbf{A})/2]$. In the relativistic case IB is nonlinear even under condition (31), because of the dependence of the relativistic mass M on the intensity of the EM field.

In order to discuss the previous results, we introduce a set of polar coordinates (Ψ,ρ) in a plane perpendicular to $\mathbf{P}$, which go through the position of the Coulomb center (see Fig. 1):

$$x_0=\frac{\rho P_x P_z}{P\sqrt{P_x^2+P_y^2}}\cos\Psi-\frac{\rho P_y}{\sqrt{P_x^2+P_y^2}}\sin\Psi,$$
$$y_0=\frac{\rho P_y P_z}{P\sqrt{P_x^2+P_y^2}}\cos\Psi+\frac{\rho P_x}{\sqrt{P_x^2+P_y^2}}\sin\Psi, \tag{36}$$
$$z_0=-\frac{\sqrt{P_x^2+P_y^2}\rho\cos\Psi}{P}.$$

In this case the impact parameter becomes $\rho=\sqrt{\mathbf{r}_0^2}$, and Ψ is the azimuthal angle corresponding to a polar axis parallel to $\mathbf{P}$. After averaging over this azimuthal angle Ψ for the LP EM wave we obtain

$$\begin{aligned}\Delta\gamma(\mathbf{P},\mathbf{r}_0)=&-\frac{K_0^2(h\rho)\gamma}{P^2}\left(g_{L,1}^2+\frac{\langle g_{L,2}^2\rangle M^2}{2\gamma^2}\right)\\&+\frac{K_1^2(h\rho)\gamma}{2P^2}\left(\frac{\langle g_{L,2}^2\rangle P^2c^2}{2\gamma^2}-g_{L,1}^2\right)\\&+\frac{K_0(h\rho)K_1(h\rho)\omega\gamma\eta\rho}{c^2P^3}\\&\times(\langle g_{L,2}^2\rangle+g_{L,1}^2)-K_0^2(2h\rho)\\&\times\left(\frac{2M^2\langle g_{L,4}^2\rangle}{\gamma P^2}+\frac{4\gamma g_{L,3}^2}{P^3}\right)\\&+K_1^2(2h\rho)\left(-\frac{2c^2\gamma g_{L,3}^2}{P^2}+\frac{2\langle g_{L,4}^2\rangle}{\gamma}\right)\\&+K_0(2h\rho)K_1(2h\rho)\\&\left(\frac{8\gamma\omega\eta\rho g_{L,3}^2}{c^2P^3}+\frac{8\eta\omega\rho\gamma\langle g_{L,4}^2\rangle}{c^2P^3}\right),\end{aligned}\tag{37}$$

where

$$\langle g_{L,2}^2\rangle=\frac{D_1^2\gamma^2(P^2\eta^2-M^2P_y^2)}{2\eta^2P^6},\quad \langle g_{L,4}^2\rangle=\frac{D_2^2M^2(P^2-P_x^2)}{32\eta^2P^6},$$

and, for the CP EM wave,

$$\begin{aligned}\Delta\gamma(\mathbf{P},\mathbf{r}_0)=&-\frac{K_0^2(h\rho)\gamma}{P^2}\left(g_{C,1}^2+g_{C,3}^2+\frac{\langle g_{C,2}^2+g_{C,4}^2\rangle M^2}{2\gamma^2}\right)\\&+\frac{K_1^2(h\rho)\gamma}{2P^2}\left(\frac{\langle g_{C,2}^2+g_{C,4}^2\rangle P^2c^2}{2\gamma^2}-g_{C,1}^2-g_{C,3}^2\right)\\&+\frac{K_0(h\rho)K_1(h\rho)\omega\gamma\eta\rho}{c^2P^3}\\&\times(\langle g_{C,2}^2+g_{C,4}^2\rangle+g_{C,1}^2+g_{C,3}^2),\end{aligned}\tag{38}$$

where

$$g_{C,1}^2+g_{C,3}^2=\frac{D_1^2M^4(P_y^2+P_z^2)}{\eta^2P^6},$$
$$\langle g_{C,2}^2+g_{C,4}^2\rangle=\frac{D_1^2\gamma^2[2P^2\eta^2-M^2(P_y^2+P_z^2)]}{2\eta^2P^6},$$
$$M^2=m^2c^4+e^2A^2.$$

The averaging over Ψ can be considered as the classical random phase approximation in the context of weak turbulence plasma theory, or as classical phase averaging in the context of beam-wave interaction theory. Note that when $\Delta\gamma$ is averaged over the azimuthal angle Ψ it has the asymptotic values $\Delta\gamma\propto 1/h^2$ at $h\to 0$. Since the perturbation theory with respect to the Coulomb potential is developed the validity condition of the used approximation is

$$\rho>\left\{\frac{e^2Z\gamma}{P^2c^2},r_\sim\right\}. \tag{39}$$

Similarly the general expression of $\Delta\gamma(\mathbf{P},\rho,\Psi,k_{\max},k_{\min})$ can be calculated for the general softened and screened Coulomb potential with $k_{\max}=1/r_{\min}$, $k_{\min}=1/r_{\max}$. However, the final result is rather cumbersome for the relativistic regime, and we present the result in the nonrelativistic limit

$$\begin{aligned}\Delta\gamma(\mathbf{P},\rho,k_{\max})=&\frac{Z^2e^6A^2}{m^3c^3\omega^2V}\frac{a^4}{2}\Big[-2K_1^2(\rho a)\cos^2\varphi-K_0^2(\rho a)(4\cos^2\varphi+\sin^2\varphi)+\rho aK_0(\rho a)K_1(\rho a)(4\cos^2\varphi+2\sin^2\varphi)\\&-2K_1^2(\rho k_{\max})\cos^2\varphi\frac{k_{\max}^2}{a^2}-K_0^2(\rho k_{\max})\frac{k_{\max}^2}{a^2}\sin^2\varphi+\rho k_{\max}K_1(\rho k_{\max})K_1(\rho k_{\max})\frac{k_{\max}^2}{a^2}\sin^2\varphi\\&+4\frac{k_{\max}}{a}K_1(\rho k_{\max})K_1(\rho a)\cos^2\varphi-2k_{\max}\rho K_1(\rho k_{\max})K_0(\rho a)-\frac{\rho k_{\max}^2}{a}K_0(\rho k_{\max})K_1(\rho a)\sin^2\varphi\\&+\frac{k_{\max}^2}{a^2}K_0(\rho k_{\max})K_0(\rho a)\sin^2\varphi\Big],\end{aligned}\tag{40}$$

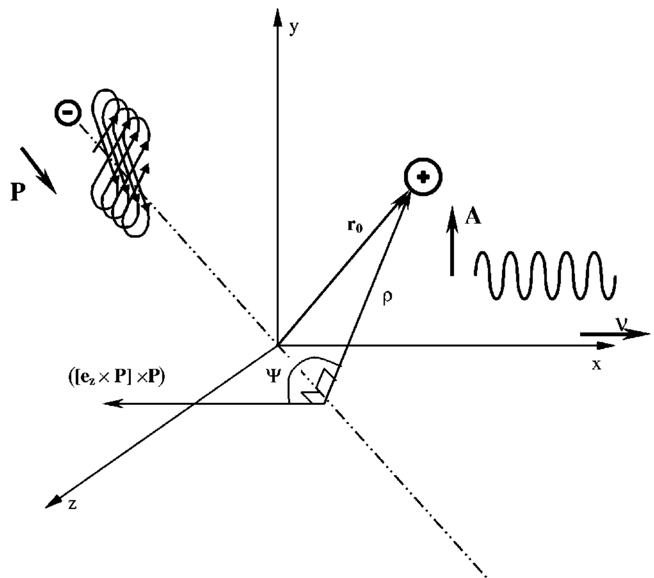

FIG. 1. Coordinate system used in the description of the scattering processes. **P** is the drift momentum of the electron, **A** is the vector potential of the LP EM wave, $\boldsymbol{\nu}$ is the direction of the wave propagation, $\mathbf{r}_0$ is the radius vector of the Coulomb center, and ρ is the distance between the Coulomb center and electron trajectory (drift) unperturbed by the Coulomb field (shown by the dashed line). In the plane perpendicular to the drift trajectory and extending through the Coulomb center, the polar coordinates (Ψ,ρ) are introduced.

where $a=\omega/V$, $V=P/m$ is the electron velocity and φ is the angle between **A** and **P**. Note that $\Delta\gamma$ does not depend on Ψ in the nonrelativistic limit because the electron motion in weak EM wave is one dimensional. We have integrated this expression for $\Delta\gamma(\mathbf{P},\rho,\Psi,k_{\max},k_{\min})$ over Ψ and ρ and the final result for the power transfer Q from the laser field to an electron population with density n_b for a LP EM wave is given by the expression

$$\begin{aligned} Q(\mathbf{P},k_{\max},k_{\min}) &= n_b V\int_0^{+\infty}\int_0^{2\pi}\Delta\gamma(\mathbf{P},\rho,\Psi,k_{\max},k_{\min})\rho d\rho d\Psi \\ &= \frac{\pi c^2 n_b}{2h^2 P}\left(\frac{g_{L,1}^2\eta\gamma}{M^2}+g_{L,3}^2-\langle g_{L,4}^2\rangle\frac{c^2P^2+c\eta P_x}{2\eta\gamma}\right) \\ &\quad+\frac{\pi c^2 n_b}{2h^2P}\{(\langle g_{L,2}^2\rangle-g_{L,1}^2)[1+\ln(k_{\max}/h)] \\ &\quad+(\langle g_{L,4}^2\rangle-g_{L,3}^2)[1+\ln(k_{\max}/2h)]\}, \end{aligned} \quad (41)$$

and, for the CP EM wave,

$$\begin{aligned} Q(\mathbf{P},k_{\max},k_{\min}) &= \frac{\pi c^2 n_b \eta}{2h^2PM^2}(g_{C,1}^2+g_{C,3}^2) \\ &\quad+\frac{\pi c^2 n_b}{2h^2P}\{[\langle g_{C,2}^2+g_{C,4}^2\rangle-(g_{C,1}^2+g_{C,3}^2)] \\ &\quad\times[1+\ln(k_{\max}/h)]\}, \end{aligned} \quad (42)$$

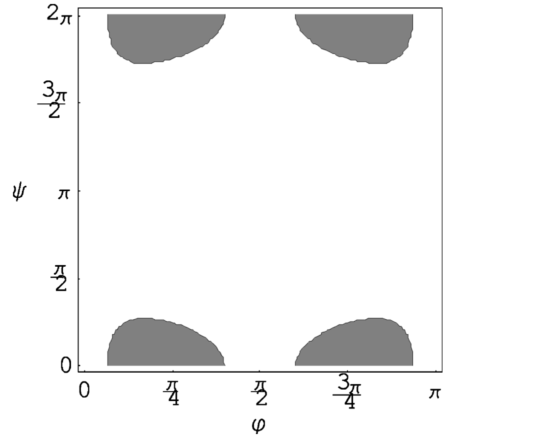

FIG. 2. The domains of the angles φ (the angle between the initial momentum of the electron and the wave polarization direction) and ψ (the azimuthal angle with the polar axis along the wave polarization direction), where the LP EM wave is amplified by the electrons (gray region), and where the LP EM wave is absorbed by the electrons (white region) for $Pc/M=0.2$ and $\ln(k_{\max}/2h)=10$. The angles are given in rad.

where $r_{\min}=1/k_{\max}$ can be either the Landau length $Ze^2\gamma/P^2c^2$ for classical collisions or the de Broglie length $\hbar/P$ for quantum collisions, and $r_{\max}=1/k_{\min}$ is the Debye length $r_{\mathrm{De}}=\sqrt{4\pi e^2 n_e/T_e}$ of a plasma with a electron density n_e and electron temperature T_e. Using this expression for Q the rate of IB in plasma with an electron distribution function $F(\mathbf{P})$ can be expressed as

$$\upsilon_{\mathrm{brem}}=\frac{n_e\int Q(\mathbf{P},k_{\max},k_{\min})F(\mathbf{P})d\mathbf{P}}{W}, \quad (43)$$

where $W=\omega^2A^2/8\pi c^2+n_e(\gamma-mc^2)$ is the energy density of an EM wave in plasma.

In the nonrelativistic limit $M^2\approx mc^2$, $D2=0$, and for $r_{\min}\approx\hbar/P$, $r_{\max}\gg V/\omega$ Eq. (41) becomes

$$Q=\frac{e^6Z^2A^2n_b}{\omega^2P^3c^2}\{2\cos^2\varphi+(1-3\cos^2\varphi)[\ln(P^2/m\omega\hbar)+1]\}, \quad (44)$$

which coincides with the quantum-mechanical formula describing the so called Marcuse effect [32], and obtained in

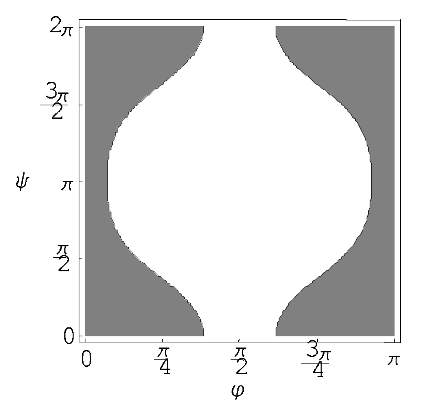

FIG. 3. The same as Fig. 2, $Pc/M=0.8$.

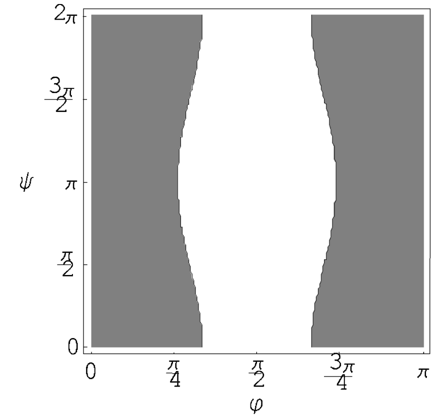

FIG. 4. The same as Fig. 2, $Pc/M=1.2$.

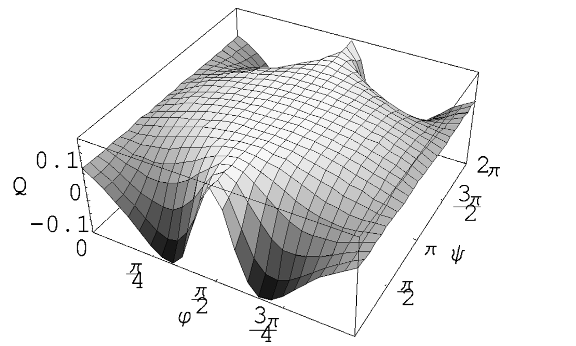

FIG. 6. The dependence of the power of the CP EM wave absorbed by the electrons, Q, scaled by $D_1^2 n_b c^2/2h^2 PM^2$, on the angles φ, and ψ for $\ln(k_{\max}/2h)=10$. The angles are given in rad.

the first Born approximation. Here φ is the angle between the vectors $\mathbf{A}$ and $\mathbf{P}$. We see from Eq. (44) that negative absorption is possible for a LP EM wave if the beam velocity $\mathbf{V}$ lies inside a cone whose axis coincides with the wave polarization direction $\mathbf{e}_y$, and whose generatrices take an angle $\varphi \approx \arccos(1/\sqrt{3})$ with this axis. For a CP EM wave this condition requires that the beam velocity lies outside a cone whose axis coincides with the wave propagation direction $\boldsymbol{\nu}$ and whose generatrices make an angle $\phi \approx \arccos(1/\sqrt{3})$ with this axis.

Formula (40) can be considered as a generalization of Marcuse's effect to take into account the dependence on the impact parameter $\boldsymbol{\rho}$. Note that in the nonrelativistic limit we recover the classical linear bremsstrahlung [14], and the condition ($n=1$, $(\mathbf{k}\cdot\mathbf{A})\ll 1$, $J_1(\mathbf{k}\cdot\mathbf{A})\approx(\mathbf{k}\cdot\mathbf{A})/2$ for Eq. (29) in $\mathbf{k}$ space is equivalent to the condition for a remote collision [Eq. (31)] in $\mathbf{r}$ space. This is quite logical, as the quiver motion is supposed to be small with respect to scale of the problem. IB in the relativistic limit is nonlinear even under condition (31) because of the dependence of the effective mass M on the intensity of EM field.

What we have found here is the relativistic formula for the Marcuse's effect. For a LP EM wave, Q also depends on the azimuthal angle ψ when we consider an electron distribution function such that $Pc>mc^2$ (see Figs. 2–4, where we introduce the spherical coordinate system $P_x=P\sin\varphi\cos\psi$,$P_y=P\cos\varphi$, and $P_z=P\sin\varphi\sin\psi$). We see in Figs. 2–4 in the φ, ψ plane that the region where negative absorption is possible decreases with respect to the classical nonrelativistic case, and is located near the direction of the wave propagation if the kinetic energy of the electron drift motion increases and becomes larger than the electron rest mass. We see in Fig. 5 that in the case of a CP EM wave, the value of the angle ϕ between the electron momentum and the direction of the wave propagation where negative absorption of a CP EM wave is possible is determined by the ratio of the electron momentum to the electromagnetic mass Pc/M. The maximum wave absorption takes place when both the relativistic electron beam and the wave are propagating in the same direction (see Figs. 6 and 7).

The relativistic Marcuse effect has been investigated for a weak EM wave [33,34] in the framework of quantum mechanics. To compare our result with the result obtained, for example, in Ref. [34], let us introduce the notation used in Ref. [34]: $P_x=P\cos\theta$, $P_y=P\sin\theta\cos\phi$, and $P_z=P\sin\theta\sin\phi$. Then the rate for IB can be presented as follows [33,34]: $\Delta\gamma=\tau_{\parallel}\cos^2\phi+\tau_{\perp}\sin^2\phi$. In the approximation used in Ref. [34]—$eA/mc^2\ll 1$, $\theta\approx\gamma/mc^2\ll 1$—we have $g_3\approx g_4\approx 0$, $\gamma\approx Pc$, and $\eta\approx m^2c^4(1+u^2)/2\gamma$, where $u=\theta\gamma/mc^2$ and, using the general expression (40), we obtain

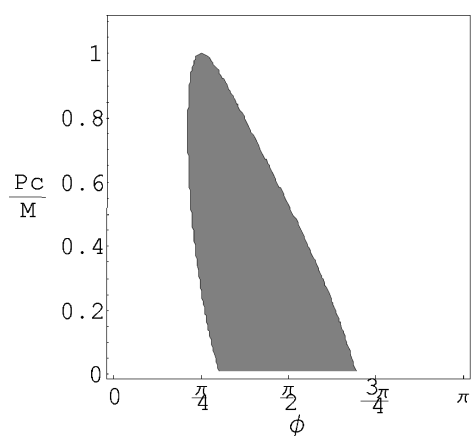

FIG. 5. The domains of the angle ϕ (the angle between the initial momentum of the electron and the wave propagation direction $\boldsymbol{\nu}$) and of the normalized momentum of the electron beam, Pc/M, where the CP EM wave is amplified by the electrons (gray region) and where the CP EM wave is absorbed by the electrons (white region) for $\ln(k_{\max}/2h)=10$. The angle is given in rad.

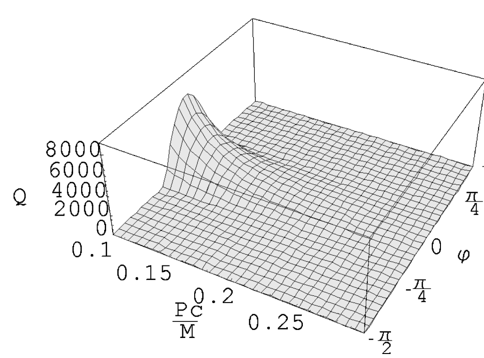

FIG. 7. The dependence of the power of the CP EM wave absorbed by the electrons, Q, scaled by $D_1^2 n_b c^2/2h^2 PM^2$, on the normalized momentum of the electrons, Pc/M, and on the angle between the initial momentum of the electrons and the CP wave propagation direction, ϕ, for $\ln(k_{\max}/2h)=10$. The angle is given in rad.

$$\tau_{\parallel} \propto -\frac{(1-u^2)^2}{(1+u^2)^4}[1+\ln(2\gamma^2/\hbar\omega mc^2)] - \frac{4u^2}{(1+u^2)^4}, \quad \tau_{\perp} \propto -\frac{1}{(1+u^2)^2}[1+\ln(2\gamma^2/\hbar\omega mc^2)], \tag{45}$$

that coincide with Eqs. (17) and (18) in Ref. [34].

If the drift energy is nonrelativistic or much less than the energy of the oscillating motion $P^2c^2 \ll M^2$, then $\eta \approx \gamma \approx M$, and Eq. (41) gives

$$Q(\mathbf{P},k_{\max},k_{\min}) = \frac{\pi D 1^2 n_b c^{12}}{4M^3V^3\omega^2(V^2/\omega^2 r_{\max}^2+1)}\left\{\frac{2\cos^2\varphi}{\sqrt{V^2/\omega^2 r_{\max}^2+1}} + (1-3\cos^2\varphi)[1-\ln(r_{\min}\omega/V)]\right\} \tag{46}$$

and, for the CP EM wave,

$$Q(\mathbf{P},k_{\max},k_{\min}) = \frac{\pi D 1^2 n_b c^{12}}{4M^3V^3\omega^2(V^2/\omega^2 r_{\max}^2+1)}\left\{\frac{2\cos^2\phi}{\sqrt{V^2/\omega^2 r_{\max}^2+1}} - (1-3\cos^2\phi)[1-\ln(r_{\min}\omega/V)]\right\}. \tag{47}$$

After averaging Q in Eqs. (46) and (47) over an isotropic electron distribution function, the terms that are proportional to the logarithm vanish. As a result the averaged rate of IB in the relativistic UHI regime is reduced M^3 times as compared with one obtained in the case of a weak EM wave.

V. RELATIVISTIC EMISSION AND ABSORPTION FOR INSTANTANEOUS COLLISIONS

Let us now calculate integral (19) for instantaneous collisions [Eq. (32)]. Within the framework of this approximation the scattering events is instantaneous and take place at time τ_0. In this case the expression for $\mathbf{r}^{\sim}[\mathbf{P}_0,\Phi_0+\tau f(\mathbf{P}_0)]$ in integral (19) can be expanded above the scattering moment τ_0:

$$\begin{aligned}\mathbf{r}^{\sim}[(\mathbf{P}_0,\Phi_0+\tau f(\mathbf{P}_0)] &= \mathbf{r}^{\sim}[\mathbf{P}_0,\Phi_0+\tau_0 f(\mathbf{P}_0)] + (\tau-\tau_0) \\ &\quad\times\frac{\partial \mathbf{r}^{\sim}[\mathbf{P}_0,\Phi_0+\tau_0 f(\mathbf{P}_0)]}{\partial\Phi_0} \\ &= \mathbf{r}^{\sim}[\mathbf{P}_0,\Phi_0+\tau_0 f(\mathbf{P}_0)] \\ &\quad + (\tau-\tau_0)\mathbf{P}^{\sim}[\mathbf{P}_0,\Phi_0+\tau_0 f(\mathbf{P}_0)].\end{aligned} \tag{48}$$

Here we will again use the unities $e=m=c=\omega=1$ for algebraic manipulations, and will distinguish among the electron momentum at the scattering, $\mathbf{P}$, the oscillating component of the electron momentum at the scattering, $\mathbf{P}^{\sim}$, and the drift component of the electron momentum at the scattering, $\mathbf{P}_0$. Then integral (19) for $T(\mathbf{P}_0,\mathbf{r}_0,\Phi_0)$ can be calculated for scattering potential (30)

$$\begin{aligned}T(\mathbf{P}_0,\mathbf{r}_0,\Phi_0) = \frac{B(\mathbf{P}_0,\Phi_0,\tau=0)}{P(\mathbf{P}_0,\Phi_0)}\{&K_0[k_{\min}\rho(\mathbf{r}_0,\mathbf{P}_0,\Phi_0)] \\ &-K_0[k_{\max}\rho(\mathbf{r}_0,\mathbf{P}_0,\Phi_0)]\},\end{aligned} \tag{49}$$

where

$$\mathbf{P}(\mathbf{P}_0,\Phi_0) = \mathbf{P}_0 + \mathbf{P}^{\sim}(\mathbf{P}_0,\Phi_0),$$

$$\rho(\mathbf{P}_0,\mathbf{r}_0,\Phi_0) = \sqrt{\mathbf{r}^2(\mathbf{P}_0,\mathbf{r}_0,\Phi_0)},$$

$$\mathbf{r}(\mathbf{P}_0,\mathbf{r}_0,\Phi_0) = \mathbf{r}_0 + \mathbf{r}^{\sim}(\mathbf{P}_0,\Phi_0) - \mathbf{P}(\mathbf{P}_0,\Phi_0)t_0(\mathbf{P}_0,\mathbf{r}_0,\Phi_0), \tag{50}$$

$$t_0(\mathbf{P}_0,\mathbf{r}_0,\Phi_0) = \frac{\mathbf{P}(\mathbf{P}_0,\Phi_0)\cdot(\mathbf{r}_0+\mathbf{r}^{\sim})}{P^2(\mathbf{P}_0,\Phi_0)},$$

$$\Phi_0 \equiv \Phi_0 + f(\mathbf{P}_0)\tau_0,$$

$\mathbf{r}(\mathbf{P}_0,\mathbf{r}_0,\Phi_0)$, $\mathbf{r}_0$, and $\mathbf{r}^{\sim}(\mathbf{P}_0,\Phi_0)$ are the total distance, drift, and oscillating component of the distance, respectively, between the electron and the Coulomb center at the scattering phase Φ_0. Then the energy exchange $\Delta\gamma(\mathbf{P},\rho,\Psi,k_{\max},k_{\min})$ can be calculated by the addition of Eq. (16).

We restrict ourselves to the nonrelativistic limit because of the complexity of the algebraic manipulation. In the nonrelativistic approximation, we obtain

$$B(\mathbf{P}_0,\Phi_0,\tau=0) = 1, \quad f(\mathbf{P}_0) = 1,$$

$$\mathbf{r}^{\sim}(\mathbf{P}_0,\Phi_0) = \mathbf{r}^{\sim}(\Phi_0), \quad \mathbf{P}^{\sim}(\mathbf{P}_0,\Phi_0) = \mathbf{P}^{\sim}(\Phi_0), \tag{51}$$

$$T(\mathbf{P}_0,\mathbf{r}_0,\Phi_0) = \frac{\{K_0[k_{\min}\rho(\mathbf{P}_0,\mathbf{r}_0,\Phi_0] - K_0[k_{\max}\rho(\mathbf{P}_0,\mathbf{r}_0,\Phi_0])\}}{P(\mathbf{P}_0,\Phi_0)},$$

In order to calculate $\Delta\gamma$ integrated over impact parameter ρ and azimuthal angle Ψ, it is more convenient to employ Eq. (29). The expression of the sum of the Bessel function with δ functions in Eq. (29) can be presented as follows:

$$\sum_{n=-\infty}^{+\infty} nJ_n^2(\mathbf{k}\cdot\mathbf{A})\left(\frac{\mathbf{P}_0}{P_0^2}\cdot\frac{\partial}{\partial\mathbf{k}}\right)\delta(\mathbf{k}\cdot\mathbf{P}_0-n)$$
$$=\left\langle\left(\frac{\mathbf{k}}{k^2}\cdot\frac{\partial^2}{\partial\mathbf{P}_0\partial\Phi_0'}\bigg|_{\Phi_0'=\Phi_0}\right)\int dt\,\exp\left\{-\frac{(\mathbf{k}\cdot\mathbf{P}_0)t+(\mathbf{k}\cdot\mathbf{A})[\sin(t+\Phi_0)-\sin\Phi_0']}{i}\right\}\right\rangle_{\Phi_0}. \tag{52}$$

Expanding $\sin(t+\Phi_0)$ above the scattering moment t_0, Eq. (52) can be rewritten as follows for instantaneous collisions:

$$\sum_{n=-\infty}^{+\infty} nJ_n^2(\mathbf{k}\cdot\mathbf{A})\left(\frac{\mathbf{P}_0}{P_0^2}\cdot\frac{\partial}{\partial\mathbf{k}}\right)\delta(\mathbf{k}\cdot\mathbf{P}_0-n)$$
$$\approx\left\langle(\mathbf{k}\cdot\mathbf{A})\cos\Phi_0\left(\frac{\mathbf{P}}{P^2}\cdot\frac{\partial}{\partial\mathbf{k}}\right)\delta(\mathbf{k}\cdot\mathbf{P})\right\rangle_{\Phi_0}. \tag{53}$$

Then inserting this expression into Eq. (29) and integrating over $\mathbf{k}$ for the scattering potential (30) the wave energy transfer to the electrons can be calculated for the instantaneous collisions in the nonrelativistic limit

$$\Delta\gamma(\mathbf{P},k_{\max},k_{\min})=-2\pi\left\langle\frac{(\mathbf{P}\cdot\mathbf{P}^{\sim})[\ln(k_{\max}/k_{\min})+1]}{P_0P^3}\right\rangle_{\Phi_0}, \tag{54}$$

where

$$\mathbf{P}=\mathbf{P}_0+\mathbf{P}^{\sim},\quad \mathbf{P}^{\sim}=\mathbf{A}\cos\Phi_0.$$

Equation (54) coincides with the expression for the energy exchange of IB obtained in the impact approximation [15,16]. The validity of the impact approximation for small-angle collisions was also shown as an asymptotic limit of the Born approximation [35].

It is interesting to note that, although the impact approximation and the approximation for small-angle collision (or ''straight path approximation'') lead to the same expression for the IB rate averaged over the impact parameter (within the overlap of the regions of their validity [see Eq. (54) and Ref. [35]]) they give different expressions for the IB rate as a function of the impact parameter and the angle between the wave polarization direction and the initial electron momentum, φ. Let us consider this fact in more detail. For simplicity, we assume that the collision is nonrelativistic, instantaneous ($\epsilon=\omega\rho/V\ll 1$), and remote ($eA/mc\omega\ll\rho$). Electrons suffer small-angle scattering $\chi=eZ/\rho mV^2\ll 1$, the LP EM wave is weak ($\xi=V_\sim/V=eA/cmV\ll 1$), and $V/\omega\ll r_{\max}$, $r_{\min}\ll\rho$. These hypothesis can be summarized in the form

$$\left\{\frac{eA}{mc\omega},r_{\min},\frac{eZ}{mV^2}\right\}\ll\rho\ll\frac{V}{\omega}\ll r_{\max},$$
$$\xi=eA/cmV\ll 1. \tag{55}$$

Using the expression derived in the impact approximation in Ref. [16] [Eqs. (2.24) and (2.25)] for the change in electron energy due to the scattering in EM and Coulomb fields,

$$\Delta\gamma(\chi,\varphi)=mV^2\frac{u(\varphi,\Phi_0)}{V}\xi\sin\Phi_0[(1-\cos\chi)$$
$$\times(\cos\varphi+\xi\sin\Phi_0)+\sin\chi\sin\varphi\cos\Psi], \tag{56}$$

where $u(\varphi,\Phi_0)=V\sqrt{1+2\xi\sin\Phi_0\cos\varphi+\xi^2\sin^2\Phi_0}$ is the total velocity of an electron in a LP EM wave, after averaging over Φ_0, and Ψ and with assumptions (55), $\Delta\gamma$ can be reduced to the form

$$\Delta\gamma(\chi,\varphi)=mV^2\chi^2\xi^2(1-3\cos^2\varphi)[1+O(\xi,\epsilon,\chi)]. \tag{57}$$

In the case of remote collisions, $\Delta\gamma(\chi,\varphi)$ can be obtained from Eq. (37) with assumptions (55)

$$\Delta\gamma(\chi,\varphi)=-mV^2\chi^2\xi^2\cos^2\varphi[1+O(\xi,\epsilon,\chi)]. \tag{58}$$

This expression can be also calculated by the straightforward integration of the equations of the electron motion [for example, by integration of Eq. (44) in Ref. [14] over $\mathbf{k}$ and $\mathbf{k}_1$ under condition (55)]. The possible reason for the difference between the dependence of $\Delta\gamma(\chi,\varphi)$ on φ in Eqs. (57) and (58) can be found by analyzing the impact approximation.

It is assumed in the context of the impact approximation [16] that the electron interacts with only an EM wave before and after the scattering, while the scattering is elastic (the electromagnetic field does not affect the scattering process) and instantaneous with respect to the period of the EM wave. The absorption from the EM field to the thermal energy is identified with the reorientation of the quiver into translational motion in an elastic collision. Although the model is simple, it cannot be considered rigorous. The condition that the collisions are instantaneous implies that scattering takes place in the static magnetic and electric fields (the ''frozen'' EM field of the wave). If these fields are sufficiently intense, they can modify the elastic scattering. This means that the condition $\omega\rho/V\ll 1$ is not a sufficient condition for the impact approximation [36] to be valid. It should be noted that the expansion in the small parameter defined as the EM frequency times on the factor which depends on the intensity of the EM field are used in various versions of the Krall-

Watson theorem [9], and that these expansions will break down at sufficiently intense EM field. Another reason is that the action of the component of the Coulomb force that is longitudinal to the electron velocity at the scattering is not taken into account in this approximation, because this component does not affect the electron deflection at the elastic scattering. As a result, in this model, the energy exchange is absent at the scattering by the one-dimensional short-range potential that is also the problem in the general case.

This difference in results [between Eqs. (57) and 58)] is not important for practical purposes, because the IB rate averaged over the impact parameter is interesting for atomic physics. It is more important that, according to Eq. (54) (see also Ref. [35]), the impact approximation gives the correct result for the IB rate averaged over the impact parameter, at least for small-angle nonrelativistic scattering in the presence of an intense EM wave.

VI. CONCLUSION

In this study we have addressed and solved several issues relevant to the problem of collisional absorption of ultrahigh intensity relativistic laser pulses. We have set up this problem with the help of a relativistic Hamiltonian formalism.

Within this framework the collisional absorption (emission) is clearly due to a set of resonances [Eq. (27)] which can be identified as harmonic Compton resonances induced by virtual photons of the Coulomb field. From a physical point of view these Compton resonances are relativistic Landau resonances between the drift motion of the electron and the beating of the laser wave with the Coulomb longitudinal virtual photon (the Fourier component of the Coulomb potential). In order to calculate the energy exchange between the wave and the electron, we have to sum and integrate all the interaction matrix elements corresponding to each of these resonances. We were able to perform such a complete calculation under two complementary hypotheses. First when the impact parameter is larger than the amplitude of the quiver oscillation (the so-called remote collision approximation), and then under the impact approximation when the interaction time is shorter than the period of the driving wave.

The expression for a frequency spectrum (energy per unit area frequency interval) as a function of the impact parameter is known for spontaneous bremsstrahlung [31]. The results we have obtained in using this allowed us to study the scaling of the stimulated bremsstrahlung with respect to the impact parameter, the quiver, and the drift momentum, and also to explore the relativistic regime. This latter study revealed some profound modifications of the gain function in this relativistic regime.

We have integrated these coefficients over the impact parameter, and derived the power transfer from a laser wave to an electron distribution. Taking the nonrelativistic limit of our results, we have recovered the well known nonrelativistic result. However, in doing so we discover a very subtle difference between the averaged and nonaveraged (over the impact parameter) coefficients in the impact approximation. We resolved this point and discussed the relevance of the various approximations.

Finally, besides the results presented, from a methodological point of view we have shown that Madey's theorem is not only useful to study free electron lasers but can be very helpful for all relativistic radiation problems. Indeed, Madey's theorem is usually used to calculate small signal gain in the theory of free electron lasers and microwave devices [18,21]. In this weak EM wave approximation, the energy gain $\Delta\gamma$ is proportional to the squared amplitude of an EM field, and is nonlinear with respect to the external static field. In this paper we have used Madey's theorem in the opposite limiting case: when the external static (or scattering) field is weak and the EM wave can have an arbitrary intensity. In this approximation $\Delta\gamma$ is proportional to the squared scattering potential and is nonlinear with respect to the amplitude of EM field [see Eqs. (27) and (29)].

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