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Average dynamics of the optical soliton in communication lines with dispersion management: Analytical results

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Applying asymptotic methods to a previously derived system of ordinary differential equations, we present an analytical description of the slow (average) dynamics of self-similar breathing pulses propagating in fiber links with dispersion management. We derive asymptotic averaged quantities (adiabatic invariants) that characterize the stable pulse propagation. In a particular, but practically important, case when the dispersion compensation period is much larger than the amplification distance we have found analytically the fixed points corresponding to the dispersion-managed solitons. [S1063-651X(98)51007-0]

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I. INTRODUCTION

Recent impressive developments in lightwave transmission systems stimulated by increasing demands for telecommunication services are based both on innovations of technology and on advances in research. In particular, the realization of soliton-based transmission has clearly demonstrated how results of the fundamental soliton theory can be successfully exploited in very important practical applications (see, e.g., [1–3] and references therein). On the other hand, the experimental implementation of optical solitons in high-bit-rate communication systems has stimulated further research in soliton theory. In this Rapid Communication we present an analytical description of the envelope soliton propagating in a medium with large periodic variations of dispersion and nonlinearity. As a specific practical application we focus on dispersion-managed soliton transmission.

Dispersion compensation has already shown its great potential both in ultralong data transmission and for the upgrading of installed terrestrial links. Dispersion management, which is a well-known technique in linear systems (for instance, low power non-return-to-zero formatted data), also allows an increase in transmission capacity of soliton-based communication lines. The traditional fundamental soliton [solution of the nonlinear Schroedinger equation (NLSE)] preserves its shape during propagation by compensating the fiber dispersion through nonlinearity; only the pulse power oscillates due to periodic amplification of the pulse to compensate for the fiber loss. Rapid oscillations of the power can be averaged out with the result that the slow pulse dynamics, in the traditional soliton-based transmission lines, is governed by the NLSE. On the other hand, numerical simulations and experiments have demonstrated that the dispersion-managed soliton differs substantially from the fundamental soliton [4–15], For example, the energy of the dispersion-managed soliton is enhanced in comparison with that of the fundamental soliton corresponding to the same path average dispersion and the same pulse width [4]. Therefore, using dispersion-managed solitons in transmission permits an increased signal-to-noise ratio with significant system performance improvement. Another feature in the dynamics is that optical pulses propagating in a link with dispersion compensation acquire a chirp (time-dependent phase) while also experiencing breathing oscillations of the pulse width. Overall, the pulse dynamics in these systems is rather complicated and typically depends on many system parameters. Numerical simulations (see, e.g., [10]) have revealed the existence of two scales in the problem: the first (fast dynamics) corresponds to a rapid oscillation of the pulse width and power due to periodic variations of the dispersion and periodic amplification, and the second (slow dynamics) which occurs due to the combined effects of nonlinearity, residual dispersion, and pulse chirping.

It has been shown in [13] that a fast (over 1 compensation period) dynamics of the dispersion-managed soliton is described by a system of two ordinary differential equations. This approximation is in agreement with a model previously obtained by means of the variational approach [9,10]. Because of the practical importance of the problem, it is of evident interest to develop analytical methods to investigate the main properties of the basic ordinary differentiated equations (ODEs).

In this Rapid Communication we present a new analytical approach based on a combination of previous work [9,10] and asymptotic methods. We show that our method can be successfully used for the description of the slow (average) dynamics of the dispersion-managed soliton and slow evolution of any input pulse. Even though the approach is rather general and can be used for any dispersion-managed system, as an illustration we consider here a particular transmission line built from two pieces of fibers with opposite dispersion.

II. BASIC EQUATIONS

The transmission of optical signals in a fiber link with dispersion compensation is governed by the basic model that we write in the following dimensionless form:

$$i\Psi_z+\frac{1}{\epsilon}d\left(\frac{z}{\epsilon}\right)\Psi_{tt}+c\left(\frac{z}{\epsilon}\right)|\Psi|^2\Psi=0, \tag{1}$$

using the notation introduced in [9,10]: time is normalized by parameter t_0, the envelope of the electric field is normalized to the input power parameter P_0, z is normalized to the nonlinear length $Z_{NL}=1/(\sigma P_0)$ (σ is the nonlinear Kerr parameter); the dimensionless dispersion compensation period is $\epsilon=L/Z_{NL}\ll 1$. Though the results will be formulated in a general form we assume at the moment, for simplicity, that the functions $d(z/\epsilon+1)=d(z/\epsilon)$ and $c(z/\epsilon+1)=c(z/\epsilon)$ are periodic with the same period, namely ϵ.

We first reduce the problem to a finite dimensional one by deriving basic ODEs for the relevant pulse parameters. Let us consider, following [13], the evolution of the integrals

$$T_{int}(z)=\left[\frac{\int t^2|\Psi|^2dt}{\int|\Psi|^2dt}\right]^{1/2},$$

$$\frac{M_{int}(z)}{T_{int}(z)}=\frac{i}{4}\frac{\int t(\Psi\Psi_t^*-\Psi^*\Psi_t)dt}{\int t^2|\Psi|^2dt}, \tag{2}$$

which are proportional to the pulse width and the chirp, respectively.

A closed system of equations for T_{int} and M_{int} can be derived (see [13]) if we assume that a propagating pulse has a self-similar structure (in the energy-bearing part)

$$\Psi(z,t)=\frac{Q[\frac{t}{T},z]}{\sqrt{T(z)}}\exp\left(i\frac{M(z)}{T(z)}t^2\right). \tag{3}$$

We assume now that, to leading order, $\frac{\partial Q}{\partial t}=0$ for the central energy-bearing part of the asymptotic pulse. This assumption is confirmed by numerical simulations of the dispersion-managed pulse dynamics, though it should be noted that the tails of the soliton (which contain a rather small part of the energy) are not self-similar [4,5,10]. Then, evaluating the first derivatives of $T_{int}(z)$ and $M_{int}(z)$, we obtain a system of ordinary differential equations for T_{int} and M_{int}, or, alternatively, $T(z)$ and $M(z)$. Relations between integrals (2) and local characteristics of the pulse width and chirp are given by

$$\frac{T_{int}(z)}{T_{int}(0)}=\frac{T(z)}{T(0)},\qquad \frac{M_{int}(z)}{T_{int}(z)}=\frac{M(z)}{T(z)}. \tag{4}$$

The evolution of the optical pulse width and chirp in the dispersion-managed line is described by the following basic set of equations

$$\frac{dT}{dz}=4\frac{1}{\epsilon}d\left(\frac{z}{\epsilon}\right)M, \tag{5}$$

$$\frac{dM}{dz}=\frac{1}{\epsilon}d\left(\frac{z}{\epsilon}\right)\frac{C_1}{T^3}-c\left(\frac{z}{\epsilon}\right)\frac{C_2}{T^2}. \tag{6}$$

The constants C_1 and C_2 are related to a structural function $Q(x)$ through

$$C_1=\frac{\int|Q_x(x)|^2dx}{\int x^2|Q(x)|^2dx},\qquad C_2=\frac{\int|Q(x)|^4dx}{\left(4\int x^2|Q(x)|^2dx\right)}. \tag{7}$$

Thus, C_1 and C_2 are determined by the shape and the energy of the dispersion-managed pulse. For instance, for the Gaussian approximation of the central part of the breathing soliton: $C_1=(1.665t_0T/T_{FWHM})^4$, $C_2=C_1^{3/4}E/(2\sqrt{2\pi}P_0t_0)$, here T_{FWHM} is a pulse width at half maximum measured in experiments and $E=E_0$ is a pulse energy at $z=0$. Note that Eqs. (5) and (6) have also been derived from a variational approach in [9,10] and applied to different practical transmission systems in [7,12]. Imposing the conditions of recovery of pulse width and chirp after one period we obtain relations that determine the properties of a dispersion-managed soliton: $\langle d(z)M\rangle=\int_0^1 d(z)M\,dz=0$ and $C_1\langle d(z)/T^3\rangle=\epsilon C_2\langle c(z)/T^2\rangle$. Below we present an analytical solution of the problem for a specific map, but these general conditions can also be used in numerical simulations. For sech-shaped soliton (that is valid for weak dispersion management) the energy enhancement is found then as $E=3.52P_0t_0^2\langle d(z)/[2\epsilon T_{FWHM}^3(z)]\rangle/\langle c(z)/T_{WFHM}^2(z)\rangle$. From here, in particular it is clear that the energy can be positive even if the dispersion-managed soliton propagates in the normal dispersion regime or at zero average dispersion. This important observation can be explained easily using Eqs. (5) and (6). Consider a phase shift that occurs due to a combined action of dispersion, nonlinearity, and chirping,

$$\frac{d(TM)}{dz}=\frac{1}{\epsilon}d\left(\frac{z}{\epsilon}\right)\frac{C_1}{T^2}-c\left(\frac{z}{\epsilon}\right)\frac{C_2}{T}+4\frac{1}{\epsilon}d\left(\frac{z}{\epsilon}\right)M^2. \tag{8}$$

In an arbitrary dispersion-managed system, a balance between the above-mentioned three effects can be achieved only on the average. Averaging of Eq. (8) immediately gives a condition on the power of DM soliton,

$$\left\langle\frac{d(z)}{T^2(z)}\right\rangle C_1+4\langle d(z)M^2\rangle=\langle d(z)\Omega^2(z)\rangle=\epsilon\left\langle\frac{c(z)C_2}{T}\right\rangle>0. \tag{9}$$

It is seen that the requirement of the anomalous average dispersion $\langle d\rangle>0$ that provides the existence of the traditional NLSE soliton is replaced for DM soliton by the condition $d_{eff}=\langle d(z)\Omega^2(z)\rangle>0$ that can be satisfied at zero and even normal path-averaged dispersion and the DM soliton can have finite energy even with $\langle d\rangle<0$. Here $\Omega^2(z)$ is nothing more but a square of the varying spectral bandwidth $\Omega(z)$ of a chirped pulse. The total phase shift resulting pulse chirping, dispersion, and nonlinearity should be zero (except the linear growth) in average for true soliton propagation. Equations (5) and (6) present a simple and useful tool for optimization of the dispersion-managed fiber transmission systems.

III. ADIABATIC INVARIANTS AND EXACT SOLUTIONS

Below we obtain the slow dynamics corresponding to Eqs. (5) and (6), using the method of averaging which forms the main part of this paper. Rescaling $z=\epsilon\tau$ and absorbing the constant C_1 in the parameters T,M by defining $T=(4C_1)^{1/4}\ b$, $M=(4C_1)^{1/4}\nu/4$ we reduce the equations to

$$\frac{db}{d\tau}=d(\tau)\nu, \quad \frac{d\nu}{d\tau}=\frac{d(\tau)}{b^3}-\epsilon\frac{C(\tau)}{b^2}, \tag{10}$$

where $C=C_2C_1^{-3/4}\sqrt{2}c(\tau)$.

An exact solution of these equations in the nonlinear case exists for the special dispersion profile corresponding to the so-called chirped quasisoliton propagating along the fiber line [15,13]. Such a dispersion-tapered fiber span provides strong confinement of the carrier pulse that is very attractive for high-bit-rate data transmission. If the dispersion is given by

$$d(\tau)=d_0C(\tau)\frac{\cosh[\alpha+\beta y(\tau)]}{\cosh[\alpha]}; \quad \frac{dy}{d\tau}=C(\tau); \quad y(0)=0, \tag{11}$$

then Eqs. (10) have an exact solution

$$b(\tau)=b_0\frac{\cosh[\alpha+\beta y(\tau)]}{\cosh[\alpha]}, \tag{12}$$

$$\nu(\tau)=\frac{\cosh[\alpha]}{b_0}\sqrt{\frac{d_0-\epsilon b_0}{d_0}}\tanh[\alpha+\beta y(\tau)]. \tag{13}$$

Here $\beta=\sqrt{d_0(d_0-\epsilon b_0)}\cosh[\alpha]/b_0^2$ and $\sinh[\alpha]=\nu_0b_0\sqrt{d_0/(d_0-\epsilon b_0)}$. This exact solution describes propagation of the quasisoliton having Gaussian tails along the fiber with specially designed dispersion. The advantage of using this carrier pulse is that soliton interaction is suppressed through chirping and fast decaying Gaussian tails. This method requires a special design of the dispersion of the transmission fiber and can be considered as a potentially attractive scheme to realize future ultra-large capacity fiber communication lines.

We now derive equations for the evolution of the adiabatic invariants. In the case $\epsilon=0$ the system has the obvious conserved quantity $I=(1/b^2)+\nu^2=(1/b_0^2)+\nu_0^2$, which we introduce as a new variable. This integral is proportional to the leading term in the Hamiltonian of Eq. (1), when evaluated at the trial function. In fact, any combination of b_0 and ν_0 can be used for describing the average nonlinear dynamics. Substituting $B=b^2$, after straightforward calculations the system takes the form $dB/d\tau=2d(\tau)\sqrt{BI-1}$. Assuming that $I=$ constant we integrate this equation and obtain a constant $G=\tilde{R}(\tau)-(1/I)\sqrt{BI-1}$, where $\tilde{R}=\int^{\tau}[d(\tau)-\langle d\rangle]d\tau$. In the case $\epsilon=0$, $G=-\nu_0b_0$ as is clear from the above solution. We now choose G as a new variable, since it should be slowly varying. Indeed, after straightforward calculations we obtain

$$\frac{dG}{d\tau}=-\epsilon\langle d\rangle-\epsilon C(\tau)\frac{I^2[\tilde{R}(\tau)-G]^2-1}{\sqrt{I}\{1+I^2[\tilde{R}(\tau)-G]^2\}^{3/2}}, \tag{14}$$

$$\frac{dI}{d\tau}=-\epsilon C(\tau)\frac{2I^3[\tilde{R}(\tau)-G]}{\sqrt{I}\{1+I^2[\tilde{R}(\tau)-G]^2\}^{3/2}}. \tag{15}$$

The advantage of using the previous transformation, is that the right-hand sides of Eqs. (15) and (16) are proportional to the small parameter ϵ and different perturbation methods can now be applied. Equations similar to Eqs. (15) and (16) have been derived in [14] by means of the variational approach. We note also that both adiabatic invariants I and G have an optical interpretation; I is related to the pulse spectral width, and G is the effective accumulative dispersion (including the effect of nonlinearity-induced spectrum evolution).

IV. AVERAGE DYNAMICS

In this section we consider the slow averaged dynamics in the general case, where no exact solutions can be found. According to the Bogolubov-Krylov averaging theorem [16], the solutions of this system are ϵ-close to the solutions of the averaged system on the interval $\Delta\tau\sim 1/\epsilon$. The averaged system is obtained by integrating the right hand-side of the equation with respect to τ. First note that the dispersion-managed soliton corresponds to a periodic solution of Eqs. (15) and (16) $[I(1)=I(0),\ G(1)=G(0)]$. Therefore, the parameters of the true periodic breathing soliton are determined by the following two conditions:

$$\int_0^1\frac{C(\tau)I^3[\tilde{R}(\tau)-G]d\tau}{\sqrt{I}\{1+I^2[\tilde{R}(\tau)-G]^2\}^{3/2}}=0, \tag{16}$$

$$\langle d\rangle=-\int_0^1\frac{C(\tau)\{I^2[\tilde{R}(\tau)-G]^2-1\}d\tau}{\sqrt{I}\{1+I^2[\tilde{R}(\tau)-G]^2\}^{3/2}}. \tag{17}$$

These conditions determine the energy and chirp required for true periodic propagation of the dispersion-managed pulse. It is evident from the equations that, for most cases this integration cannot be carried out analytically. A particular case where this can be done is when C is a constant. This assumption is valid when the compensation period is much larger than the amplification distance. This is an important practical limit, corresponding to the so-called lossless model, which is well justified in the long-haul transmission systems. In addition, we gain much intuition from the results in this limit. In the case that C is a constant, the averaged equations read

$$\frac{d\bar{G}}{d\tau}=-\langle d\rangle+\frac{C}{\bar{I}^{3/2}}\frac{d_1-d_2}{d_1d_2}\Bigg(\frac{2\bar{I}(d_1L_1-\bar{G})}{[1+\bar{I}^2(d_1L_1-\bar{G})^2]^{1/2}}+\frac{2\bar{I}\bar{G}}{(1+\bar{I}^2\bar{G}^2)^{1/2}}-\ln\frac{\bar{I}(d_1L_1-\bar{G})+[1+\bar{I}^2(d_1L_1-\bar{G})^2]^{1/2}}{-\bar{I}\bar{G}+(1+\bar{I}^2\bar{G}^2)^{1/2}}\Bigg), \tag{18}$$

$$\frac{d\overline{I}}{d\tau}=2C\overline{I}^{1/2}\frac{d_2-d_1}{d_1\ \ d_2}\left(\frac{1}{[1+\overline{I}^2(d_1L_1-\overline{G})^2]^{1/2}}-\frac{1}{(1+\overline{I}^2\overline{G}^2)^{1/2}}\right), \tag{19}$$

where $\overline{I}=\langle I\rangle=\int_0^1 I(\tau)d\tau$ and similarly for $\overline{G}$; $d(\tau)=d_1+H(\tau-L_1)(d_2-d_1)$; here H is the Heaviside function, $d_{(1,2)}$ is varying part of the dispersion coefficient for the piece of fiber of length $L_{(1,2)}$, respectively. It is easy to check that $d_1L_1+d_2L_2=0$ and $L_1+L_2=1$.

These averaged equations describe slow secondary oscillations of the breathing pulse propagating in the system with dispersion management. The stationary solutions of the averaged equations corresponding to the dispersion-managed soliton are characterized as the fixed points of Eqs. (19) and (20); namely,

$$\overline{G}_c=\frac{1}{2}d_1L_1, \tag{20}$$

$$-\langle d\rangle\frac{\overline{I}_c^{3/2}}{C}\frac{d_1d_2}{d_2-d_1}=2\ln[\overline{I}_c\overline{G}_c+(1+\overline{I}_c^2\overline{G}_c^2)^{1/2}]-\frac{4\overline{I}_c\overline{G}_c}{(1+\overline{I}_c^2\overline{G}_c^2)^{1/2}}. \tag{21}$$

Rewriting C in terms of the energy E and introducing the parameter $K=0.25|d_1L_1-d_2L_2|=0.5|d_1L_1|$ that characterizes the strength of the map yields for the second condition

$$-\frac{\langle d\rangle}{E}\sqrt{\pi}P_0t_0\overline{I}_c^{3/2}(2K)=\ln[\overline{I}_cK+(1+\overline{I}_c^2K^2)^{1/2}]-\frac{2\overline{I}_cK}{(1+\overline{I}_c^2K^2)^{1/2}}. \tag{22}$$

This analytical expression gives the energy enhancement discovered in [4]. For small K the enhancement factor is parabolic in accordance with [5]. Another interesting observation is that a periodic solution is possible even if average dispersion is zero or normal ($\langle d\rangle<0$). For the case $\langle d\rangle=0$ we get $\overline{I}_c=\sinh(\xi_c)2/K$; here, ξ_c is a solution of the equation $\xi=2\tanh(\xi)$ with a corresponding sign. Note that we have obtained the averaged equations that also describe the slow secondary oscillations observed in [10] (as well as similar phenomena reported in other papers).

V. CONCLUSIONS

We have studied the slow (average) dynamics of dispersion-managed optical pulses in fiber transmission lines. By applying asymptotic methods to an approximate system of the ordinary differential equations, we have presented an analytical description of the average evolution of the self-similar breathing pulse in fiber links with periodic amplification and dispersion compensation. We have derived asymptotic, averaged quantities (adiabatic invariants) which characterize stable pulse propagation. For the practically important limit where the dispersion compensation period is much larger than the amplification distance, we have found analytically the fixed points corresponding to stable dispersion-managed soliton propagation. The analytical results we obtained then provide asymptotic features of the pulse.

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