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Synchronization of time-delay systems

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We present the linear stability analysis of synchronized states in coupled time-delay systems. There exists a synchronization threshold, for which we derive upper bounds, which does not depend on the delay time. We prove that at least for scalar time-delay systems, synchronization is achieved by transmitting a single scalar signal, even if the synchronized solution is given by a high-dimensional chaotic state with a large number of positive Lyapunov exponents. The analytical results are compared with numerical simulations of two coupled Mackey-Glass equations. [S1063-651X(98)51210-X]

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The problem of synchronization of dynamical systems is one of the classical fields in engineering science [1]. Recently, renewed interest in this field was stimulated in connection with the synchronization of chaotic motion. Especially, the potential applicability for communication has attracted much research in recent years [2]. Yet, there are a lot of results available concerning the synchronization of low-dimensional chaotic systems, theoretical as well as experimental [3]. Contrary, the synchronization of high-dimensional chaotic systems with possibly a large number of positive Lyapunov exponents remains open. From the point of view of numerical simulations the synchronization of specific high-dimensional chaotic systems has been achieved [4], while to our best knowledge only very few general results are available (cf. [5]). For that reason we address in this paper the question of the synchronization of coupled identical time-delay systems. We focus on time-delay systems, since on the one hand it is well established that these systems are prominent examples of high-dimensional chaotic motion with a large number of positive Lyapunov exponents [6], and on the other hand synchronization of Mackey-Glass-type electronic oscillators has been reported from the experimental point of view [7].

Let us consider a fairly general theoretical model and investigate the stability problem of a synchronized state. For that purpose consider two identical arbitrary scalar time-delay systems with a symmetric coupling

$$\begin{aligned} \dot{x} &= F(x,x_\tau) - K(x-y), \\ \dot{y} &= F(y,y_\tau) - K(y-x), \end{aligned} \tag{1}$$

where we adopt the notation $x_\tau := x(t-\tau)$ to indicate the time-delayed variables. We specialize from the beginning to the frequently analyzed case that the coupling is bidirectional and acts additive to the single dynamical system. However, we stress that the subsequent considerations apply with minor modifications to much more general situations, e.g., to vector-type variables, to systems with much more general delay terms, or to a nonadditive coupling, as long as the coupling vanishes in the synchronized state $x(t) \equiv y(t)$. But we think, that the choice made in Eq. (1) makes our arguments more transparent.

Let z denote the synchronized solution, i.e., $\dot{z} = F(z,z_\tau)$. Considering deviations from that state according to $x = z + \delta x$, $y = z + \delta y$ and performing a linear stability analysis, we obtain for the deviation $\Delta := \delta y - \delta x$ from the synchronized state the linear differential-difference equation

$$\dot{\Delta} = \alpha(t)\Delta + \beta(t)\Delta_\tau. \tag{2}$$

Here, the time-dependent coefficients are given in terms of the synchronized solution as $\alpha(t) = \partial_1 F(z,z_\tau) - 2K$ and $\beta(t) = \partial_2 F(z,z_\tau)$, where the symbol $\partial_{1/2}$ denotes the derivative with respect to the first/second argument. An inspection of Eq. (2) might suggest that the synchronized solution is stable if $\alpha(t)$ is "sufficiently negative," since the delay term can be neglected. Although the outcome of this superficial argument will turn out to be correct, our analysis will reveal that the delay term cannot be neglected at all. For that reason we are carrying out a rigorous stability analysis. Suppose the coefficients are bounded in the sense that $\alpha(t) \leq -a < 0$ and $|\beta(t)| \leq b$ holds for some fixed values a and b. Since the equation is linear, it is sufficient to analyze the solution with the special initial condition $\Delta(0) = 1, \Delta(t) \equiv 0, t < 0$. The general case follows by a simple integration. There are different ways to estimate the stability of the trivial solution, $\Delta(t) \equiv 0$, of Eq. (2). Here we use the fact that for scalar quantities a simple closed analytical formula for the solution can be written down. One just integrates the linear equation (2) in the time intervals $[N\tau,(N+1)\tau]$ and considers the delay term as an inhomogeneous part. By this continuation method (cf. [8], p. 45) the full solution is obtained as

$$\begin{aligned} \Delta(t) = {} & e^{\int_0^t \alpha(t')dt'} + \int_\tau^t dt_1 \beta(t_1) e^{\int_{I_1}\alpha(\theta)d\theta} \\ & + \int_{2\tau}^t dt_1 \int_{2\tau}^{t_1} dt_2 \beta(t_1)\beta(t_2-\tau) e^{\int_{I_2}\alpha(\theta)d\theta} + \cdots \\ & + \int_{N\tau}^t dt_1 \int_{N\tau}^{t_1} dt_2 \cdots \int_{N\tau}^{t_{N-1}} dt_N \\ & \times \beta(t_1)\beta(t_2-\tau)\cdots\beta(t_N-(N-1)\tau) e^{\int_{I_N}\alpha(\theta)d\theta} \\ & \text{for } N\tau \leq t \leq (N+1)\tau. \end{aligned} \tag{3}$$

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Here, the domains of integration for the exponents are given by $I_k:=[0,t]/([t_1,t_1-\tau]\cup[t_2-\tau,t_2-2\tau]\cup\ldots\cup[t_k-(k-1)\tau,t_k-k\tau])$. An upper bound for $|\Delta(t)|$ is obtained, if the maximal values $\alpha(t)=-a$ and $\beta(t)=b$ are inserted into Eq. (3). But then, the expression reduces to a solution Γ of the differential-difference equation with constant coefficients

$$\dot{\Gamma}=-a\Gamma+b\Gamma_\tau. \tag{4}$$

Hence, a solution of Eq. (4) yields an upper bond for $|\Delta(t)|$. But the last equation is easily solved by a Laplace transformation (cf., Ref. [8]) or loosely speaking by an exponential ansatz $\Gamma(t)=e^{st}$. Since the corresponding eigenvalues obey

$$s=-a+b\ \exp(-s\tau), \tag{5}$$

negative real parts, i.e., stability, occur if and only if $a>b$. This inequality yields an upper bond K_+ for the critical coupling strength beyond which synchronization is achieved. If we take the definitions of $\alpha(t)$ and $\beta(t)$ into account it reads explicitly

$$K_+=1/2[\max_t\ \partial_1F(z,z_\tau)+\max_t|\partial_2F(z,z_\tau)|]. \tag{6}$$

We note as a by-product that Eq. (4) may be viewed as a kind of Grownwall-like lemma [9] for the time-dependent equation (2).

Before we proceed let us comment on our estimate for the stability. It cannot be improved without taking details of the equation of motion (2) into account, since the estimate becomes exact for time independent coefficients. Of course, for concrete mathematical models of the type (1) one may succeed in deriving better estimates. Furthermore, our estimate (5) yields for large values of a the asymptotic behavior $s\simeq-\ln(a/b)/\tau+\mathcal{O}(1)$ for the dominant eigenvalue, which governs the transient behavior. Hence, one cannot neglect the delay term in general, even for sufficiently negative values of $\alpha(t)$. In that respect our estimate (6) is a nontrivial consequence of the delay dynamics. In fact, for several delay times the situation may become even more intricate, since the problem of stable operators [10] may become important.

In what follows, we compare our analytical result to numerical simulations. We specialize to the Mackey-Glass system, i.e.,

$$F(x,x_\tau)=-x+\frac{Ax_\tau}{1+x_\tau^{10}}. \tag{7}$$

In order to investigate the properties of the synchronization mechanism by numerical methods, we chose the distance between trajectories as a suitable measure. For that reason the quantity

$$D_T(t)=\int_t^{t+T}|x(t')-y(t')|dt', \tag{8}$$

which of course depends on the range of averaging T and the point of reference t, was analyzed.

We used a Runge-Kutta algorithm of fourth order with step size 0.1. The simulations have been performed for the parameter value $A=3$, starting with $K=0.35$. A constant initial condition for x and y, which differs by an amount of 10^{-3}, has been chosen. The system was allowed to relax for a time $t=80\tau$. After that, the distance D was integrated on a trajectory of the length $T=80\tau$. For the next value of coupling strength K we again distorted the last state of the trajectory by adding an amount of 10^{-3} to the y coordinates and used it as initial condition. We performed the computation for increasing as well as decreasing coupling constant. Figure 1 summarizes our findings.

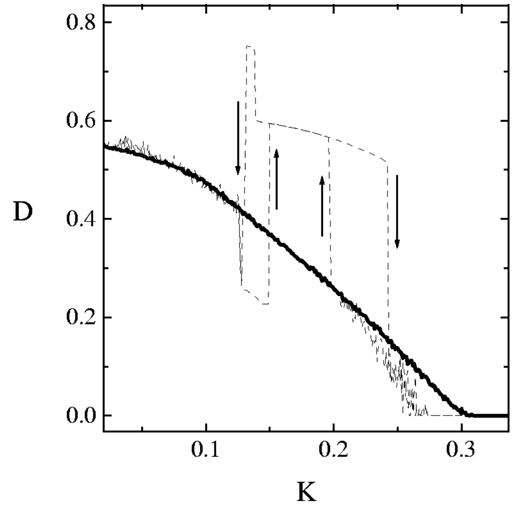

FIG. 1. Distance D for two coupled Mackey-Glass systems for $\tau=10$ (dashed line) and $\tau=100$ (solid line).

For $\tau=10$ we observe distinct jumps, indicating that the system switches between coexisting periodic states with a pronounced hysteresis. For $\tau=100$, the overall behavior of the system appears to be quite similar as for $\tau=10$, except that no switching and no hysteresis is observed. Within the resolution of the graphics the same behavior has already been observed for a smaller value $\tau=50$. From the numerical simulations the synchronization threshold is estimated as $K_c(\tau=10)\approx0.24$, $K_c(\tau=100)\approx0.28$. If we evaluate our analytical estimate Eqs. (6) and (7) using upper bounds for the derivatives we obtain values which differ by an order of magnitude but are independent of the delay time τ, $K_+=(81A/40-1)/2=2.53$. Since we have applied a rather graceful rigorous estimate, such a discrepancy is far from being astonishing.

In order to understand the dynamics in the vicinity of the synchronization threshold, time traces of the difference $x(t)-y(t)$ have been computed (cf. Fig. 2). Slightly below the synchronization threshold K_c we observe an intermittent behavior very similar to on-off intermittency [11]. Additionally, we investigated the distribution of laminar and turbulent phases under variation of the coupling strength K and the delay time τ. To this end, the distance $D_T(t)$ of the two systems in the phase space has been computed on a time series of length $\tau\times10^6$. With the threshold value $D_T\leqslant0.10$ ($D_T>0.10$) the laminar (turbulent) phases have been recorded. In the vicinity of the critical coupling strength K_c we observe a power-law scaling with exponential cutoff for the distribution P_l of the laminar phases over a wide range, P_l

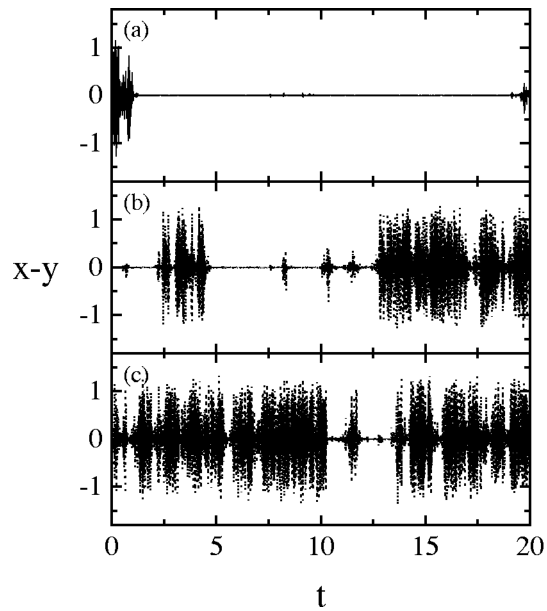

FIG. 2. Time series for $\tau=10$ and different values of the coupling (a) $K=0.2595$, (b) $K=0.250$, (c) $K=0.240$.

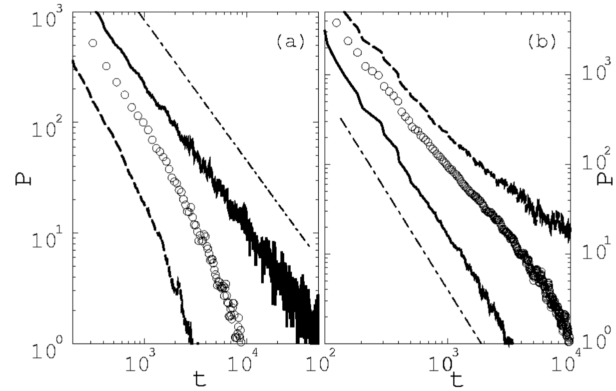

FIG. 3. (a) Distribution of laminar phases; (b) distribution of turbulent phases for $\tau=100$ and $K=K_c=0.28$ (solid line), $K=0.26$ (circles), and $K=0.24$ (dotted line). The dashed line indicates a shifted power-law fit with $\alpha_l=1.27$ and $\alpha_t=2.19$.

$\propto t^{-\alpha_l(\tau)}$, where the exponent α_l depends slightly on the delay time. In the low-dimensional chaotic case, for $\tau=10.0$, we observe $\alpha(10)=1.50$ in agreement with the value predicted by the scenario of on-off intermittency. For increasing delay time, we observe a decreasing exponent: $\alpha(30.0)=1.41, \alpha(50.0)=1.38, \alpha(100.0)=1.27$, indicating that there might be deviations from the simple on-off intermittency scenario (cf. Fig. 3). Note that in the latter cases the dynamics is high-dimensional. From the estimation of the Lyapunov exponents of the synchronized solution we estimated the Kaplan-Yorke dimensions to be $D_{KY}(\tau=30.0)\approx 26.5$, $D_{KY}(\tau=50.0)\approx 48.0$, $D_{KY}(\tau=100.0)\approx 96.0$. Therefore, the synchronization transition corresponds to a high-dimensional chaos-chaos transition. Even the distribution of turbulent phases seems to follow a power-law scaling in this τ region.

At the moment we have no theoretical explanation at the hand for our findings in the case of large τ, i.e., in the case of high-dimensional dynamics.

We conclude with how our results depend on noise or other imperfections that are present in realistic systems. In fact, in order to apply a concept like synchronization such perturbations have to be small and we may assume a general linear dependence. Formally such contributions are introduced into Eq. (1) by adding the two terms $G(x,x_\tau)\xi$ and $G(y,y_\tau)\eta$, where ξ and η denote, for example, realizations of a noise. Considering the perturbations of the same order of magnitude like the deviation from the unperturbed synchronized state and proceeding as above we finally end up with

$$\dot{\Delta}=\alpha(t)\Delta+\beta(t)\Delta_\tau+G(z,z_\tau)(\eta-\xi), \qquad (9)$$

which differs from Eq. (2) just by an inhomogeneous contribution. The theory of linear difference-differential equations tells us [8] that Eq. (9) inherits its stability properties from the corresponding homogeneous system (2) except that the perturbations cause fluctuations around the unperturbed synchronized state. Whenever the perturbations are so large that contributions beyond the linear order have to be taken into account, one has to resort to different methods. One of these cases, which are also relevant from the experimental point of view, is given by the synchronization of nearly identical time-delay systems. Since, in this case, no strict synchronized solution $x\equiv y$ exists, one has to rely on more general concepts, such as the generalized synchronization [12].

In summary, we emphasize that an analytical upper bond for the solution of Eq. (2) is obtained if one replaces the time dependency of the coefficients by their extreme values. One might get better estimates in special cases. In particular one might argue, that Eq. (4) already determines the stability, if the time averages of the coefficients are inserted. This statement is in fact true if either the coefficients are periodic functions of time with the delay τ being an integer multiple of the period or, if the coefficients are almost constant (cf. [8], p. 277). Whether or not the general case can be treated by this refined estimate remains open. Nevertheless, we have shown that for sufficiently large coupling constant K the synchronized solution of Eq. (1) becomes stable whenever $|\delta_{1/2}F(z,z_\tau)|$ are uniformly bounded. In particular the critical coupling strength remains bounded even in the limit of large delay times, i.e., it does not increase with the dimension of the attractor. In fact, our numerical simulations indicate only a weak dependence of the actual critical coupling strength on the delay time. Last but not least our approach clearly demonstrates that the success of the synchronization is independent of the number of positive Lyapunov exponents, even if our coupling uses one scalar variable only, illustrating the results of Stojanovski *et al.* [5].

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[1] I. Blekhman, P. S. Landa, and M. G. Rosenblum, Appl. Mech. Rev. **48**, 733 (1995).

[2] K. Cuomo and A. V. Oppenheim, Phys. Rev. Lett. **71**, 65 (1993); G. Perez and H. A. Cerdeira, *ibid.* **74**, 1970 (1995); U. Parlitz, L. Kocarev, T. Stojanovski, and H. Preckel, Phys. Rev. E **53**, 4351 (1996).

[3] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); H. G. Winful and L. Rahman, *ibid.* **65**, 1575 (1990); T. L. Carroll and L. M. Pecora, IEEE Trans. Circuits Syst. **38**, 453 (1991); L. O. Chua, L. Kocarev, K. Eckert, and M. Itoh, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 705 (1992); R. Roy, and K. S. Thornburg, Jr., Phys. Rev. Lett. **72**, 2009 (1994).

[4] J. H. Peng, E. J. Ding, M. Ding, and W. Yang, Phys. Rev. Lett. **76**, 904 (1996); L. Kocarev and U. Parlitz, *ibid.* **77**, 2206 (1996); A. Tamasevicius and A. Cenys, Phys. Rev. E **55**, 297 (1997); U. Parlitz, and L. Kocarev, Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**, 407 (1997).

[5] T. Stojanovski, U. Parlitz, L. Kocarev, and R. Harris, Phys. Lett. A **233**, 347 (1997).

[6] J. D. Farmer, Physica D **4**, 366 (1982); K. Ikeda and K. Matsumoto, *ibid.* **29**, 223 (1987); F. T. Arecchi, G. Giacomelli, A. Lapucci, and R. Meucci, Phys. Rev. A **43**, 4997 (1991); I. Fischer, O. Hess, W. Elsäßer, and E. O. Göbel, Phys. Rev. Lett. **73**, 2188 (1994); Q. L. Williams, J. Garcia-Ojalvo, and R. Roy, Phys. Rev. A **55**, 2376 (1997).

[7] A. Tamasevicius (unpublished).

[8] R. Bellman and K. L. Cooke, *Differential-Difference Equations* (Academic Press, London, 1963).

[9] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1986).

[10] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations* (Springer, New York, 1993), p. 284.

[11] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **74**, 918 (1985); N. Platt, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993).

[12] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996); K. Pyragas, Phys. Rev. E **54**, R4508 (1996).