PHYSICAL REVIEW E VOLUME 58, NUMBER 2 AUGUST 1998

Determination of dynamical critical exponents from hysteresis scaling

G. P. Zheng and J. X. Zhang
Department of Physics, Zhongshan University, GuangZhou 510275, People's Republic of China

(Received 17 February 1998)

A method is proposed to determine the dynamical critical exponent. The method is based on the scaling for dynamical hysteresis resulted from a linearly swept field. We prove that in model A dynamical hysteresis scaling at critical temperature is universal. The nearest-neighbor Ising models are used to demonstrate such concepts and the dynamical critical exponents can be determined accurately. We also propose a universal relation between static and dynamical critical exponents in Ising class in single-spin-flip dynamics. [S1063-651X(98)51608-X]

PACS number(s): 64.60.Ht, 75.60.Ej, 75.40.Gb

In the past two decades, dynamical critical phenomena in classical spin systems were extensively studied [1]. Determination of dynamical critical exponent z play an important part of these studies. Though static critical exponents can be determined accurately by renormalization-group theory [2], the renormalization-group method in dynamical version has some difficulties in calculating the dynamical critical exponent z, even in evaluating z of a two-dimensional Ising model [4]. In brief, the methods that had been used to estimate the value of z might include the following: high-temperature expansion [3], dynamical renormalization-group methods [4], Monte Carlo simulations [5], and nonequilibrium relaxation analysis [6]. For nearest-neighbor Ising system with ferromagnetic interaction in two-dimensional lattice, the values obtained from the methods mentioned above cover the range from 1.73 to 2.34 [3–6]. Moreover, for such Ising systems on fractal structures with infinite ramification [7], z can hardly be calculated because of the uncertainty of the dynamical recursive relation [8]. Therefore, the value of dynamical critical exponent of Ising system is still an open question.

Some reasons may be accounted for the systematic errors appeared in the above-mentioned estimates, e.g., long-time tail in finite system and the critical slowing down [9]. Such difficulties can hardly be overcome by simulational methods. Recently, two simulational studies have been given to evaluate the dynamical critical exponent, in two-dimensional Ising model. Both methods depended on the dynamical scalings for some thermodynamic quantities in short-time region. One of them [10] was based on the scaling of initial growing of magnetization [11]. The z exponent determined in Ref. [10] by means of Monte Carlo (MC) simulation is $z=2.132$. Unfortunately, during the heat-bath MC simulation, the initial magnetization, which is inversely proportional to the characteristic time of the growing region of the evolving process, must be zero and the initial configuration should be well prepared, otherwise the early time scaling relation collapses. But in such cases the short-time region overlaps with the long-time region. The other method [12] was free from the effect of initial condition and was based on the scaling of particular quantities that are independent of system size. $z=2.16$ was given by this approach. However, the scaled quantities, pertained to internal ones, and could not match with bulk variables of the systems (e.g., magnetization). This may prevent it from confirming by experiments. On the other hand, the quantities constructed in this method need quite many statistical samples for nonequilibrium averaging.

We propose here a method to determine the z exponent by means of hysteresis scaling. Since Tome and de Oliviera [13] used mean-field type kinetic Ising model to study the dynamic phase transition under cosine external field, the concept of hysteresis scaling and its universality in Ising models [14] under periodic field had been proposed and extended to other phenomenological models [15]. Unfortunately, a universal relation for the scaling exponents is still absent. In this Rapid Communication, dynamic hysteresis response caused by a linearly swept field is studied. We show that at critical temperature, the area of hysteresis loop A can be scaled with respect to the sweeping rate h of the field: $A=g(T_C)h^b$, and b depends on the static and dynamical critical exponents. Finite-size scaling for A is used to determine z accurately, using standard MC simulations. This method may lead to the following advantages. First, the system will begin with all spins up and evolve toward the configuration of all spins down under a large-amplitude unfavorable field. The scaling relation and exponent will be independent of the initial condition. Second, long-time tail and critical slowing down can both be dramatically refrained due to the applying field. Third, the simulation result may be able to be compared with experiments of dynamical hysteresis measurements. Dependence of the areas of hysteresis loop on the rates of a sweeping field had been reported, in hysteresis measurements of ferromagnet and ferroelectric sample [16].

Now we analyze the field-theoretic model with scalar order parameter field $\phi(\mathbf{x})$, Landau-Ginzburg Hamiltonian is given by [17]

$$\mathcal{H}=\int d^dx\left(\frac{r}{2}\,\phi^2(x)+\frac{g}{4!}\,\phi^4(x)-\frac{H}{k_BT}\,\phi(x)+\frac{1}{2}\,[\nabla\phi(x)]^2\right), \tag{1}$$

where $r,g>0$ are coefficients. Here $r=K-K_c$ with $K\propto 1/k_BT$, T is the temperature and k_B the Boltzman constant. The system described by Eq. (1) has a first-order phase transition (FOPT) driven by the external field H. To study the kinetics of FOPT at $r\approx 0$, we start from the Langevin equation

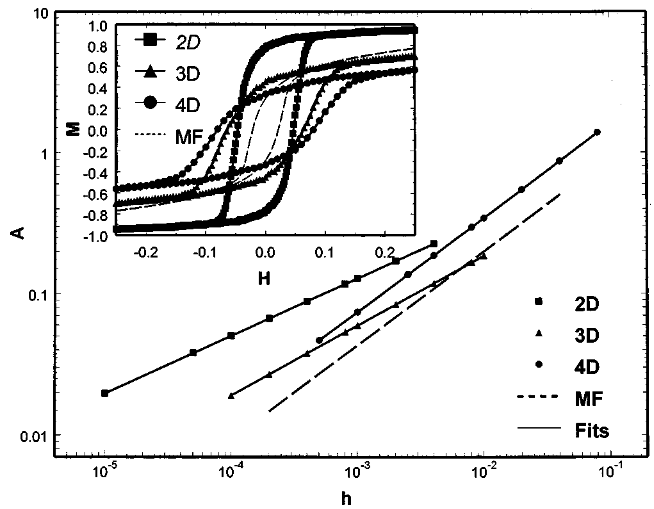

FIG. 1. Scaling of the areas of hysteresis loops with respect to the sweeping rates at critical temperature. The inset shows the hysteresis loops for 2D ($h=0.0001$), 3D ($h=0.001$), and 4D ($h=0.01$) Ising models. MF approximation ($h=0.0005$) is plotted as a dashed line.

$$\frac{\partial}{\partial t}\phi(x,t)=-\lambda\frac{\delta\mathcal{H}(\phi)}{\delta\phi}+\xi(x,t), \tag{2}$$

where ξ is Gaussian white noise with the following correlations:

$$\langle\xi(x,t)\xi(x',t')\rangle=2k_BT\lambda\,\delta(x-x')\,\delta(t-t'), \tag{3}$$

with λ the dynamic constant. Following the field theoretical treatment for stochastic process by means of path integral description [17], we can use the perturbational expansions near the critical fixed point T_c (the Curie point), and safely arrive at the following equations up to one-loop order [18]:

$$\frac{d}{dt}M(t)=-\lambda[r+gC(t)/2]M(t)-(\lambda g/6)M^3(t)+\lambda H/k_BT, \tag{4a}$$

$$C(t)=\frac{1}{(2\pi)^d}\int C_k(t,t)d^dk,$$

$$C_k(t,t)=-\exp[-2\lambda(r+k^2)t]/(r+k^2). \tag{4b}$$

Here $C_k(t,t)$ are correlators, $M(t)=\langle\phi_k(t)\rangle$ is magnetization, $\phi_{\mathbf{k}}(t)$ are the Fourier components of $\phi(\mathbf{x},t)$.

We now apply the general technique of renormalization group to the hysteresis scaling. First, we may introduce a new variable,

$$R(t)=r+\frac{g}{2}C(t)+\frac{g}{6}M^2(t). \tag{5}$$

Therefore, Eq. (4a) becomes the following equation if we set $\lambda=1$:

$$\frac{d}{dt}M(t)=-R(t)M(t)+H/k_BT. \tag{6}$$

In the coarse-grain procedure in the momentum space, we can eliminate the evolving modes which are governed by $\Lambda/\mathcal{L}<k<\Lambda(\mathcal{L}>1$, Λ is the cutoff parameter) in Eqs. (4) by calculating the integral in Eq. (4b). Subsequently, no new correlations appear in Eq. (6) after this procedure, except the adjustment of the coefficient g in Eq. (5).

Then we carry the scale transformation: $\mathbf{k}\to\mathbf{k}'=\mathcal{L}^{-1}\mathbf{k},\mathbf{x}\to\mathbf{x}'=\mathcal{L}^{-1}\mathbf{x}$ with $\mathcal{L}$ the rescale factor. We assume the quasi-equilibrium growth of magnetic domain under a slow varying external field ($h\to 0$), the renormalization transformation of the order parameter may become

$$\phi'=\mathcal{L}^{\beta/\nu}\phi,\quad \phi'_{\mathbf{k}}=\mathcal{L}^{d/2+\beta/\nu}\phi_k.$$

We also assume that the system has been initialized with $M=1$ at $t=0$ and is allowed to develop in a time-dependent external field $H(t)=-ht$. If we rescale time as

$$t\to t'=\mathcal{L}^{-z}t,$$

the renormalization transformation of the bulk variables will obey the following relations:

$$M'(r',h',t',L')=\mathcal{L}^{\beta/\nu}M(r,h,t,L),$$
$$C'_k(t')=\mathcal{L}^{d+\beta/\nu}C_k(t),\quad C'(t')=\mathcal{L}^{2\beta/\nu}C(t) \tag{7a}$$

If the Hamiltonian (1) and equation of motion (6) remain unchanged, we can get the following recursive relations:

$$R'(t')=\mathcal{L}^zR(t),$$
$$r'=\mathcal{L}^{1/\nu}\mathbf{r},\quad T'=\mathcal{L}^{-1/\nu}T,$$
$$g'=\mathcal{L}^{-(z+2\beta/\nu)}g,$$
$$h'=\mathcal{L}^{(2z+\beta/\nu-1/\nu)}h. \tag{7b}$$

Therefore we get the renormalization transformation for h,

$$h\to h'=\mathcal{L}^{2z+\beta/\nu-1/\nu}h. \tag{8}$$

The area of hysteresis loop due to a swept cycle of H has the following finite-size scaling:

$$A=\oint MdH$$
$$=2L^{-\beta/\nu}hx\int M(rL^{1/\nu)},L^{2z+\beta/\nu-1/\nu}h,L^{-z}t)dt$$
$$=L^{z-\beta/\nu}h\tilde{A}(rL^{1/\nu},L^{2z+\nu/\nu-1/\nu}h), \tag{9a}$$

where $\tilde{A}$ a universal function. At the critical temperature $r=0$, we should have the relationship between the hysteresis in two systems with size L and L',

$$A=\mathcal{L}^{z-\beta/\nu}hA'(\mathcal{L}^{2z+\beta/\nu-1/\nu}h). \tag{9b}$$

Given a fixed sweeping rate h, the rescale factor can be chosen as $\mathcal{L}=h^{-1/(2z+\beta/\nu-1/\nu)}$. Equation (9b) will yield

TABLE I. Numerical results of the hysteresis scaling exponents b. The Ising systems with increasing sizes are denoted as ascending order. From 1 to 4, $L=50$, 100, 200, 400 ($d=2$), $L=10,20,40,60$ ($d=3$), and $L=10,20,30,40$ ($d=4$), respectively.

	$d=2$	$d=3$	$d=4$	MF
		Exponent b		
1	0.41 ± 0.01	0.50 ± 0.01	0.661 ± 0.005	
2	0.41 ± 0.01	0.496 ± 0.005	0.663 ± 0.005	0.667
3	0.410 ± 0.008	0.495 ± 0.005	0.665 ± 0.004	
4	0.408 ± 0.008	0.495 ± 0.005	0.666 ± 0.004	
z	2.137 ± 0.008	2.020 ± 0.005	2.000 ± 0.005	2

$$A \sim h^b, \quad b = 1 - \frac{z-\beta/\nu}{2z+\beta/\nu-1/\nu}. \tag{10}$$

Here we get a simple power-law scaling relation between A and h at $T=T_c$, with another dynamical exponent $b=1-(z-\beta/\nu)/(2z+\beta/\nu-1/\nu)$. Therefore, we have the universal relation for the static and dynamical critical exponents: $(z\nu+2\beta-1)/(2z\nu+\beta-1)=b$. Now the scaling exponent b is an intrinsic parameter reflecting the critical dynamics.

According to Eq. (10), the dynamical exponent z can be determined by a scaling of macroscopic response of order parameter to a linearly driven field. To demonstrate the validity of such method, we study the spin- $\frac{1}{2}$ Ising models on two-dimensional (2D), three-dimensional (3D), and four-dimensional (4D) supercubic lattices.

Now we consider a system of N Ising spins with ferromagnetic interaction. The Hamiltonian of this system is given by

$$\mathcal{H}_{\text{Ising}} = -K\sum_{\langle i,j\rangle} S_i S_j - H'(t)\sum_i S_i, \tag{11}$$

where the spin variables are represented by $\{S_i\}$ with $S_i=\pm1$; $\langle i,j\rangle$ is the sum extending over all nearest-neighbor spins, $H'(t)=H(t)/k_BT$ is a linearly swept magnetic field, T is the temperature of the spin system, $K=J/k_BT>0$ is the reduced coupling, $K_c=0.44\,069$, 0.22 166, and 0.14 966 for 2D, 3D, and 4D Ising models, respectively [19,20]. The dynamics of Eq. (11) is simulated by the Metropolis single-spin-flip MC algorithm, which has been found to be consistent with Langevin dynamics described by Eq. (2) [2]. To produce a hysteresis loop, the field $H(t)=H_0-ht$ is applied to the system with all spins up and then $H(t)=-H_0+ht$ is applied to the same system with all spins down. H_0 is the amplitude of magnetic field. Contrary to hysteresis studies on the same systems under a small amplitude cosine field by Acharyya and Chakrabarti [14], we use linear field with large amplitude [21], and find that H_0 does not affect hysteresis. Detail results have been published elsewhere [21]. The observation time t is measured in MC step per site (MCS), corresponding to all spins update. The magnetization $M(t)=N^{-1}\sum_{i=1}^{N} S_i(t)$. $A=\oint M dH$ is averaged over various H_0.

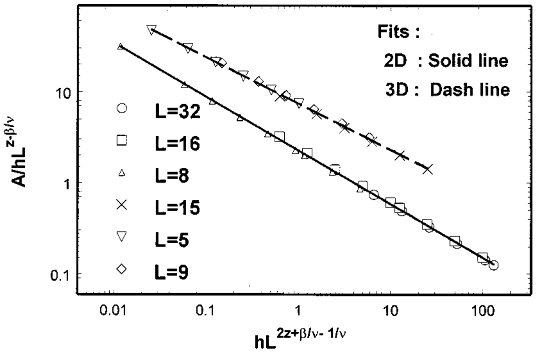

FIG. 2. Finite-size scaling for loop areas, in 2D and 3D Ising models. The fits are negative power-law functions.

Figure 1 is the relations between A and h. Though we simulated not very large Ising systems, we have found the good power-law relation between the areas of hysteresis loops and the sweeping rates. The scaling exponents b given by Fig. 1 are $b=0.408\pm0.008$, 0.495 ± 0.005, and 0.666 ± 0.004 for 2D, 3D, and 4D Ising models respectively. Table I lists the dynamical exponents obtained from hysteresis scaling for d-dimensional Ising models with different sizes. The kinetic Ising model is also studied using mean-field (MF) approximation. The equation of motion for magnetization is given by [2]

$$M(t)/dt = -M(t) + \tanh\{K[M(t)+H(t)]\}. \tag{12}$$

At MF critical temperature $K_c=J/k_BT_c=1$, the hysteresis loop is obtained by solving the differential equation numerically. The scaling exponent at MF approximation is $b\cong\frac{2}{3}$. For 4D Ising model and in MF approximation, z are consistent with exact results with high accuracy.

Figure 2 shows the finite-size scaling for A, in small Ising systems. The scaling function $\tilde{A}$ at $r=0$ in Eqs. (9a) is universal: $\tilde{A}(x)\sim x^{b-1}$. The dynamical critical exponents can also be determined, the results are consistent with z listed in Table I.

In conclusion, scaling for hysteresis with respect to the sweeping rate of a linear driving field is studied by renormalization-group theory. The exponent in a power-law scaling relation is found to connect with the static and dynamical exponents of a scalar model. The universality of this scaling is demonstrated by MC simulation in nearest-neighbor Ising models. Using the scaling relation, we obtain a method to evaluate the critical dynamical exponents z in 2D, 3D, and 4D Ising models. Compared with other methods, the method we propose is effective and can be testified experimentally. Moreover, such methods can be extended to determine z of other complex systems, for example, spins on Sierpinski Carpet [7,8], and the N-vector model in the large-N limit [15]. We hope this method may be confirmed by experiments. To avoid avalanche jump and Barkhausen effect during hysteresis, high quality samples, for example, a ferromagnet, which has a single domain and is free from defects, is needed.

[1] See, e.g., *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 8; see also H. Furukawa, Adv. Phys. **34**, 703 (1985); P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[2] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, Massachusetts, 1976); see also *Phase Transitions and Critical Phenomena*, Ref. [1], Vol. 2.

[3] B. Dammann and J. D. Reger, Europhys. Lett. **21**, 157 (1993); J. Wang, Phys. Rev. B **47**, 869 (1993).

[4] D. Kandel, Phys. Rev. B **38**, 486 (1988).

[5] S. Miyashita and H. Takano, Prog. Theor. Phys. **73**, 1122 (1985).

[6] D. Stauffer, Physica A **184**, 201 (1992); N. Ito, *ibid.* **196**, 591 (1993).

[7] Y. Gefen, Y. Meir, B. B. Mandelbrot and A. Aharony, Phys. Rev. Lett. **50**, 145 (1983).

[8] Y. Achiam, Phys. Rev. B **33**, 7762 (1986); **32**, 1796 (1985).

[9] M. Suzuki, Prog. Theor. Phys. **58**, 1142 (1977).

[10] Z. B. Li, L. Schulke and B. Zheng, Phys. Rev. Lett. **74**, 3396 (1995).

[11] See, e.g., H. K. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B **73**, 539 (1989), and references therein.

[12] M. S. Soares, J. K. L. da Silva, and F. C. Sa Barreto, Phys. Rev. B **55**, 1021 (1997).

[13] T. Tome and M. J. de Oliviera, Phys. Rev. A **41**, 4251 (1990).

[14] M. Acharyya and B. K. Chakrabarti, Phys. Rev. B **52**, 6550 (1995); and references therein. With small amplitude cosine field, they get the frequency scaling exponents for A at any temperatures in 2D, 3D, and 4D Ising models: $b=0.36$, 0.47, and 0.5, respectively. However, in our MC studies, we found b is universal only at temperatures below the Curie temperature.

[15] M. Rao, H. R. Krishnamurthy, and R. Pandit, Phys. Rev. B **42**, 856 (1990); A. M. Somoza and R. C. Desai, Phys. Rev. Lett. **70**, 3279 (1993).

[16] J. S. W. Rayleigh, Philos. Mag. **23**, 225 (1887), Y. L. He and G. C. Wang, Phys. Rev. Lett. **70**, 3279 (1993); K. Carl, Ferroelectrics **9**, 23 (1975).

[17] See, e.g., J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford, Clarendon, 1989); see also D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (World Scientific, Singapore, 1984).

[18] H. K. Janssen, in *From Phase Transition to Chaos*, edited by G. Gyorgyi *et al.* (World Scientific, Singapore, 1992).

[19] B. K. Chakrabarti, H. G. Baumgartel, and D. Stauffer, Z. Phys. B **44**, 333 (1981).

[20] K. Binder and D. P. Landau, Phys. Rev. B **30**, 1477 (1984).

[21] G. P. Zheng and J. X. Zhang, J. Phys.: Condens. Matter **10**, 1863 (1998).