Electron oscillations in a plasma slab

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(Received 2 June 1998)

We have found that new nonlinear volume plasma modes can exist in cold plasma slabs with particular density profiles. The wave trapping disappears in the linear limit. [S1063-651X(98)05711-0]

PACS number(s): 52.35.Mw

The science of low-temperature bounded plasmas is not as well developed as that of high-temperature fusion plasmas. Because of many possible industrial applications [1] it is, however, desirable that plasma physicists focus more attention on cold bounded plasmas. Although the linear behavior of such plasmas has been fairly well understood for a long time (see, e.g., the review papers [2] and [3]) it has been difficult to secure a thorough understanding of their nonlinear properties [4,5]. Thus, there are still some important basic, but nontrivial, theoretical problems that remain to be solved.

In the present Brief Report we are going to consider a very simple geometry, namely a plasma slab of width $2d$. The electrons are assumed to be cold, and the ions are regarded as an immobile background with density $n_0(x)$. We shall for simplicity restrict our analysis to *one-dimensional* volume plasma oscillations. Our electron fluid is thus governed by the equations of continuity and momentum, and the Poisson equation, i.e.,

$$\partial_t n + \partial_x(nv) = 0, \tag{1}$$

$$\partial_t v + v\,\partial_x v = (q/m)E, \tag{2}$$

and

$$\partial_x E = q(n - n_0)/\varepsilon_0, \tag{3}$$

where $E(x,t)\hat{\mathbf{x}}$, $n(x,t)$, $v(x,t)\hat{\mathbf{x}}$, q, and m represent the electric field, electron density, velocity, charge, and mass, respectively.

In a previous paper [6] a plasma slab with constant ion density was considered. The slab was bounded by a dielectric at $x=-d_0$ and $x=+d_0$. For piecewise constant initial density perturbations [6], e.g., $n(x,0)=n_0(1+\Delta)$ for $-d_0 \leq x < -d_0/2$ and $d_0/2 < x \leq d_0$, and $n(x,0)=n_0(1-\Delta)$ for $-d_0/2 \leq x \leq d_0/2$, where Δ is a constant parameter describing the initial electron density perturbation, it turned out to be possible to find exact solutions, with boundary condition $v(\pm d_0,t)=0$, for any amplitude. The solutions had an explosive character for sufficiently large values of Δ. In the present paper, where the ion density is a more general function of x, one may similarly find the electron motion by replacing the ion density profile $n_0(x)$ by piecewise constants n_{0j} for $d_j < x \leq d_{j+1}$. In each short interval j, one can then look for solutions with electron densities $n_j(t)$ and velocities $v_j(x,t)=u_{0j}(t)+u_{1j}(t)x$, and match the velocities at the boundaries. Due to very lengthy algebra it is, however, in reality impossible to proceed in this way. Thus, in the present Brief Report we shall propose an alternative method where a perturbation expansion in the wave amplitudes is used.

We first eliminate n and E in Eqs. (1)–(3) to obtain the equation for the electron fluid velocity v

$$\partial_t^2 v + \omega_p^2(x)v + \partial_t\partial_x v^2/2 + v\,\partial_t\partial_x v + v\,\partial_x^2 v^2/2 = 0 \tag{4}$$

where $\omega_p=(n_0(x)q^2/\varepsilon_0 m)^{1/2}$ is the electron plasma frequency.

Next, we look for a solution $v=\Sigma_j v_j(x)\exp(-ij\omega t)$, where $v_1\exp(-i\omega t)$ is the linear solution. Keeping only the lowest order terms, i.e., limiting our analysis to a small amplitude expansion, we then calculate v_2 from

$$\partial_t^2 v_2 + \omega_p^2(x)v_2 + \partial_t\partial_x v_1^2/2 + v_1\partial_t\partial_x v_1 = 0, \tag{5}$$

which means that

$$v_2 = \tfrac{3}{2} i\omega\,\partial_x v_1^2/(\omega_p^2(x) - 4\omega^2) \tag{6}$$

The equation for v_1 is accordingly

$$\partial_t^2 v_1 + \omega_p^2(x)v_1 + \partial_t\partial_x(v_1^* v_2) + v_1^*\partial_t\partial_x v_2 + v_2\partial_t\partial_x v_1^* + v_1\partial_x^2(v_1 v_1^*) + v_1^*\partial_x^2 v_1^2/2 = 0, \tag{7}$$

i.e.,

$$v_1 = [3i\omega v_1^*\partial_x v_2 - v_1\partial_x^2(v_1 v_1^*) - v_1^*\partial_x^2 v_1^2/2]/(\omega_p^2(x) - \omega^2). \tag{8}$$

Inserting Eq. (6) for v_2 into Eq. (8), and noting that here $v_1=v_1^*$, it follows from (8) that

$$\omega^2 - \omega_p^2(x) = 3\,\partial_x(\theta^{-1}\partial_x v_1^2/2), \tag{9}$$

where

$$\theta = (\omega_p^2(x) - 4\omega^2)/(\omega_p^2(x) - \omega^2). \tag{10}$$

The electric field and electron fluid velocity are zero at the turning points which are denoted by the coordinates $x=-d$ and $x=+d$. It follows from Eq. (9) that θ is also zero at these points. Consequently we choose the slab boundaries to be at $x=\pm d$. The solution of Eq. (9) is accordingly

$$v_1^2(x)=\tfrac{2}{3}\int_{-d}^{x}dx'\,\theta(x')\int_{-d}^{x'}dx''[\omega^2-\omega_p^2(x'')], \quad (11)$$

which, together with the condition $v_1^2(d)=0$, yields the dispersion relation

$$\omega^2=\left[\int_{-d}^{d}dx\ \theta(x)\int_{-d}^{x}dx'\,\omega_p^2(x')\right]\Bigg/\int_{-d}^{d}dx(x+d)\,\theta(x). \quad (12)$$

Equation (12), together with Eq. (10), is the main result of this Brief Report. By using our specific slab profile function $n_0(x)$, or $\omega_p^2(x)$, we have thus deduced an equation from which ω^2 can be calculated. It should then be stressed that only slabs with particular density profiles can support our resonant volume oscillations, for which the fitting of the boundary conditions can occur. The wave amplitudes should also be so small that the omission of fourth order terms in the expansion below Eq. (4) is justified. This means that v_1 has to be much smaller than $\omega_p(0)d$.

In order to shed some light on Eq. (9), it may first be instructive to inspect the linear limit. As Eq. (9) then reduces to the impossible equality $\omega^2-\omega_p^2(x)=0$ we have to slightly generalize our basic equations by adding a pressure term $-v_t^2(\partial_x n)/n$ to the right hand side of Eq. (2), where v_t is a parameter representing the thermal velocity. The *linearized* equation which corresponds to Eq. (9) is then [7,8]

$$\omega^2-\omega_p^2(x)=-(\omega^2/v_1)\partial_x(\lambda_D^2\partial_x v_1), \quad (13)$$

where $\lambda_D(x)=v_t/\omega_p$ is the electron Debye length.

Thus, whereas Eq. (9) has been derived for a cold and slightly nonlinear plasma, Eq. (13) is the corresponding result for a low-temperature and linear plasma. In both cases, the right hand side of the deduced equations [(9) and (13)] is necessary to determine the two turning points for our inhomogeneous plasma. The waves are trapped between these turning points.

We have tried to generalize our problem to consider a slightly nonlinear low-temperature plasma. Due to mathematical difficulties we did not derive a useful generalization of Eqs. (9) and (13), however.

With a slowly varying wave number $k(x)$ one has, from Eq. (13),

$$\omega^2-\omega_p^2(x)-\omega^2k^2(x)\lambda_D^2(x)=0. \quad (14)$$

As an example [7], if $\omega_p^2=\omega_0^2(1+x^2/x_0^2)$ where ω_0 and x_0 are constants, Eq. (14) determines the turning points $x_{1,2}=\pm(\omega^2/\omega_0^2-1)x_0$, which together with the Bohr-Sommerfeld quasiclassical quantization rule [8] $\int_{x_2}^{x_1}dx\ k(x)=\pi(j+1/2)$ where j is an arbitrary integer, yields

$$\omega^2=\omega_0^2+(2j+1)\omega_0 v_t/x_0. \quad (15)$$

In our cold plasma case, which obviously differs significantly from those described in the linear limit [7,8], we cannot find any turning points unless the nonlinear terms are included. A basic formula (12), that corresponds to Eq. (15), was therefore derived. In evaluating that eigen frequency formula, one has in general to face some rather lengthy algebra. As a comparatively simple example, we choose a slab with $\omega_p^2=\omega_0^2(1+x/x_0)$ for $x<0$ and $\omega_p^2=\omega_0^2(1-x/x_0)$ for $x>0$, where $x_0>d$. Equation (12) can for this particular profile, after some straightforward algebra, be reduced to the equation

$$1+8p-12p(1+3p)\ln(1+1/3p)=0. \quad (16)$$

for $p\equiv\omega^2/\omega_0^2$. Equation (16) has a solution $p\sim0.05$, which demonstrates that this cold slab can support resonant volume oscillations. The frequency is slightly changed if, by means of very lengthy algebra, fourth order terms in the amplitude expansion are included. Adopting other slab density profiles one can similarly calculate the eigenfrequencies by means of Eq. (12).

This work was supported by the Russian Academy of Sciences and the Royal Swedish Academy of Sciences.

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