Nonlinear modulation of periodic waves in the small dispersion limit of the Benjamin-Ono equation

Y. Matsuno*
Department of Applied Science, Faculty of Engineering, Yamaguchi University, Ube 755-8611, Japan
(Received 12 May 1998)

The Whitham modulation theory is used to construct large time asymptotic solutions of the Benjamin-Ono (BO) equation in the small dispersion limit. For a wide class of initial data, asymptotic solutions are represented by a single-phase periodic solution of the BO equation with slowly varying amplitude and wave number. The Whitham system of modulation equations for these wave parameters has a very simple structure, and it can be solved exactly under appropriate boundary conditions. It is found that the oscillating zone expands with time, and eventually evolves into a train of solitary waves. In the case of localized initial data, the number density function of solitary waves is derived in a closed form. The resulting expression coincides with the corresponding formula obtained from the asymptotic theory based on the conservation laws of the BO equation. For steplike initial data, the total number of created solitary waves increases without limit in proportion to time. [S1063-651X(98)04412-2]

PACS number(s): 03.40.Kf, 03.40.Gc, 02.90.+p

I. INTRODUCTION

Since the pioneering work of Whitham [1,2], who developed the modulation theory of periodic waves for certain class of water wave equations such as the Korteweg–de Vries (KdV) and Boussinesq equations, a large number of studies has been devoted to analyzing the mathematical structure of the modulation equations. Gurevich and Pitaevsky [3] investigated the onset and development of the nondissipative shock waves caused by the breaking of a wave front within the framework of the Whitham modulation theory for the KdV equation. Lax and Levermore [4] considered the small dispersion limit of the KdV equation on the basis of the inverse scattering transform (IST), and obtained the same equations as Whitham's modulation equations. As a result, their theory provided a justification of Whitham's modulation theory. Tsarev [5] proved the complete integrability of the modulation equations as a Hamiltonian system. See a review paper [6] on this topic. In regard to recent progress of the Whitham modulation theory for various nonlinear evolution equations, one may refer to Refs. [7,8].

Recently, the author investigated the modulation problem [9] of the periodic wave described by the Benjamin-Ono (BO) equation [10–12]

$$u_t+uu_x+\epsilon Hu_{xx}=0,\quad u=u(x,t), \tag{1.1a}$$

where u represents the wave profile, H is the Hilbert transform defined by

$$Hu(x,t)=\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{u(y,t)}{y-x}\,dy, \tag{1.1b}$$

and ϵ is a positive parameter characterizing the magnitude of the dispersion. In particular, we considered the behavior of the solution in the small dispersion limit. Using Whitham's theory, we derived modulation equations for the wave parameters characterizing the single-phase periodic wave solution of the BO equation, and then constructed an asymptotic solution for the initial value problem of the BO equation with a step initial condition. In the context of water waves, this solution is relevant to modeling the nonlinear evolution of internal bore waves in deep fluids.

Although our previous work [9] dealt with a specific initial value problem, here we shall construct the asymptotic solutions of the BO equation for a wide class of initial conditions while employing the Whitham modulation theory. In the case of the BO equation, the Whitham averaged system was seen to possess a very simple structure compared with the corresponding system for the KdV equation. In this paper, the modulation equations for the BO equation will be solved explicitly for both localized and steplike initial conditions to obtain the main feature of the solution in the small dispersion limit. The material presented here will provide a simple approximate method for constructing solutions of the BO equation without recourse to IST.

In Sec. II, we summarize the modulation equations for the BO equation. In Sec. III, we seek solutions of the modulation equations and investigate their asymptotic properties. An explicit calculation is performed for two different types of initial conditions which will help one to understand the technical details. Section IV is devoted to concluding remarks.

II. MODULATION EQUATIONS

In this section, we shall briefly describe the Whitham modulation equations which can be derived using a variational principle. The BO equation (1.1) can be derived by means of the variational principle

$$\delta\int\int L(\phi_t,\phi_x,\phi)dx\;dt=0, \tag{2.1}$$

where the Lagrangian L is given by

*Electronic address: matsuno@po.cc.yamaguchi-u.ac.jp

$$L=\frac{1}{2}\ \phi_t\phi_x+\frac{1}{6}\ \phi_x^3+\frac{\epsilon}{2}\ \phi_x H\phi_{xx}, \tag{2.2}$$

with $u=\phi_x$. In the following analysis, we consider the modulation problem of the single-phase periodic wave solution of the BO equation. To be more specific, we take the uniform wavetrain of the form

$$\phi=\psi+\Phi(\theta/\epsilon), \tag{2.3a}$$

with

$$\psi=\beta x-\gamma t, \tag{2.3b}$$

$$\theta=kx-\omega t. \tag{2.3c}$$

It then turns out from Eq. (2.3) that

$$u=\phi_x=\beta+\frac{k}{\epsilon}\ \Phi'(\theta/\epsilon), \tag{2.4}$$

where the prime appended to Φ denotes the differentiation with respect to its argument. The explicit single-phase periodic solution is given by [10]

$$u=\frac{4k^2}{\sqrt{a^2+4k^2}-\alpha\ \cos(\theta/\epsilon)}+\beta, \tag{2.5}$$

where a is the amplitude of the wave defined by

$$a=\tfrac{1}{2}(u_{\max}-u_{\min}). \tag{2.6}$$

The phase velocity c of the wave is then expressed in terms of a, k, and β as

$$c\equiv\frac{\omega}{k}=\frac{1}{2}\ \sqrt{a^2+4k^2}+\beta. \tag{2.7}$$

It should be remarked that the periodic wave (2.5) reduces to a solitary wave of algebraic type in the limit of zero wavenumber, i.e., $k\to 0$.

In accordance with the Whitham modulation theory, the next step is to derive the time evolution of the parameters β, γ, k, ω, and c, which are assumed to be slowly varying functions of x and t. For this purpose, consider the average Lagrangian

$$\overline{L}\equiv\frac{1}{2\pi}\int_0^{2\pi}L\ d\tilde{\theta},\quad(\tilde{\theta}=\theta/\epsilon). \tag{2.8}$$

If we use Eqs. (2.2), (2.5), (2.7), and (2.8), we can evaluate the integral with respect to $\tilde{\theta}$, and obtain the result

$$\overline{L}=\frac{k^3}{3}-k\left(\frac{\omega^2}{k^2}-\frac{\beta\omega}{k}+\gamma\right)+\frac{1}{6}\ \beta^3-\frac{1}{2}\ \beta\gamma. \tag{2.9}$$

The Euler equations for the pairs (β,γ) and (k,ω) are given, respectively, by

$$\frac{\partial}{\partial t}\ \frac{\partial\overline{L}}{\partial\gamma}-\frac{\partial}{\partial x}\ \frac{\partial\overline{L}}{\partial\beta}=0, \tag{2.10}$$

$$\frac{\partial}{\partial t}\ \frac{\partial\overline{L}}{\partial\omega}-\frac{\partial}{\partial x}\ \frac{\partial\overline{L}}{\partial k}=0. \tag{2.11}$$

On substituting Eq. (2.9) for $\overline{L}$, the above equations become

$$\left(k+\frac{\beta}{2}\right)_t+\left(\omega+\frac{\beta^2}{2}-\frac{\gamma}{2}\right)_x=0, \tag{2.12}$$

$$\left(-2\ \frac{\omega}{k}+\beta\right)_t-\left(k^2+\frac{\omega^2}{k^2}-\gamma\right)_x=0. \tag{2.13}$$

In addition to these equations, we must supplement the equations

$$\beta_t+\gamma_x=0, \tag{2.14}$$

$$k_t+\omega_x=0, \tag{2.15}$$

which follow from the compatibility conditions $\psi_{tx}=\psi_{xt}$ and $\theta_{tx}=\theta_{xt}$. Note that in the modulation theory, the wave parameters are assumed to be local quantities and the functions ψ and θ must be defined by the relations $\beta=\psi_x$, $\gamma=-\psi_t$, $k=\theta_x$ and $\omega=-\theta_t$. If the effect of the wave modulation is negligible, these equations are readily integrated to yield Eqs. (2.3b) and (2.3c). Substituting Eqs. (2.14) and (2.15) into Eq. (2.12), we find that $[(\beta^2/2)-\gamma]_x=0$, so that we can take

$$\gamma=\frac{\beta^2}{2}, \tag{2.16}$$

without loss of generality. Thus only the three equations are seen to be independent. In terms of β, k and c, these equations are written in the forms

$$\beta_t+\beta\beta_x=0, \tag{2.17}$$

$$k_t+(kc)_x=0, \tag{2.18}$$

$$c_t+kk_x+cc_x=0. \tag{2.19}$$

The system of equations (2.17)–(2.19) describes the slow change of the parameters characterizing the wave, and they are called modulation equations. An important feature of the above system of equations is that the equation for β is completely decoupled from other equations and may be solved independently. It is interesting to observe that the system of equations (2.18) and (2.19) coincides with the one-dimensional gas dynamic equations for isentropic flow when the ratio of the specific ratio is equal to 3 [2].

In conclusion, it is worthwhile to show that the system of equations (2.17)–(2.19) can also be derived by averaging the local conservation laws of the BO equation, the first three of which are obtained directly from Eq. (1.1) in a simple manner. They may be written in the forms

$$u_t+\left(\frac{u^2}{2}+\epsilon H u_x\right)_x=0, \tag{2.20}$$

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + \epsilon u H u_x\right)_x - \epsilon u_x H u_x = 0, \tag{2.21}$$

$$\left(\frac{u^3}{3} + \epsilon u H u_x\right)_t + \left[\frac{u^4}{4} + \epsilon(u^2 H u_x + u H(u u_x)) + \epsilon^2\left(\frac{1}{2} u_x^2 - u u_{xx} + \frac{1}{2}(H u_x)^2\right)\right]_x + \epsilon(u u_x H u_x + u_x H(u u_x)) = 0. \tag{2.22}$$

Substituting Eq. (2.5) into the above equations and then averaging, we obtain the following system of modulation equations for k, c, and β:

$$(2k+\beta)_t + \left(2kc + \frac{\beta^2}{2}\right)_x = 0, \tag{2.23}$$

$$\left(2kc + \frac{\beta^2}{2}\right)_t + \left(\frac{2}{3}k^3 + 2kc^2 + \frac{\beta^3}{3}\right)_x = 0, \tag{2.24}$$

$$\left(\frac{2}{3}k^3 + 2kc^2 + \frac{\beta^3}{3}\right)_t + \left[2kc(k^2+c^2) + \frac{\beta^4}{4}\right]_x = 0. \tag{2.25}$$

In deriving Eqs. (2.24) and (2.25), we used the fact that for any 2π-periodic functions f and g, there follow the relations $\overline{fHf}=0$ and $\overline{fHg}+\overline{gHf}=0$, which may be proved by expanding f and g in the Fourier series and using the formula $He^{ikx} = i\ \mathrm{sgn}\ k e^{ikx}$. One can easily confirm that the above system of equations is equivalent to the system of equations (2.17)–(2.19).

III. SOLUTIONS OF THE MODULATION EQUATIONS

A. Statement of the problem

In this section, we shall construct the asymptotic solutions for the initial value problem of the BO equation when the dispersion parameter ϵ is very small. Here, we consider the initial value

$$u(x,0) = f(x), \tag{3.1}$$

(i) $f(x) \geqslant 0$, $f(x) \to 0$ as $|x| \to \infty$, and (ii) f has a single maximum. The initial value with nonvanishing boundary values can be treated in the same way, for which we shall describe only the final result at the end of this subsection. The main subject here is to construct an asymptotic solution within the framework of the Whitham modulation theory described in Sec. II.

In the limit of $\epsilon \to 0$, the initial evolution of the wave profile will be governed by the Hopf equation

$$u_t + u u_x = 0, \tag{3.2}$$

which simply stems from Eq. (1.1) by neglecting the dispersive term. The solution of Eq. (3.2) with the initial condition (3.1) can be found in an implicit form as

$$u(x,t) = f(x - u(x,t)t). \tag{3.3}$$

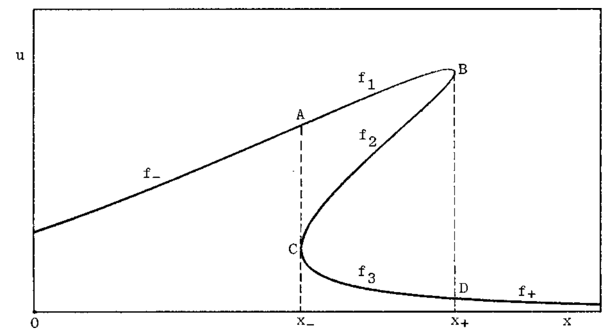

FIG. 1. The three-valued solution of the Hopf equation after the breaking time. The function f_+ (f_-) represents the branch of the solution in the range $x > x_+$ ($x < x_-$), where $x_+(x_-)$ is the leading (trailing) edge determined by the condition $\partial x/\partial u = 0$. The functions f_1, f_2, and f_3 represent, respectively, the branches *AB*, *BC*, *CD* of the solution.

As time evolves, the above solution will become multivalued function after a breaking time t_b. This situation is depicted schematically in Fig. 1, where the multivalued region lies between the trailing edge x_- and the leading edge x_+.

At a later time after t_b, however, one cannot neglect the effect of dispersion. The basic assumption in applying the Whitham modulation theory to the present problem is that after t_b the multivalued region ($x_- \leqslant x \leqslant x_+$ in Fig. 1) may be replaced by an oscillating zone whose profile is described by a periodic wave (2.5) with the slowly varying parameters a, k, and β. This solution must be joined smoothly at the boundaries $x_\pm$ with the solution (3.3) of the Hopf equation. The problem under consideration is thus reduced to finding explicit functional forms of these parameters as well as $x_\pm$ in terms of the initial condition.

In the following analysis, we shall treat the case where solution (3.3) becomes a three-valued function between x_- and x_+. In this situation, the single-phase periodic solution (2.5) will be found to be appropriate to describe the oscillating characteristic of the solution in the multivalued region.

B. Solutions

The general solutions of the modulation equations (2.17)–(2.19) can be found immediately, and they are written in implicit forms as [9]

$$c + k = g_1(x - (c+k)t), \tag{3.4}$$

$$c - k = g_2(x - (c-k)t), \tag{3.5}$$

$$\beta = g_3(x - \beta t), \tag{3.6}$$

where g_1, g_2, and g_3 are arbitrary functions. To specify the unknown functions $g_j (j=1,2,3)$, we must impose appropriate boundary conditions. These conditions are the same as those introduced by Gurevich and Pitaevsky [3] in their study of the modulation problem for a cnoidal wave solution of the KdV equation. We shall now detail it.

At the trailing edge $x = x_-$ the wave amplitude vanishes, since at this point the oscillation would begin with an infinitesimal amplitude. Then we require that the average value of

u should be joined smoothly with the solution of the Hopf equation. Explicitly, these conditions can be written as

$$a=0,\quad \overline{u}=f_-,\quad \text{at } x=x_-. \tag{3.7}$$

At the leading edge $x=x_+$, on the other hand, the wave number should vanish, since, near this point, the wave profile would be approximated by a solitary wave. In addition, the averaged solution must be joined with the solution of the Hopf equation, namely,

$$k=0,\quad \overline{u}=f_+,\quad \text{at } x=x_+. \tag{3.8}$$

We shall now apply these conditions to the general solutions (3.4)–(3.6). For this purpose, one first needs the average of u which is now easily calculated using Eq. (2.5) and the definition (2.8) of the average over one period. The result is

$$\overline{u}=2k+\beta. \tag{3.9}$$

At the trailing edge $x=x_-$, from Eqs. (2.7) and (3.7) we obtain the relation $c=k+\beta$. Substituting this into Eqs. (3.4) and (3.5), one has

$$2k+\beta=g_1(x_- -(2k+\beta)t), \tag{3.10}$$

$$\beta=g_2(x_- -\beta t). \tag{3.11}$$

It also follows from Eqs. (3.7) and (3.9) that

$$2k+\beta=f_-(x_-,t). \tag{3.12}$$

On the other hand, Eq. (3.6) yields, at $x=x_-$,

$$\beta=g_3(x_- -\beta t). \tag{3.13}$$

Combining Eqs. (3.10)–(3.13), we can see that

$$g_1(x_- -f_- t)=f_-(x_-,t),$$

$$g_2(x_- -\beta t)=g_3(x_- -\beta t),\quad \text{at } x=x_-. \tag{3.14}$$

Applying a similar procedure at the leading edge $x=x_+$, we find that

$$c=g_1(x_+ -ct),\quad c=g_2(x_+ -ct),$$

$$\beta=g_3(x_+ -\beta t), \tag{3.15}$$

$$\beta=f_+(x_+,t), \tag{3.16}$$

which enable us to take

$$g_1(x_+ -ct)=g_2(x_+ -ct),$$

$$g_3(x_+ -f_+ t)=f_+(x_+,t)\quad \text{at } x=x_+. \tag{3.17}$$

The explicit functional forms of the solutions (3.4)–(3.6) satisfying conditions (3.14) and (3.17) are readily found, and they are simply expressed as follows:

$$g_1(x-(c+k)t)=f_1(x,t),$$

$$g_2(x-(c-k)t)=f_2(x,t), \tag{3.18}$$

$$g_3(x-\beta t)=f_3(x,t),\quad (x_-\leqslant x\leqslant x_+).$$

In view of the definition (see Fig. 1), f_j are single-valued functions of x in the range $x_-\leqslant x\leqslant x_+$ for a fixed $t(>t_b)$. Once the initial value f is specified, f_j are constructed from solution (3.3), as shown in Fig. 1. Using Eqs. (3.4)–(3.6) and (3.18), the wave parameters c, k, and β are expressed in terms of f_j as follows:

$$c(x,t)=\tfrac{1}{2}[f_1(x,t)+f_2(x,t)], \tag{3.19}$$

$$k(x,t)=\tfrac{1}{2}[f_1(x,t)-f_2(x,t)], \tag{3.20}$$

$$\beta(x,t)=f_3(x,t). \tag{3.21}$$

It now remains to determine $x_\pm$ in terms of the initial condition. However, this can be done simply by solving the equation $\partial x/\partial u=0$ with $x=ut+f^{-1}(u)$, where f^{-1} denotes the inverse function of f. Since we are concerned here with the three-valued function, there exist only two solutions $x_\pm$, as seen from Fig. 1. Thus we have completed the construction of the solutions.

Remark: Although the result presented here is applicable to localized initial data, we can also construct solutions for initial data with nonvanishing boundary conditions. One example is a monotonically decreasing function $f(x)$ with boundary conditions such that $f(x)\to u_0$ as $x\to-\infty$ and $f(x)\to 0$ as $x\to+\infty$, where u_0 is a positive constant. Obviously, the solution of the Hopf equation for this initial value breaks down in a finite time, and becomes a three-valued function. A special case for such initial data is a step initial condition already treated in Ref. [9]. Applying a similar procedure to that developed here for the localized initial data, one can show that solutions to the modulation equations take exactly the same forms as those given by Eqs. (3.19)–(3.21).

C. Asymptotic behavior of solutions for large time

Here we shall investigate the behavior of solutions (3.19)–(3.21) for large time. The periodic solution u given by Eq. (2.5) will be found to evolve into a train of solitary waves in the range $x_-\leqslant x\leqslant x_+$. Suppose, for simplicity, that $f(x)$ has a maximum u_0 at $x=0$. When t tends to infinity, the functions f_1 and f_2 in Fig. 1 will approach the straight line $u=x/t$ in the range $x_-\leqslant x\leqslant u_0 t$. Let $V=x/t(x_-/t\leqslant V\leqslant u_0)$, $u_1=f_1(Vt,t)$, and $u_2=f_2(Vt,t)$. Then it follows from Eq. (3.3) that

$$Vt-u_1 t=f_-^{-1}(u_1),\quad Vt-u_2 t=f_+^{-1}(u_2), \tag{3.22}$$

where $f_+^{-1}(f_-^{-1})$ denotes the positive (negative) branch of the inverse function f^{-1}. See Fig. 2.

If we use the approximation $u_1\sim u_2\sim V=x/t$ as $t\to\infty$, we find, from Eqs. (3.20) and (3.22) and the relations $u_1=f_1$ and $u_2=f_2$, that $k(x,t)$ behaves in the limit of $t\to\infty$ as

$$k(x,t)\sim\frac{1}{2t}\left[f_+^{-1}\left(\frac{x}{t}\right)-f_-^{-1}\left(\frac{x}{t}\right)\right]\left(0<\frac{x}{t}\leqslant u_0\right). \tag{3.23}$$

If we denote the two solutions of the equation $f(x)=V(0<V\leqslant u_0)$ by x_1 and $x_2(x_1<0<x_2)$, i.e., $x_1=f_-^{-1}(V)$ and $x_2=f_+^{-1}(V)$ (see Fig. 2), we can rewrite Eq. (3.23) as

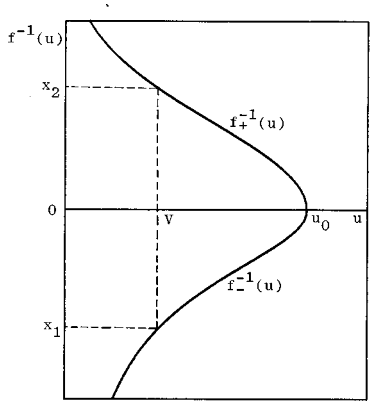

FIG. 2. The inverse function $f^{-1}(u)$. The function f_+^{-1} (f_-^{-1}) represents the positive (negative) branch of the inverse function f^{-1}. The ordinates x_1 and x_2 are given, respectively, by $x_1 = f_-^{-1}(V)$ and $x_2 = f_+^{-1}(V)$.

$$k(x,t) \sim \frac{1}{2t}(x_2 - x_1). \tag{3.24}$$

Since both x_1 and x_2 are finite for V of order 1, relation (3.24) shows that the local wave number vanishes in the limit of $t \to \infty$ except for a narrow region near the trailing edge. It follows from these observations that the oscillating zone expands with time, and evolves into a train of solitary waves after the elapse of a long time from the wave breaking. To estimate the asymptotic distribution of the amplitude, we introduce the number density function $F(A)$, which gives the number dN_s of solitary waves with amplitudes within the interval $(A, A+dA)$. Then

$$dN_s = F(A)dA = \frac{dx}{\lambda} = \frac{k}{2\pi\epsilon}dx, \tag{3.25}$$

where λ is the local wavelength defined by $\lambda = 2\pi\epsilon/k$. Using the fact that the amplitude of the BO solitary wave is related to the velocity by the relation $A = 4V = 4x/t$, from Eqs. (3.24) and (3.25) we obtain the formula

$$F(A) \sim \frac{1}{16\pi\epsilon}(x_2 - x_1) = \frac{1}{16\pi\epsilon}\int_{A<4f(x)} dx, \tag{3.26}$$

where, in the last expression, the integration interval is subjected to the condition $A < 4f(x)$. The total number N_s of solitary evolving from the initial condition $u(x,0) = f(x)$ is then given by

$$N_s = \int_0^\infty F(A)dA \sim \frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty} f(x)dx. \tag{3.27}$$

Expressions (3.26) and (3.27) completely coincide with the corresponding formulas obtained from the asymptotic theory based on the conservation laws of the BO equation [13,14].

D. Examples

Here we shall apply the procedure developed in Sec. III B to the two different types of initial conditions, and investigate the asymptotic behavior of the solutions.

1. Localized initial condition

We first consider the localized initial condition. As an example, we suppose the following initial profile

$$f(x) = \frac{u_0}{x^2+1} \quad (u_0 > 0). \tag{3.28}$$

In this example, solutions (3.19)–(3.21) to the modulation equations can be written in closed form. In fact, each branch of the three-valued function in Fig. 1 is obtained by solving the cubic equation $u[(x-ut)^2+1] = u_0$. Since we are concerned with the behavior of the solutions for large time, we shall describe only the leading terms of the large time asymptotics of various quantities. The leading and trailing edges of the solution behave like

$$x_+ \sim u_0 t, \quad x_- \sim 3 \times 2^{-2/3}(u_0 t)^{1/3}. \tag{3.29}$$

In the range $x_- \leqslant x \leqslant x_+$, k, c, ω, and a have the asymptotic forms

$$k(x,t) \sim \frac{1}{t}\left(\frac{1}{z} - 1\right)^{1/2}, \tag{3.30a}$$

$$c(x,t) \sim u_0 z, \tag{3.30b}$$

$$\omega(x,t) \sim \frac{u_0}{t} z\left(\frac{1}{z} - 1\right)^{1/2}, \tag{3.30c}$$

$$a(x,t) \sim 2\left[(u_0 z)^2 - \frac{1}{t^2}\left(\frac{1}{z} - 1\right)\right]^{1/2}, \tag{3.30d}$$

where $z \equiv x/u_0 t$. At the edges $x_\pm$, β behaves like

$$\beta(x_+, t) \sim u_0^{-1} t^{-2}, \quad \beta(x_-, t) \sim 2^{-3/2} u_0^{1/3} t^{-2/3}. \tag{3.31}$$

Integrating the relations $\theta_x = k$ and $\theta_t = -\omega$ with k and ω given, respectively, by Eqs. (3.30a) and (3.30c), we can determine the phase of the wave as

$$\theta(x,t) \sim u_0\left[\sqrt{z(1-z)} + \sin^{-1}\sqrt{z} - \frac{\pi}{2}\right], \tag{3.32}$$

where the integration constant has been chosen such that the phase function vanishes at the leading edge $z = 1$. The envelopes of the maximum and minimum values of u are found from Eqs. (2.5), (3.30), and (3.31), and they take the forms

$$u_{\max} = \frac{4k^2}{\sqrt{a^2+4k^2} - a} + \beta \sim \frac{4x}{t}, \tag{3.33a}$$

$$u_{\min} = \frac{4k^2}{\sqrt{a^2+4k^2} + a} + \beta = O(t^{-2/3}). \tag{3.33b}$$

Relations (3.33) show that the amplitude of each solitary wave varies linearly with distance. In particular, at the leading edge $x = x_+ \sim u_0 t$, the amplitude attains four times the maximum amplitude of the initial profile. Using Eqs. (3.26), (3.27), and (3.28), the number density function $F(A)$ and the total number N_s of solitary waves evolved from the initial profile are given, respectively, by

$$F(A)\sim\begin{cases}\frac{1}{4\pi\epsilon}\left(\frac{4u_0}{A}-1\right)^{1/2} & \text{for } 0<A\leqslant 4u_0\\ 0, & \text{for } A>4u_0,\end{cases} \tag{3.34}$$

$$N_s\sim\frac{u_0}{4\epsilon}. \tag{3.35}$$

These results are completely in agreement with the corresponding formulas derived on the basis of the conservation laws of the BO equation in the small dispersion limit [13,14].

2. Steplike initial condition

The second example is concerned with the initial condition with a nonvanishing boundary value. We consider a steplike profile of the form

$$f(x)=\frac{u_0}{2}(1-\tanh x),\quad (u_0>0). \tag{3.36}$$

Without entering into the detail, we shall describe only the final results. The leading terms for the asymptotic expansions of various quantities take the following forms:

$$x_+\sim u_0 t,\quad x_-\sim\frac{1}{2}\ln(u_0 t), \tag{3.37}$$

$$k(x,t)\sim\frac{u_0}{2}(1-z), \tag{3.38a}$$

$$c(x,t)\sim\frac{u_0}{2}(1+z), \tag{3.38b}$$

$$\omega(k,t)\sim\frac{u_0^2}{4}(1-z^2), \tag{3.38c}$$

$$a(k,t)\sim 2u_0 z, \tag{3.38d}$$

$$\beta(x_+,t)\sim u_0 e^{-2u_0 t},\quad \beta(x_-,t)\sim(2t)^{-1}, \tag{3.39}$$

where $z=x/u_0 t$. Expressions (3.38) are valid within the range $(1/2u_0 t)\ln(u_0 t)\leqslant z\leqslant 1$. The phase θ is now expressed as

$$\theta(x,t)\sim-\frac{u_0^2 t}{4}(1-z)^2. \tag{3.40}$$

Also, u_{max} and u_{min} are given, respectively, by

$$u_{max}\sim u_0(1+z+2\sqrt{z}),\quad u_{min}\sim u_0(1+z-2\sqrt{z}). \tag{3.41}$$

As in the case of the localized initial condition exemplified above, the amplitude of the leading solitary wave is the four times the maximum amplitude of the initial profile. The number of created solitary waves is estimated from Eqs. (2.5) and (3.40), and it gives

$$N_s\sim\frac{u_0^2 t}{8\pi\epsilon}. \tag{3.42}$$

One can see from Eq. (3.42) that the total number of solitary waves increases indefinitely in proportion to time. This fact is in striking contrast to the corresponding result for the localized initial data, in which case the total number is found to be definite as long as the integral given in Eq. (3.27) converges. We also note that these leading order asymptotics coincide with those corresponding to a step initial condition [9] for which expressions (3.38), (3.40), (3.41), and (3.42) become exact.

Remark: The results derived above are not the specific features of the solution depending on the initial condition (3.36). In fact, we can obtain the same asymptotic expressions for a wide class of monotonically decreasing initial data.

IV. CONCLUDING REMARKS

In this paper, we developed an approximate method for solving the initial value problem of the BO equation by means of the Whitham modulation theory. In particular, we were concerned with the wave profile evolving from the localized initial data when the effect of dispersion is very small. We then find that the corresponding problem for the steplike initial data can be dealt with in the same way. For the localized initial data, the large time asymptotic of the solution consists of a train of solitary waves whose amplitude distribution can be determined in a closed form in terms of the initial condition. The result obtained here is completely in agreement with the formula [13,14] derived independently from the asymptotic theory based on the conservation laws of the BO equation.

A few numerical analyses have been performed on the initial value problem of the BO equation. Christie [15] calculated an asymptotic profile for a steplike initial condition which simulates the evolution of an internal bore wave in deep fluids. The result is qualitatively in agreement with the analytical result presented in Sec. III D. In this respect, it should be remarked that IST for the BO equation [16,17] has not yet been applied to the nonvanishing initial condition like a step profile. Miloh *et al.* [18] performed a similar calculation for localized initial conditions under the small dispersion parameter. Their result for the number of the created solitary waves coincides completely with the analytical expression [Eq. (3.27)].

In contrast to a large number of studies devoted to the modulation problem for the periodic waves of the KdV equation, there appeared few works dealing with the corresponding problem for the BO equation. One reason for this may be attributed to the nonlocal nature of the dispersive term expressed by the Hilbert transform. The present study shows that the Whitham modulation theory is also applicable to the BO equation, and gives the main feature of the solution. Surprisingly, the mathematical structure of the modulation equations for the BO equation is seen to be extremely simple compared with that for the KdV equation. From a rigorously mathematical point of view, however, the various results presented in this paper should be justified on the basis of an exact method of solution such as IST, or an analog of the Lax-Levermore theory for the KdV equation. These problems will be left for future studies.

[1] G. B. Whitham, Proc. R. Soc. London, Ser. A **283**, 238 (1965).
[2] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[3] A. V. Gurevich and L. P. Pitaevsky, Zh. Eksp. Teor. Fiz. **65**, 590 (1973) [Sov. Phys. JETP **38**, 291 (1974)].
[4] P. D. Lax and C. D. Levermore, Commun. Pure Appl. Math. **36**, 253 (1983); **36**, 571 (1983); **36**, 809 (1983).
[5] S. P. Tsarev, Sov. Math. Dokl. **31**, 488 (1985).
[6] B. A. Dubrovin and S. P. Novikov, Russ. Math. Surveys **44**, 35 (1989).
[7] *Singular Limits of Dispersive Waves*, edited by N. M. Ercolani *et al.* (Plenum, New York, 1994).
[8] A. M. Kamchatnov, Phys. Rep. **286**, 199 (1997).
[9] Y. Matsuno, J. Phys. Soc. Jpn. **67**, 1814 (1998).
[10] T. B. Benjamin, J. Fluid Mech. **29**, 559 (1967).
[11] R. E. Davis and A. Acrivos, J. Fluid Mech. **29**, 593 (1967).
[12] H. Ono, J. Phys. Soc. Jpn. **39**, 1082 (1975).
[13] Y. Matsuno, Phys. Lett. **87A**, 15 (1981).
[14] Y. Matsuno, J. Phys. Soc. Jpn. **51**, 667 (1982).
[15] D. R. Christie, J. Atmos. Sci. **46**, 1462 (1989).
[16] A. S. Fokas and M. J. Ablowitz, Stud. Appl. Math. **68**, 1 (1983).
[17] D. J. Kaup and Y. Matsuno, Stud. Appl. Math. **101**, 73 (1998).
[18] T. Miloh, M. Prestin, L. Shtilman, and M. P. Tulin, Wave Motion **17**, 1 (1993).