Potential of interaction between two- and three-dimensional solitons

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A general method to find an effective potential of interaction between far separated two-dimensional (2D) and 3D solitons is elaborated, including the case of 2D vortex solitons. The method is based on explicit calculation of the overlapping term in the full Hamiltonian of the system (*without* assuming that the ''tail'' of each soliton is not affected by its interaction with the other soliton). The result is obtained in an explicit form that does not contain an artificially introduced radius of the overlapping region. The potential applies to spatial and spatiotemporal solitons in nonlinear optics, where it helps to solve various dynamical problems: collisions, formation of bound states (BS's), etc. In particular, an orbiting BS of two solitons is always unstable. In the presence of weak dissipation and gain, the effective potential can also be derived, giving rise to bound states similar to those recently studied in 1D models. [S1063-651X(98)02912-2]

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I. INTRODUCTION

Recent progress in studies of two-dimensional (2D) solitons in models of non-Kerr nonlinear optical media has attracted a lot of interest to their interactions. 2D vortex solitons and interactions between them in the quintic nonlinear Schrödinger (QNLS) equation were studied in Ref. [1], nonplanar interactions between 2D solitons in a medium with the quadratic ($\chi^{(2)}$) nonlinearity were considered, numerically and analytically, in Ref. [2], and various features of the interaction between 2D solitons in photorefractive media were revealed by numerical simulations and direct experiments [3–5]. The nonlinearity must be non-Kerr because the usual cubic (Kerr) self-focusing term gives rise to collapse in 2D and 3D cases. As it was demonstrated in Ref. [6], the collapse does not take place in any physical dimension in the model with the $\chi^{(2)}$ nonlinearity. This opens the way to stable 2D and 3D *spatiotemporal* solitons, or ''light bullets'' (LB's) [7]. The $\chi^{(2)}$ LB's were recently studied in detail in Refs. [8] and [9].

The objective of this work is to find an effective potential of interaction between 2D and 3D solitons in isotropic media (note that, as it was demonstrated in a very recent experimental work [4], the interaction of 2D solitons in intrinsically anisotropic photorefractive media is, in effect, practically isotropic too). The interaction potential is necessary to solve various dynamical problems, such as collisions, formation of bound states of solitons, etc., including a practically important problem of designing all-optical switching by means of interaction between 2D optical solitons. It will be demonstrated that a universal effective potential can be obtained analytically by means of a technique which generalizes that developed for the 1D solitons in Ref. [10]. As a paradigm model, one can take the multidimensional quintic Ginzburg-Landau (QGL) equation,

$$i v_t + \frac{1}{2}\nabla^2 v + |v|^2 v - \alpha |v|^4 v = -iv + i\gamma_1 \nabla^2 v + i\gamma_2 |v|^2 v - i\gamma_3 |v|^4 v, \quad (1)$$

where the coefficients α and $\gamma_{1,2,3}$ are positive. The QNLS equation is a conservative version of Eq. (1), without its right-hand side. The quintic defocusing term $\sim\alpha$ is included in order to prevent the collapse. Note that this term is not merely the simplest one that stabilizes the model: according to experimental data [11], the combination of the focusing cubic and defocusing quintic terms adequately models the nonlinear optical properties of some real materials. The first two terms on the right-hand side of Eq. (1) take into regard linear losses, the cubic term γ_2 accounts for *nonlinear gain* which compensates the losses, and the quintic dissipation term $\sim\gamma_3$ provides for the overall stabilization of the model. The QGL equation was first introduced in Ref. [12] (in the 2D form), and its 1D (one-dimensional) version later attracted a great deal of interest (see, e.g., Ref. [13] and references therein). In particular, stable localized pulses in the 1D QGL equation were found in Ref. [14] for the case of weak dissipation (relevant for the applications to nonlinear optics), $0 \leqslant \gamma_{1,2,3} \ll 1$, that will also be assumed here. The existence of the stable pulses in the opposite limit of strong dissipation was independently shown in three different works [15]. Actually, the model (1) is selected just for the reference, as the one that certainly gives rise to stable multidimensional solitons; as will be seen below, the derivation of the effective potential for the interaction between the solitons, presented in this work, is quite universal and may be applied to any conservative or weakly dissipative model that supports multidimensional solitons.

Note that stable 2D solitons, as well as two-soliton bound states, were also found numerically in a model with the quintic nonlinearity similar to that in Eq. (1), in which, however, the linear part is of a higher order, containing the operators $\partial^2/\partial t^2$ and ∇^4 [16]. However, that model is essentially more complicated than Eq. (1), and its physical applications are less clear.

The paper is organized as follows. In Sec. II, the 2D and 3D soliton solutions are briefly considered, with emphasis on the form of their asymptotic ''tails,'' which determine the effective interaction potential. In the same section, the model (1) is also reformulated in terms of nonlinear optics, where it finds applications of two types: the description of spatial

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cylindrical solitons in the bulk medium, and 2D and 3D LB's in the 2D nonlinear wave guide or 3D bulk, respectively. The multidimensional solitons in the model with the $\chi^{(2)}$ nonlinearity have their own peculiarities, which are summarized in a separate subsection in Sec. II. In Sec. III, the interaction potential is analytically derived, in a general form, for the 2D and 3D solitons. In the same section, the interaction potential for LB's is also considered. In particular, the potential may be *spatiotemporally anisotropic* for the $\chi^{(2)}$ LB's, while in the other models it can always be cast into an effectively isotropic form. Concluding remarks are collected in Sec. IV, including a discussion of a possibility of existence of bound states of the solitons. In particular, it is concluded, in accord with the recent results obtained for the $\chi^{(2)}$ spatial solitons in Ref. [2], that a bound state of two solitons orbiting around each other may exist (in the dissipationless model), but it is always unstable. In the presence of the weak dissipation and gain, there are bound states of quiescent solitons, quite similar to those recently studied in the 1D model (that may be both unstable and almost stable). New possible states in the 2D and 3D cases are soliton lattices and ''molecules.''

II. TWO- AND THREE-DIMENSIONAL SOLITONS

A. The general case

A general stationary solution to Eq. (1) is $v=\exp(-i\omega t)V(\mathbf{r})$, where $V(\mathbf{r})$ satisfies the equation

$$\frac{1}{2}\nabla^2 V+|V|^2V-\alpha|V|^4V+\omega V =-iV+i\gamma_1\nabla^2V+i\gamma_2|V|^2V-i\gamma_3|V|^4V. \quad (2)$$

In the 2D case, the solution is restricted to the form

$$V(\mathbf{r})=\exp(is\theta)\mathcal{V}(r), \quad s=0,\pm1,\pm2,\ldots, \quad (3)$$

where r and θ are the polar coordinates, $s\neq0$ corresponding to a *vortex soliton*, and $\mathcal{V}(r)$ exponentially decays at $r\to\infty$. From the consideration of Eq. (1) it follows that the asymptotic form of the soliton at $r\to\infty$ is

$$\mathcal{V}(r)\approx A_s r^{-1/2}\exp(-\kappa r), \quad (4)$$

$$\kappa=\sqrt{-\frac{\omega+i}{1/2+i\gamma_1}}\approx\sqrt{-2\omega}-iq, \quad q=\frac{1}{\sqrt{-2\omega}}+\gamma_1\sqrt{-2\omega}, \quad (5)$$

and, at $r\to0$,

$$\mathcal{V}(r)\approx a_s r^{|s|} \quad (6)$$

(i.e., the vortex soliton has a hole in its center), with unknown constants A_s and a_s. The expansion of κ in Eq. (5) employs the fact that, in the weakly dissipative regime, γ_1 is small, and $\omega\gg1$, as the dissipation coefficient in front of the term $-v$ in Eq. (1) is 1.

The stability of the $s=0$ soliton in the model (1) is very plausible, and, in the conservative version of Eq. (1), the stability of the vortex soliton with $|s|=1$ was numerically demonstrated in Ref. [1]. It is not known if the solitons with $|s|>1$ are stable (note that all the bright vortex solitons are unstable in the $\chi^{(2)}$ model, see, e.g., Ref. [17]). Below, an arbitrary integer value of s will be kept, as the potential can be derived in the general case, provided that the two solitons have $s_1=\pm s_2$.

Description of 3D solitons with the internal ''spin'' is a rather complicated problem, therefore only the 3D solitons with the zero spin will be considered here. The corresponding solution is sought for in the form of Eq. (3) with $s=0$, and with the difference that r is now the radial variable in the 3D space, hence

$$\mathcal{V}(r)\approx A_s r^{-1}\exp(-\kappa r) \quad (7)$$

at $r\to\infty$.

In the conservative version of the model, the frequency $\omega<0$ is an arbitrary parameter of the soliton, while the amplitude A_s, that can be found numerically, is a function of ω [as well as a_s in Eq. (6)]. In the presence of the weak dissipation and gain, an actual soliton solution is selected from the continuous family as the one providing for a balance of the ''number of photons,'' $\int_0^\infty|V(r)|^2r^{D-1}dr$ [14]. In that case, the value of ω should also be found numerically. Below, ω and A_s will be treated as given parameters.

In the application to the nonlinear optics, the QNLS version of the 2D model (1) describes time-independent light distributions in a 3D medium, so that the variable t is not time, but the propagation coordinate. The dynamics of LB's in 2D and 3D optical media is governed by an equation that is also similar to Eq. (1). Neglecting the dissipative part, the corresponding QNLS equation is

$$iv_z+\frac{1}{2}(\nabla_\perp^2 v+v_{\tau\tau})+|v|^2v-\alpha|v|^4v=0, \quad (8)$$

where v is the envelope of the electromagnetic waves, z and $\tau\equiv t-z/c_{\mathrm{gr}}$ are the propagation coordinate and the so-called retarded time, c_{gr} being the mean group velocity of the carrier wave, and the operator $\nabla_\perp^2$ acts on the transverse coordinate(s). In Eq. (8), *anomalous* temporal dispersion (accounted for by the term $v_{\tau\tau}$) is assumed. A spatiotemporal-soliton solution to Eq. (8) (i.e., LB) can be sought for in the form [cf. Eq. (3)]

$$v=\exp(ikz)\mathcal{V}(\xi), \quad \xi\equiv\sqrt{r_\perp^2+\tau^2}. \quad (9)$$

Here, k is the *propagation constant* and $r_\perp$ is the transverse coordinate in the 2D model, or the radial variable in the transverse plane in the 3D model. In the latter case, a more general solution with a ''spatiotemporal spin'' can be looked for in the form

$$v=\exp(ikz+is\theta)\mathcal{V}(\xi), \quad (10)$$

where this time θ is the formal angular coordinate on the plane $(r_\perp,\tau)$. The solution (10) has a ''hole'' in its center, cf. Eq. (6). The asymptotic form of all the LB solutions at $\xi\to\infty$ is similar to that given above by Eqs. (4), (5), and (7):

$$\mathcal{V}(\xi)\approx A\xi^{-(D-1)/2}\exp(-\sqrt{2q}\,\xi). \quad (11)$$

B. The model with the quadratic nonlinearity

An allied physically important model is that which describes multidimensional $\chi^{(2)}$ media [8]:

$$iv_z+\frac{1}{2}(\nabla_\perp^2 v+v_{\tau\tau})-v+v^*w=0, \quad (12)$$

$$2iw_z+\frac{1}{2}(\nabla_\perp^2 w+\delta w_{\tau\tau})-\gamma w+\frac{1}{2}v^2=0, \quad (13)$$

where v and w are envelopes of the fields at the fundamental harmonic (FH) and second harmonic (SH), $\gamma>0$ is a mismatch parameter, and δ is a *relative coefficient* of the temporal dispersion. In the real physical situations, $\delta<1$, including negative values (which correspond to the normal dispersion at the second harmonic). As was shown in Refs. [8] and [9], the spatiotemporal soliton solutions to Eqs. (12) and (13) can only exist if $\delta\geqslant 0$. However, the solution cannot be sought for in the form (9), except for the unrealistic special case $\delta=1$. The asymptotic form of the soliton can be nevertheless easily found from the linearized versions of Eqs. (12) and (13), cf. Eqs. (4)–(7) and (9):

$$v_s\approx A\xi^{-(D-1)/2}\exp(-\sqrt{2}\xi), \quad (14)$$

$$w_s\approx B\tilde{\xi}^{-(D-1)/2}\exp(-\sqrt{2\gamma}\tilde{\xi}), \quad \tilde{\xi}^2\equiv r_\perp^2+\delta^{-1}\tau^2. \quad (15)$$

Here, only the case $s=0$ is considered, and the propagation constant is not explicitly introduced, as it may be absorbed by the mismatch parameter γ.

The consideration of Eq. (13) readily demonstrates that, while the asymptotic expression (14) for FH is always relevant, the expression (15) makes sense only if it decays at $r_\perp,\tau\to\infty$ *not faster* than v_s^2. Further straightforward analysis shows that this condition is always satisfied, provided that $\gamma<4\delta$, and never satisfied, if $\gamma>4$. In the intermediate case

$$4\delta<\gamma<4 \quad (16)$$

(recall that the physical constraint is $\delta<1$, and $\gamma=4$ has a special meaning corresponding to the *exact matching* between FH and SH [8]), the condition holds, on the plane of the variables r and τ, inside the sector

$$(\tau/r_\perp)^2<\frac{4-\gamma}{\gamma-4\delta}\delta, \quad (17)$$

and does not hold outside this sector. The $\chi^{(2)}$ solitons for which this condition holds, i.e., the shape of their asymptotic ''tails'' in each harmonic is *independently* determined by the corresponding linearized equations, may be naturally called *free-tail solitons*.

In the case when the above condition does not hold, i.e., v_s^2, as given by Eq. (14), decays slower than w_s as per Eq. (15), the latter expression does not apply. In this case, the actual decay of the SH tail is governed by the quadratic term in Eq. (13), while Eq. (14) remains valid. The final result which, in this case, replaces Eq. (15) is

$$w_s\approx\frac{1}{2}A^2\frac{\xi^{3-D}}{(\gamma-4)\xi^2+4(1-\delta)\tau^2}\exp(-2\sqrt{2\gamma}\xi) \quad (18)$$

[note that the underlying conditions that define the present case guarantee that the expression (18) is not singular]. The solitons of this type may be called, on the contrary to the free-tail ones defined above, *tail-locked* solitons. Note that, in the intermediate case (16), the free-tail asymptotic expression (15) holds inside the sector (17), while the tail-locked expression (18) is valid outside the sector.

III. THE INTERACTION POTENTIAL

A. The general case

Coming back to the paradigm model (1), one notes that the conservative (left-hand) side of its stationary version (2) can be derived from the Hamiltonian

$$H=\int\left(\frac{1}{2}|\nabla V|^2-\frac{1}{2}|V|^4+\frac{1}{3}\alpha|V|^6-\omega|V|^2\right)d\mathbf{r}. \quad (19)$$

The Hamiltonian allows one to define an effective interaction potential for two separated solitons [18]. In the original works, the wave field corresponding to the two-soliton configuration was postulated to be a linear superposition of two isolated solitons. This was substituted into the Hamiltonian, and a term produced by the overlapping of the ''body'' of each soliton with a weak ''tail'' of the other one was identified as an effective interaction potential. This approach requires actual calculation of the corresponding integral term in Eq. (19), a drawback being that a distortion of the ''tail'' due to its interaction with the other soliton is ignored. In this work, a more consistent approach will be developed, following that elaborated for the 1D solitons in Ref. [10]. The method is based on representing the wave field in a vicinity of each soliton in the form

$$v(\mathbf{r},t)=\exp(-i\omega t)[V_s(\mathbf{r})+V_t(\mathbf{r})], \quad (20)$$

where $V_s(\mathbf{r})$ is the isolated soliton (3), $V_t(\mathbf{r})$ is a small tail generated by the second soliton, and the influence of a given soliton on the other soliton's ''tail'' is *not* neglected. The distance R between the centers of the two solitons is assumed to be essentially larger than the soliton's size $\sim\kappa^{-1}$, see Eq. (4). A similar structure of the wave field is assumed near the center of the second soliton.

Only the case when the interacting solitons are identical is considered [in particular, $s_1=\pm s_2$ for the 2D vortex solitons, and the amplitudes $|A_s|$, defined as per Eq. (4), are equal], hence the solitons have the same frequency ω, which allows one to define a phase difference ψ between them. The case of the identical solitons is the most relevant one, as parameters of the solitons are, in a real physical situation, uniqely selected by the above-mentioned balance between the gain and dissipation.

The next step is, as was said above, to insert the expression (20) into the Hamiltonian (19) and calculate the overlap term in an area around the first soliton, adding then a symmetric contribution from the vicinity of the other soliton. In the first approximation, only the terms linear in V_t are to be kept, which yields the following expression for the effective interaction potential

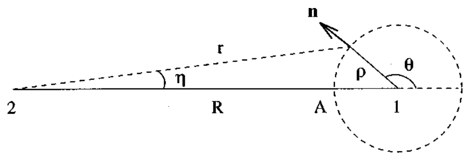

FIG. 1. The two-soliton configuration in the two- and three-dimensional cases (in the latter case, the figure shows the central cross section of the 3D configuration). The points 1 and 2 are centers of the solitons.

$$U_D(R,\psi)=\left[\int\left(\frac{1}{2}\nabla V_s\cdot\nabla V_t^*-|V_s|^2V_sV_t^*+\alpha|V_s|^4V_sV_t^*-\omega V_sV_t^*\right)d\mathbf{r}+\text{c.c.}\right]+\{1\rightleftharpoons 2\},\tag{21}$$

where the subscript D pertains to the dimension. Here, c.c. stands for the complex conjugate expression, the integration is assumed over the overlapping region in a vicinity of the first soliton, and $\{1\rightleftharpoons 2\}$ is the symmetric contribution from the second soliton. Applying the Gauss theorem to the first term in Eq. (21), one transforms, in the 2D case, the expression (21) into the form

$$U_D(R,\psi)=\left\{\left[-\int\left(\frac{1}{2}\nabla^2V_s+|V_s|^2V_s-\alpha|V_s|^4V_s+\omega V_s\right)V_t^*d\mathbf{r}+\frac{1}{2}\int V_t^*(\mathbf{n}\cdot\nabla)V_sdl\right]+\text{c.c.}\right\}+\{1\rightleftharpoons 2\},\tag{22}$$

where the surface integral term is taken over a closed contour surrounding the first soliton, $\mathbf{n}$ being a local vector normal to the contour. As the contour, one can choose a circumference whose center coincides with that of the first soliton (Fig. 1). The radius ρ is chosen so that

$$\kappa^{-1}\ll\rho\ll R,\tag{23}$$

i.e., it is much larger than the size of the soliton, but much smaller than the separation between the two solitons. The final objective will be to obtain an expression that does not depend on the auxiliary radius ρ. To this end, it will be necessary to supplement the condition (23) by the additional one

$$\rho^2/R\ll\kappa^{-1},\tag{24}$$

which is obviously compatible with Eq. (23).

In the 3D case, the difference is that the surface integral in Eq. (22) is taken over a sphere of the radius ρ, so that Fig. 1 shows the central cross section of the 3D situation. The conditions (23) and (24) pertain equally well to the 3D case.

At this stage of the analysis, the dissipative terms in Eq. (2) are still neglected. Because V_s is an exact single-soliton solution to Eq. (2), the first integral term in Eq. (22) vanishes. The conditions (23) allow one to substitute both V_t and V_s in the surface integral term in Eq. (22) by the asymptotic expressions (4), which yields, in the 2D case

$$U_2(R,\psi)=-\sqrt{-\frac{\omega}{2}}|A_s|^2\sqrt{\rho}e^{-\kappa\rho}\times\left[\int_0^{2\pi}r^{-1/2}\exp(i\psi+is_1\theta-is_2\eta)\times\exp(-\kappa r)d\theta+\text{c.c.}\right]+\{1\rightleftharpoons 2\}.\tag{25}$$

The angles η and θ and the radius

$$r=\sqrt{(R+\rho\cos\theta)^2+\rho^2\sin^2\theta}=R+\rho\cos\theta+\frac{1}{2}(\rho^2/R)\sin^2\theta+\cdots\tag{26}$$

are defined in Fig. 1, the condition (23) being used to expand the radical in Eq. (26). Substituting the expansion into Eq. (25) and taking into regard the condition (24), in the first approximation it is enough to keep the first two terms from Eq. (26) in $\exp(-\kappa r)$, and only the first term in $r^{-1/2}$. Additionally, in the same approximation one may set $\eta=0$, which leads to

$$U_2(R,\psi)=-\sqrt{-\frac{\omega}{2}}|A_s|^2R^{-1/2}\sqrt{\rho}e^{-\kappa\rho}\times\left[e^{-\kappa R}\int_0^{2\pi}\exp(i\psi+is_1\theta)\times\exp(-\kappa\rho\cos\theta)d\theta+\text{c.c.}\right]+\{1\rightleftharpoons 2\}.\tag{27}$$

The integral in Eq. (27) can be calculated exactly in terms of the Bessel functions, but this is not necessary. Indeed, taking into regard, in line with the previous approximations, that $\kappa\rho\gg 1$, the Laplace approximation can be applied to the integral, a dominant contribution coming from a vicinity of the point $\theta=\pi$ (point A in Fig. 1):

$$\int_0^{2\pi}\exp(-\kappa\rho\cos\theta)d\theta\approx\sqrt{2\pi}(\kappa\rho)^{-1/2}e^{+\kappa\rho}.\tag{28}$$

Substituting this into Eq. (27), one sees that the ρ-dependent multipliers in Eq. (27) are *exactly* cancelled by $\rho^{-1/2}e^{+\kappa\rho}$ from Eq. (28). This cancellation (in the lowest-order approximation considered here) is a crucial result, as it makes the effective potential independent of the auxiliary radius ρ.

Of course, the dependence on ρ will not disappear if one tries to calculate higher-order corrections (with respect to R^{-1}) to the effective potential. Actually, this implies that the effective interaction potential, treating the solitons as particles, can be consistently defined only in the lowest-order approximation. At the higher orders, it is necessary to explicitly take into regard deformation of the solitons by the interactions, which is not an objective of the present work.

In the term $\{1 \rightleftharpoons 2\}$ in Eq. (27), ψ is replaced, according to its definition, by $-\psi$, and the dominant point in the surface integral is $\theta=0$. This means that the term $\{1 \rightleftharpoons 2\}$ is obtained by the change $\psi \to -\psi \quad s_1\pi \to -s_1\pi$. Finally, in the multiplier $e^{-\kappa R}$ in Eq. (27), small $q=-\mathrm{Im}\kappa$ should be also taken into regard [see Eq. (5)], as it gives rise to an important effect, viz., long-period oscillations in the exponentially decaying tail of the interaction potential [10]. Note that the potential does not directly take into account the model's small dissipative part; however, that part indirectly affects the potential, inducing the oscillations in the solitons' tails in Eq. (4) through $\mathrm{Im}\kappa$.

With regard to what was said above, the final expression for the potential (27) is

$$U_2(R,\psi)=-2\sqrt{2\pi}|A_s|^2(-1)^s \cos\psi(\sqrt{-2\omega}/R)^{1/2} \times \exp(-\sqrt{-2\omega}R)\cos(qR), \tag{29}$$

where s is either s_1 or $s_2 \equiv \pm s_1$, both giving the same value. Except for the factors $(\sqrt{-2\omega}/R)^{1/2}$ and $(-1)^s$, which are specific for the 2D case, the potential (29) is essentially the same as that obtained in the similar 1D models in Ref. [10].

In the 3D case, the consideration is also limited to the interaction of identical solitons (as it was said above, only the spinless solitons are considered in the 3D case). The above expression (22) yields the interaction potential in the 3D case too (recall that, in this case, the integration in the surface term is over the sphere). As well as in the 2D case, the first term in Eq. (22) vanishes in the approximation that neglects the direct influence of the dissipation, and the integration over the sphere is dominated by a contribution from a small vicinity of the point A (Fig. 1). Substituting into Eq. (22) the 3D asymptotic expressions (7) for V_s and V_t and the expansion (26), one arrives, instead of the integral (28), at

$$2\pi\int_0^\pi \exp(-\kappa\rho\cos\theta)\sin\theta\, d\theta = 2\pi(\kappa\rho)^{-1}(e^{+\kappa\rho}-e^{-\kappa\rho}) \approx 2\pi(\kappa\rho)^{-1}e^{+\kappa\rho}. \tag{30}$$

With regard to Eq. (30), the final expression for the effective interaction potential in the 3D case becomes [cf. Eq. (29)]

$$U_3(R,\psi)=-4\pi|A_s|^2\cos\psi R^{-1}\exp(-\sqrt{-2\omega}R)\cos(qR). \tag{31}$$

Note that the auxiliary radius ρ is completely canceled out in the final expressions (29) and (31).

The potentials (29) and (31) can be as well applied to the description of the interaction between the 2D and 3D spatiotemporal solitons (LB's), given by Eqs. (9) and (4), (7). The differences from the above results are that $q=0$ (recall the dissipation was completely neglected in the LB models), ω must be replaced by $-k$, and the separation R between the solitons is replaced by the *spatiotemporal separation* Ξ defined according to Eq. (9):

$$\Xi=\sqrt{R_\perp^2+T^2}, \tag{32}$$

$R_\perp$ and T being, respectively, the separation between the solitons in the transverse direction and the temporal delay between them.

B. The model with the quadratic nonlinearity

The interaction potential for the $\chi^{(2)}$ solitons has its own peculiarities. For the spatial (stationary) 2D $\chi^{(2)}$ solitons, the exponentially decaying potential with two components, generated by the FH and SH fields, was postulated in Ref. [2]. The interaction between the $\chi^{(2)}$ LB's is more complicated, because the nonstationary model (12) and (13) is, effectively, *spatiotemporally anisotropic*, as it was explained in detail in the previous section, see Eqs. (14), (15), and (18). A straightforward consideration demonstrates that, in both 2D and 3D cases, the SH-generated interaction potential dominates at $\gamma<\delta$, so that the potential is given by Eqs. (29) and (31), with ω replaced by $-\gamma$, $q=0$, and R replaced by $\tilde{\Xi} \equiv \sqrt{R_\perp^2+\delta^{-1}T^2}$, cf. Eqs. (15) and (32). On the contrary to this, at $\gamma>1$, the FH-generated interaction always dominates, which means that one should use the potentials (29) and (31), with $\omega=-1$ and R replaced by Ξ defined as per Eq. (32). In the intermediate case $\delta<\gamma<1$ [cf. Eq. (16); recall that the physically relevant case is $\delta<1$], the interaction potential turns out to be truly anisotropic in the plane $(R_\perp, T)$: the SH-generated interaction dominates inside the sector [cf. Eq. (17)]

$$(T/R_\perp)^2<\frac{1-\gamma}{\gamma-\delta}\delta, \tag{33}$$

and the FH-generated interaction dominates outside the sector (33). Accordingly, one should substitute R in the expressions (29) and (31) for the interaction potential by $\tilde{\Xi}$ inside the sector (33), and by Ξ outside of it.

IV. CONCLUDING REMARKS

The effective interaction potentials (29) and (31) can give rise to bound states (BS's) of two solitons. In the presence of the dissipation and gain, it makes sense to consider only BS's of quiescent solitons, as any motion is suppressed by a friction force. Because the form of the potentials is essentially the same as in 1D, the situation is not different from the 1D case, which was recently studied in detail [19]. There are two types of BS's, with the phase difference between the solitons $\psi=0$ or π, and with $\psi=\pi/2$. The BS's of the former type are saddles, while the BS's of the latter type have imaginary stability eigenvalues. The fact that the BS's with $\psi=0$ or π are saddles is related to a fundamental property of the interacting solitons: while an effective mass m_R of the two-soliton system corresponding to the radial degree of freedom R is positive, an effective mass m_ψ of the phase degree of freedom is *negative* [19].

Thus, these two types of the BS's are, respectively, unstable and stable, in the first approximation. In Ref. [19], it has also been demonstrated that the BS with $\psi=\pi/2$ is subject, in the next approximation, to an extremely weak instability, which transforms it into a very slowly unwinding spiral. However, it was also demonstrated that, even if this next-order instability can be observed, it does not destroy the BS, but, instead, makes it dynamical, with R and ψ very slowly oscillating in a *limited* range. Note that the same mechanism gives rise, in the 1D case, to (almost) stable chains of the bound solitons; in the 2D and 3D cases, a new possible pattern is a *lattice* of the bound solitons. There may also exist ''covalent soliton molecules,'' in the form of triangles and tetrahedrons in the 2D and 3D cases, respectively.

In the absence of the dissipation, BS of mutually orbiting solitons is possible in the 2D and 3D cases (in the latter case, it is assumed that the two solitons move in one plane). Orbiting of incoherently interacting 2D solitons was experimentally observed in a photorefractive medium [3]. Numerical simulations and analytical arguments presented in Ref. [2] demonstrate that the orbiting BS states of the 2D solitons in the $\chi^{(2)}$ model are unstable. In the present class of the models, the orbiting BS cannot be stable either. Indeed, for the orbiting state the interaction potential (29) or (31) must be supplemented by the centrifugal energy $E_{\mathrm{cf}} = (M^2/2m_R)R^{-2}$, where M is the angular moment of the soliton pair, and m_R is the above-mentioned effective mass. Thus, the net effective energy of the orbiting state is

$$\begin{aligned} E_{\mathrm{eff}} &= U_D(R,\psi) + E_{\mathrm{cf}} \\ &\equiv C_D \cos\psi R^{-(D-1)/2} \exp(-\sqrt{-2\omega}R) \\ &\quad + (M^2/2m_R)R^{-2}, \end{aligned} \tag{34}$$

where, according to Eqs. (29) and (31), the constant C_D depends on the dimension D and the soliton's spin s. It is easy to check that the effective energy (34) gives rise to a stationary state with $\sin\psi = 0$, $C_D \cos\psi < 0$, provided that M^2 is small enough. However, this stationary state always has $\partial^2 E_{\mathrm{eff}}/\partial R^2 < 0$, i.e., it is a *maximum* of the effective energy, consequently, the orbiting BS is unstable against variation of R. Moreover, one can check that the same state always has $\partial^2 E_{\mathrm{eff}}/\partial \psi^2 > 0$. With regard to the above mentioned $m_\psi < 0$, this BS is also unstable against variation of ψ.

In conclusion, a general method to find the effective potential of interaction between two-dimensional and three-dimensional solitons was elaborated, including the case of the two-dimensional vortex (spinning) solitons. The method is based on calculation of the overlapping term in the full Hamiltonian of the system. The main technical point that makes the calculation possible is that the bulk integral reduces to a surface one and, in the lowest-order approximation, the final expression does not contain the auxiliary radius of the overlapping region. The result applies to spatial solitons and ''light bullets'' (spatiotemporal solitons) in nonlinear optics (in the model with the quadratic nonlinearity, the interaction between the ''bullets'' may be spatiotemporally anisotropic). The interaction potential predicts that an orbiting bound state of two solitons exists, but is always unstable. In the presence of weak dissipation and gain, the effective potential can also be derived, giving rise to bound states of the solitons (both unstable and almost stable) similar to those recently studied in the one-dimensional model.

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