

Thermodynamic and relativistic aspects of thermal inertia in fluids

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We present a generalized relativistic Lagrangian which yields the effect of heat flux \mathbf{q} and nonequilibrium stress τ in the energy-momentum tensor of an extended reversible fluid exhibiting thermal inertia. The actual momentum of heat (thermal momentum related to the entropy flow) follows to be many orders of magnitude larger than \mathbf{q}/c^2 (c is the light speed) but it is consistent with Grad's kinetic theory [in *Principles of the Theory of Gases*, edited by S. Flugge, Handbüch der Physik Vol. 12 (Springer, Berlin, 1958)] and with experiments in heat conduction. On the other hand, the net momentum of heat remains \mathbf{q}/c^2 , in agreement with the standard relativistic result, this net momentum being the result of incomplete compensation of the actual thermal momentum and the momentum associated with self-diffusion of particles. The classical densities of mass and entropy, ρ and ρ_s , cease to be natural variables of energy density E in the sense of Callen [*Thermodynamics and an Introduction to Thermostatistics* (Wiley, New York, 1988)] whenever inertial effects prevail. This fact necessitates the use of what may be called the thermal potential T^- , a new quantity replacing the classical temperature T . Changes in thermodynamic formalism are related to the replacement of T by T^- . The admission of a freely varied four-flux of entropy in an extended Hamilton principle implies all nonequilibrium corrections (\mathbf{q} and τ) to the energy-momentum tensor, making it possible to investigate the effect of nonequilibrium phenomena on the properties of associated gravitational fields. [S1063-651X(98)01912-6]

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I. INTRODUCTION: AIMS AND SCOPE

Thermal inertia is a gradual change of the heat flux under a rapid change of the temperature gradient. It manifests itself in both nonrelativistic and relativistic cases. The purpose of this work is to develop a relativistic theory of thermal inertia based on an appropriate Lagrangian L which needs to be determined. To begin with we consider the simplest possible extension of the standard relativistic Lagrangian of the adiabatic fluid, which is the negative rest energy density E^0 . This L is made applicable to nonadiabatic fluids by distinguishing between the absolute velocity of the transferred entropy $\mathbf{u}_s = \mathbf{J}_s/\rho_s$ (where ρ_s is the entropy density and \mathbf{J}_s the total entropy flux) and the hydrodynamic velocity $\mathbf{u} = \mathbf{J}/\rho$ (where \mathbf{J} is the mass flux and ρ is the total fluid density). Allowing for an unconstrained entropy flow, with the diffusive component \mathbf{j}_s , in the corresponding Hamilton principle our L yields the energy-momentum tensor which contains the heat flux $\mathbf{q} = T\mathbf{j}_s$, associated with the momentum density \mathbf{q}/c^2 , and the nonequilibrium stress $\tau = \mathbf{q}\mathbf{q}/(T\rho_s c^2)$. The obtained theory is what we call a "quasiclassical extension" of the theory of relativistic adiabatic fluid. The resulting and residual nature of the momentum density \mathbf{q}/c^2 (disappearing in the frame of the relativistic energy and at the limit $c \rightarrow \infty$) is a distinctive feature of the transfer process considered. All energy results are valid, however, such a theory can predict thermal inertia solely as a relativistic effect (Secs. II–IV).

In Secs. V–VII, reference is made to the problem of thermal inertia, implied by kinetic theories of Grad's type which predict momentum densities of heat to be of many orders of

magnitude larger than \mathbf{q}/c^2 ; they are of the order \mathbf{q}/c_0^2 , where c_0 is close to the thermal speed $k_B T/m$ for the particle of mass m . In view of unpredictability of these large momenta and related nonequilibrium corrections to the internal energy by the (described above) quasiclassical extension of the (relativistic) adiabatic fluid theory, it is shown that a hypothesis of a "thermal mass," associating a part of the observed rest mass with the entropy rather than with particles, furnishes a simple "nonclassical extension" capable of predicting these large momenta and the related energy terms. The principal property of this extension is that it preserves finite thermal inertia in nonrelativistic limit. We pursue here the simplest model with the constant amount of thermal mass per unit entropy (the coefficient θ adjusted so that it fits Grad's kinetic data in their broad plateau regime, where $\theta \approx m/3k_B$) that allows us to handle the kinetic effects with reasonable accuracy. The inclusion of θ does not change the chemical potential μ , but requires a redefinition of the usual statistical temperature according to the simple formula $T^- = T + \theta\mu$. Yet, this inclusion adds to the modified temperature the relativistic component θc^2 , analogous to the c^2 component of the relativistic chemical potential μ^* . The relativistic intensities $T^* = T^- + \theta c^2$ and $\mu^* = \mu + c^2$ obey the Planck-Einstein equations for the relativistic transformation of T [1].

The nonclassical extension shows that the actual momentum density (the thermal momentum density), associated with the entropy flux \mathbf{j}_s , equals, in fact, $T^- \mathbf{j}_s (1/c^2 + 1/c_0^2)$, where T^- is the redefined statistical temperature and $c_0 = (T^-/\theta)^{1/2}$. For the unchanged definition of heat, $\mathbf{q} = T\mathbf{j}_s$, the net momentum of heat remains \mathbf{q}/c^2 , in agreement with the well-established result, the net momentum being the result of compensation of the thermal momentum and the momentum associated with the self-diffusion of bare particles.

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The partial momenta, thermal and material, are then interpreted as internal momenta of extended thermodynamics. The analysis explains terms like \mathbf{q}/c_0^2 in some formulas for internal momenta and their squares in the internal energy function. On the other hand, the analysis shows that these thermal momenta are still compatible with the standard rest energy, momentum density, enthalpy, and pressure; yet improvement is achieved for the components of the energy-momentum tensor. Using the extended model in the context of Ray's [2] variational principle, a nonequilibrium energy-momentum tensor is obtained in general relativity. This tensor takes into account heat and nonequilibrium stress and allows one to investigate their influence on properties of resulting gravitational fields. Our results are consistent with the recent findings obtained in the so-called generic formalism for nonequilibrium dynamics [3].

II. STANDARD DESCRIPTION OF RELATIVISTIC FLUIDS

The basic ingredients of the standard relativistic theory of a one-component fluid are the symmetric energy-momentum tensor G^{ik} , the particle four-flux J^i , and the entropy flux S^i ($i, k = 1, \dots, 4$). The central problem of thermodynamics is how to interrelate these primary variables. The quantities G^{ik} and J^i are conserved, i.e.,

$$G_{,i}^{ik} = 0, \quad J_{,i}^i = 0. \quad (1)$$

The entropy four-flux S^i obeys the second law constraint: the positive entropy production S^i , $i \geq 0$. From Eq. (1) one obtains

$$U_k G_{,i}^{ik} = 0 \quad (2)$$

as the energy balance, and

$$h_{kj} G_{,i}^{ik} = 0 \quad (3)$$

as the momentum balance [4]. The signature convention is $(+++ -)$. U^i is the particle frame four-velocity and $h^{ik} = g^{ik} + c^{-2} U^i U^k$ is the projection tensor, the operator which projects onto the 3-space orthogonal to an arbitrary timelike vector U^i . If we are interested in dissipative effects in a heat conducting viscous fluid, we can define the energy-momentum tensor

$$G^{ik} = c^{-2} (E^0 U^i U^k + q^j U^k + q^k U^i) + \tau^{ik} + P h^{ik}, \quad (4)$$

where P is the pressure, q^i is the heat flux density, q_i/c^2 is the related momentum density, and τ^{ik} is the total viscous stress. $E^0 = n^0 m^0 c^2 + n^0 e^{0'}$ is the total density of the rest energy, n^0 is the number density, $e^{0'}$ is the internal energy per particle ($e' = e/n$). The zero superscript pertains to the rest frame. We use also the rest densities of mass $\rho^0 = n^0 m^0$ and internal energy $\rho^0 e^0 = n^0 e^{0'}$. Heat is understood here as the energy flow relative to particle stream. The quantities q^i and τ^{ik} satisfy

$$q^i U_i = U_i \tau^{ik} = 0. \quad (5)$$

Equations (2)–(5) refer to Eckart's [5] relativistic thermodynamic theory where use is made of the particle frame. To an

observer at rest in this frame, the flux of the particles appears to be zero and a heat flux is the flow of the energy relative to the particle stream. Another scheme, which we do not use here, is based on the energy frame (Landau and Lifschitz's frame [6,7]). In that frame the flux of the energy appears to vanish to an observer and the heat flux appears as a drift of the particles.

In the framework of Eckart's theory of heat flow [5], thermal inertia is possible only as a relativistic phenomenon. However, Israel's nonstationary theory [8–10], which takes into account the second order terms in the entropy four-flux, is sufficient to solve the paradox of infinite propagation speeds of thermal signals. Israel's theory involves the energy-momentum tensor G^{ik} as the quantity generalizing mass in both a covariant form of the Gibbs equation and a generalized Gibbs-Duhem formula containing the four-velocity [9]. It follows that the corrections to the classical entropy and entropy flux must be of, at least, second order in dissipative fluxes to make the theory compatible with standard thermodynamics in the quasistatic limit. However, that theory, while general and yielding a more correct qualitative description than the previous theories, offers only a rather formal view of thermal inertia.

In Sec. V of this work another way is chosen to interpret and then investigate the effects of thermal inertia. It is based on our earlier recognition of the importance of the free entropy flow (independent of the flow of the particles) in Hamilton's principle of nonrelativistic systems [11,12]. Using this principle we test a hypothesis that a part of the observed *rest* mass of the fluid is of purely thermal origin, in the sense that it is associated with the entropy rather than with the particles. While this hypothesis does not change the background rest energy $\mathcal{E} = M c^2$, we are able to show that the large kinetic terms of Grad's theory and extended irreversible thermodynamics (EIT, [13]) can be quite easily obtained. The resulting theory, which is certainly in the spirit of extended thermodynamics, is easily applied to Ray's [2] construction of the energy-momentum tensor in general relativity G^{ik} , showing not only that the general formula (4) can be derived in a direct way but also that definitions of heat \mathbf{q} and stress τ can be furnished in G^{ik} . These results are of value, since the contemporary relativistic thermodynamics sets the structure of Eq. (4) only, on the basis of general covariance principles, but without any information regarding the nature of \mathbf{q} and τ terms in this equation.

III. TRANSFORMATION OF THERMODYNAMIC INTENSITIES IN A CLASSICAL CASE

The problem of relativistic transformation of thermodynamic intensities, such as T and μ , will play a role in our analysis, hence a brief introduction is appropriate. Exact relativistic transformations of thermodynamic parameters were determined by Planck and Einstein [1] soon after the establishment of relativity theory. In their work, the temperature T^0 and the amount of heat Q^0 in the proper frame moving with the velocity \mathbf{v} with respect to a laboratory are connected with the quantities T and Q in the laboratory frame by the noncovariant transformations

$$T = T^0 \sqrt{(1 - \mathbf{v}^2/c^2)}, \quad Q = Q^0 \sqrt{(1 - \mathbf{v}^2/c^2)}, \quad (6)$$

i.e., moving bodies appear cooler. However, in 1963 Ott's [14] work appeared implying that the true transformations have the form

$$T = T^0 / \sqrt{(1 - \mathbf{v}^2/c^2)}, \quad Q = Q^0 / \sqrt{(1 - \mathbf{v}^2/c^2)}. \quad (7)$$

These quantities are time components of the related four-vectors. (For example, the temperature four-vector $T^i = T^0 U^i/c$.) This controversy divided the researchers into three groups, each applying different transformation laws [Eqs. (6), (7) or $T = T^0$]. All researchers agree that the entropy is a relativistic invariant, so that the equality $S = S^0$ should hold in any case. The assumption $T = T^0$ follows from the position that the temperature has a most natural meaning when assigned to its value in the proper frame [15,16]. We have omitted here a comprehensive discussion of the abundant literature referring the reader to several reviews [16–19]. A summarizing remark is appropriate, however. Some authors conclude that equilibrium statistical mechanics cannot provide an unambiguous answer to the transformation formulas of thermodynamic quantities and, therefore, all three kinds of transformations are acceptable [20]. In view of our results here and some earlier ones [21,22] we accept the classical formulas (6) as the most appropriate for reversible processes in the energy representation, which is of interest to us. Staruszkiewicz's [23] work solves exactly the problem of the transformation of T in the energy representation, leading towards acceptance of Eq. (6).

In the framework of the variational formalism developed here and in our previous work [11,12] the solution to the problem is straightforward. Both kinds of temperatures are eligible to appear in the dynamical formulas, but only the Planck-Einstein “cold” intensities T^- and μ^- are the ingredients of the *thermodynamic transformation formalism* that involves, in the nonequilibrium case, the velocities or fluxes as additional variables. The temperature T must be the negative partial derivative of the kinetic potential density L with respect to the entropy density ρ_s at the constant transfer velocities, \mathbf{u} and \mathbf{u}_s , or the partial derivative of the energy density E with respect to ρ_s , at the constant momenta. We shall show here that this requirement leads to the Planck-Einstein temperatures.

Let us compute the cold intensities corresponding to the classical equilibrium and nonequilibrium theories. The relativistic kinetic potential is the negative of the rest energy density. The latter is usually written as the sum $E^0(\rho_s^0, \rho^0)$ of the thermal (internal) energy density $\rho^0 e^0$ and the relativistic rest energy density $\rho^0 c^2 = n^0 m^0 c^2$. Working with the total quantity $E^0(\rho_s^0, \rho^0)$ leads to a transparent picture. Expressing rest densities in terms of the laboratory frame densities yields

$$L = -E^0(\rho_s \sqrt{1 - \mathbf{v}^2/c^2}, \rho \sqrt{1 - \mathbf{v}^2/c^2}). \quad (8)$$

The negative partial derivative of L with respect to ρ_s at constant \mathbf{v} is

$$T = -(\partial L / \partial \rho_s)_v = (\partial E^0 / \partial \rho_s^0)(\partial \rho_s^0 / \partial \rho_s) = T^0 \sqrt{1 - \mathbf{v}^2/c^2}. \quad (9)$$

Similarly, for the chemical potential,

$$\begin{aligned} \mu^* &= -(\partial L / \partial \rho)_v = (\partial E^0 / \partial \rho^0)(\partial \rho^0 / \partial \rho) \\ &= \mu_*^0 \sqrt{1 - \mathbf{v}^2/c^2} = (\mu^0 + c^2) \sqrt{1 - \mathbf{v}^2/c^2}. \end{aligned} \quad (10)$$

The asterisk is used to indicate relativistic quantities. On the other hand, when the kinetic potential L is expressed in terms of fluxes (or—in the relativistic case—momentum densities) and pertinent differentiations are made, Ott's result is obtained. While the theory cannot answer the question of which T 's are closer to the indications of a thermometer, it identifies the Planck-Einstein T 's as those measures of the frequency of thermal agitation which furnish the canonical formalism of thermomechanics based on the Legendre transformation and thermohydrodynamic potentials, and also recognizes Ott's T 's as different but specific differential measures of the energy.

Equations (8)–(10) can be easily generalized to nonequilibrium situations. When the process is a (reversible) nonequilibrium process, different transfer velocities \mathbf{u}_s and \mathbf{u} must be introduced into the kinetic potential L , Eq. (8). We may note that the effect of the entropy velocity vanishes in the nonrelativistic approximation of L ,

$$\begin{aligned} L &= -E^0(\rho_s \sqrt{1 - \mathbf{u}_s^2/c^2}, \rho \sqrt{1 - \mathbf{u}^2/c^2}) \\ &\cong \frac{1}{2} \rho \mathbf{u}^2 - \rho c^2 - \rho e(\rho_s, \rho). \end{aligned} \quad (11)$$

The nonequilibrium intensities follow immediately as

$$\begin{aligned} T &= -(\partial L / \partial \rho_s)_{\rho, \mathbf{u}_s, \mathbf{u}} = (\partial E^0 / \partial \rho_s^0)(\partial \rho_s^0 / \partial \rho_s) \\ &= T^0 \sqrt{1 - \mathbf{u}_s^2/c^2}, \end{aligned} \quad (12)$$

$$\begin{aligned} \mu^* &= -(\partial L / \partial \rho)_{\rho_s, \mathbf{u}_s, \mathbf{u}} = (\partial E^0 / \partial \rho^0)(\partial \rho^0 / \partial \rho) \\ &= \mu_*^0 \sqrt{1 - \mathbf{u}^2/c^2}. \end{aligned} \quad (13)$$

In this (“quasiclassical”) case the entropy of the continuum is exclusively associated with the internal part of the energy E^0 . In other words, in the formula for the rest energy $E^0(\rho_s^0, \rho^0) = \rho^0 c^2 + \rho^0 e^0$, the only term containing the entropy is the $\rho^0 e^0$ term. This causes identical temperatures T and T^* , meaning that the temperature, as opposed to the chemical potential, has no relativistic component. Therefore any inertia acquired by the entropy can be linked only with a small internal energy term, and is practically negligible. This is seen explicitly below when the corresponding thermal momentum is computed.

IV. PARTIAL MOMENTA OF CLASSICAL NONEQUILIBRIUM THERMODYNAMICS

The method developed earlier [12] allows us to compute partial momenta for an arbitrary kinetic potential L , so relativistic L 's can be applied as well. The final expressions of our equations will contain low-velocity approximations. The thermal momentum density $\partial L / \partial \mathbf{u}_s$ is

$$\begin{aligned} \mathbf{p}_s &= (\partial L / \partial \mathbf{u}_s)_{\rho, \rho_s, \mathbf{u}} = -(\partial E^0 / \partial \rho_s^0)(\partial \rho_s^0 / \partial \mathbf{u}_s) \\ &= c^{-2} T^0 \rho_s \mathbf{u}_s / \sqrt{1 - \mathbf{u}_s^2/c^2} \cong T \mathbf{J}_s / c^2. \end{aligned} \quad (14)$$

This quantity vanishes in the nonrelativistic limit and, in this sense, one can say that the entropy of classical thermodynamics is inertialess or weightless. We stress that this is not the case of any extended thermodynamics, where the quantity \mathbf{p}_s is much larger than that predicted by Eq. (14) and does not vanish for an infinite c [24]. In Eq. (14) the temperature T^0 equals $\partial E^0/\partial \rho_s^0 = \partial \rho^0 e^0/\partial \rho_s^0$, the classical temperature of the resting medium, and the relativistic mass term present in E^0 has no effect on T^0 . On the other hand, it is the relativistic term that strongly influences the chemical potential. Therefore the relativistic chemical potential must be distinguished from the nonrelativistic one. The rest frame chemical potential of the relativistic theory is two component: $\mu^{0*} = \partial E^0/\partial \rho^0 = c^2 + \mu^0$. Thanks to the very large c^2 the momentum density of the matter predominates,

$$\begin{aligned} \mathbf{p} &= (\partial L/\partial \mathbf{u})_{\rho, \rho_s, u_s} = -(\partial E^0/\partial \rho_s^0)(\partial \rho_s^0/\partial \mathbf{u}) \\ &= c^{-2}(c^2 + \mu^0)\rho \mathbf{u} / \sqrt{1 - \mathbf{u}^2/c^2} \cong \mathbf{J} + \mu \mathbf{J}/c^2. \end{aligned} \quad (15)$$

The sum of \mathbf{p} and \mathbf{p}_s is the total momentum density Γ [12],

$$\begin{aligned} \Gamma &= \mathbf{p}_s + \mathbf{p} = c^{-2} T^0 \rho_s^0 \left(\frac{\mathbf{u}_s}{1 - \mathbf{u}_s^2/c^2} - \frac{\mathbf{u}}{1 - \mathbf{u}^2/c^2} \right) \\ &\quad + (1 + c^{-2} h^0) \frac{\rho^0 \mathbf{u}}{1 - \mathbf{u}^2/c^2} \\ &\cong c^{-2} T \mathbf{j}_s + (1 + c^{-2} h) \mathbf{J}. \end{aligned} \quad (16)$$

When the heat flux $\mathbf{q} = T \mathbf{j}_s$ is introduced,

$$\Gamma \cong c^{-2} \mathbf{q} + (1 + c^{-2} h) \mathbf{J}. \quad (17)$$

Summing up, the simplest ‘‘quasiclassical’’ extension of the standard theory of adiabatic fluid, which allows thermal degrees of freedom, implies that the momentum density of heat is \mathbf{q}/c^2 . In the nonrelativistic approximation the total momentum density is the mass flow \mathbf{J} . Our distinguishing the paths of the matter and the entropy leads to the prediction of the heat in the form $\mathbf{q} = T \rho_s (\mathbf{u}_s - \mathbf{u}) = T \mathbf{j}_s$, and the resulting momentum of this heat, \mathbf{q}/c^2 , follows consistently. The effect of accompanying viscosity is $\tau = \mathbf{q}\mathbf{q}/(T \rho_s c^2)$, a new result.

It can also be shown that the relativistic energy $E = \mathbf{p}_s \cdot \mathbf{u}_s + \mathbf{p} \cdot \mathbf{u} - L$ simplifies into the well-known equilibrium result when $\mathbf{u} = \mathbf{u}_s$ [2,7]. The energy E in the nonrelativistic approximation ($c \rightarrow \infty$) is the sum of the kinetic, internal, and rest energies in agreement with the standard theory of equilibrium fluid [2]. The low-velocity L takes a familiar form of the kinetic potential of perfect fluid,

$$\begin{aligned} L &= -\rho^0 c^2 + \rho^0 e(\rho^0, \rho_s^0) \\ &= -(\rho \sqrt{1 - \mathbf{u}^2/c^2}) [c^2 + e(\rho_s \sqrt{1 - \mathbf{u}_s^2/c^2}, \rho \sqrt{1 - \mathbf{u}^2/c^2})] \\ &\cong \frac{1}{2} \rho (1 + h/c^2) \mathbf{u}^2 - \rho c^2 - \rho e(\rho_s, \rho) \\ &\cong \frac{1}{2} \rho \mathbf{u}^2 - \rho c^2 - \rho e(\rho_s, \rho). \end{aligned} \quad (18)$$

While all energy results are valid, and the improvement in comparison with the adiabatic fluid theory is significant, it would be hard to say that the last word in the theory of heat

flow has been said. In this regard the problem lies with the heat momentum itself: no thermal inertia can be predicted in a nonrelativistic limit. In other words, the improvement provided by ‘‘quasiclassical’’ Lagrangians is insufficient to correctly describe thermal inertia. This situation calls for further research; we shall see that equations can be obtained, which contains the c -free kinetic terms. They will appear in an extended thermodynamics description when a part of the background energy $\rho^0 c^2$ is the energy associated with the entropy.

V. A HYPOTHESIS OF THERMAL MASS AND EXTENDED THERMODYNAMICS

The kinetic potential L , whose Legendre transform is E , depends on the assumption on how the observed inertia is divided between the thermal and substantial degrees of freedom, represented by the entropy and mass. Any classical thermodynamics is consistent with the assumption that the contribution of the entropy to the rest mass density ρ^0 is zero; which means that the entropy or heat flows are inertialess in the nonrelativistic limit. This is an assumption which leads to the known paradoxes in the Fourier heat transfer theory and quantitative disagreement with implications of the kinetic theory. *The hypothesis of a thermal mass* is based on the assumption that part of the observed rest mass of a macroscopic body is of purely thermal origin, meaning that it should be attributed to entropy rather than to particles [24]. Assumption of a finite θ , which is the amount of the ‘‘thermal mass’’ per unit of the entropy, leads to an extended thermodynamics in which a part of the observed inertia is linked with the entropy flux. In view of differences in values of θ implied by various models [24] we assume here a constant θ in the plateau regime of Grad’s model; this θ is well approximated by the ratio $m/3k_B$. The finiteness of θ eliminates the paradox of infinite propagation speed and leads to kinetic terms in the nonequilibrium energy formula which are of the same order of magnitude as those found from the nonequilibrium statistical mechanics (Grad’s 1958 moment analysis [25,26]). In fact, Grad’s theory as well as experiments in heat conduction show that a reasonable inertia should be attributed to the entropy four-flux $S^i = (\rho_s, \mathbf{J}_s)$ and the heat flux $\mathbf{q} = T \mathbf{j}_s$. The associated thermal momentum \mathbf{p}_s is then many orders of magnitude larger than \mathbf{q}/c^2 of Eq. (14), yet the *resulting* momentum in the resting fluid frame ($\mathbf{J} = 0$) is still \mathbf{q}/c^2 .

Consequently, a Lagrangian L is stated below which is based on the split of the background relativistic energy $\rho^0 c^2$ into the ‘‘bare matter’’ part $\rho_m^0 c^2$ and the ‘‘thermal part’’ $\rho_s^0 \theta c^2$, such that their sum remains equal to $\rho^0 c^2$. Such L preserves, of course, the same observed rest density $\rho^0 = \rho_m^0 + \theta \rho_s^0$ and the total rest energy $\rho^0 c^2$ but admits that the inertial responsibilities are shared by the mass and the entropy in a balanced way. It is interesting [27] that any constant choice of θ does not influence the value of the pressure P derived from L . Since the pressure is the extremum value of a field Lagrangian [11,24], the result proves that there is no preference for $\theta = 0$ from the standpoint of action-based criteria. Nonetheless, the magnitude of θ affects the temperature definition and the limiting classical dynamics of the fluid system.

The difference between the classical and extended description may be seen as a consequence of the choice of variables more appropriate than the classical. The traditional kinetic potential $L(\rho, \rho_s, \mathbf{u}, \mathbf{u}_s)$ is replaced by $L(\rho_m, \rho_s, \mathbf{u}_m, \mathbf{u}_s)$, where the latter is obtained from the same rest energy E^0 but expressed in terms of the densities ρ_m and ρ_s rather than the traditional densities. In the rest frame the kinetic potentials in the old and new variables are equal, i.e., $L(\rho^0, \rho_s^0) = L(\rho_m^0, \rho_s^0)$. However, the L changes its form when passing to the laboratory frame; whereas in the traditional variables (ρ, ρ_s) the L takes the nearly traditional form (18), in the variables (ρ_m, ρ_s) , it takes a different form

$$\begin{aligned}
L &= -E^0(\rho_s \sqrt{1 - \mathbf{u}_s^2/c^2}, \rho_m \sqrt{1 - \mathbf{u}_m^2/c^2}) \\
&= -\rho^0 c^2 + \rho^0 e(\rho^0, \rho_s^0) \\
&= -(\theta \rho_s^0 + \rho_m^0)[c^2 + e(\rho_s^0, \rho_m^0)] \\
&= -(\theta \rho_s \sqrt{1 - \mathbf{u}_s^2/c^2} + \rho_m \sqrt{1 - \mathbf{u}_m^2/c^2}) \\
&\quad \times [c^2 + e(\rho_s \sqrt{1 - \mathbf{u}_s^2/c^2}, \rho_m \sqrt{1 - \mathbf{u}_m^2/c^2})] \\
&\cong \frac{1}{2} \theta \rho_s \mathbf{u}_s^2 + \frac{1}{2} \rho_m \mathbf{u}_m^2 - (\theta \rho_s + \rho_m) c^2 - (\theta \rho_s + \rho_m) e(\rho_s, \rho_m).
\end{aligned} \tag{19}$$

Let us compare Eqs. (18) and (19), in which the last lines describe the limiting nonrelativistic approximations. The transformation of the internal energy is neglected in Eq. (18) as it yields extremely small corrections to L and its derivatives. The inertialess property of the ‘‘nonrelativistic’’ entropy was seen in the quasiclassical Eq. (18). On the other hand, the choice of a finite θ in Eq. (19) changes the form of the kinetic energy, or switches the responsibility of carrying a part of the inertial effect from mass to entropy. Clearly, the limiting nonrelativistic kinetic potentials of Eqs. (18) and (19) are *not* equivalent. The nonrelativistic L in the new variables (ρ_s, ρ_m) has the kinetic energy larger than L evaluated for the corresponding conventional variables (ρ_s, ρ) .

Let us consider the differential of the relativistic internal energy density in terms of the ‘‘canonical intensities’’ (ρ_s, ρ_m) . In the rest frame

$$\begin{aligned}
dE^0 &= \frac{\partial E^0}{\partial \rho_s^0} d\rho_s^0 + \frac{\partial E^0}{\partial \rho_m^0} d\rho_m^0 \\
&\equiv T_*^{-0} d\rho_s^0 + \mu_*^{-0} d\rho_m^0 \\
&= T_*^{-0} d\rho_s^0 + \mu_*^{-0} d(\rho^0 - \theta \rho_s^0) \\
&= (T_*^{-0} - \theta \mu_*^{-0}) d\rho_s^0 + \mu_*^{-0} d\rho^0.
\end{aligned} \tag{20}$$

In the first line of this equation ‘‘canonical intensities’’ are defined as quantities in which the minus superscript stresses their relation to the ‘‘cold’’ Planck-Einstein temperatures. The asterisks stress the relativistic origin of both intensities. The third line describes transformation to the classical variables, the mass density ρ and the entropy density ρ_s . Comparison of Eq. (20) with the classical Gibbs equation (operating with ρ_s^0 and ρ^0)

$$dE^0 = T^0 d\rho_s^0 + \mu^0 d\rho^0 = T^0 d\rho_s^0 + (\mu^0 + c^2) d\rho^0 \tag{21}$$

yields the desired connection between the relativistic intensities

$$\mu_*^{-0} = \mu^0 \quad (= \mu^0 + c^2), \tag{22}$$

$$T_*^{-0} = T^0 + \theta \mu_*^{-0} \quad (= T^0 + \theta \mu^0 + \theta c^2). \tag{23}$$

The canonical intensities represent then the usual relativistic chemical potential and its thermal analog, which we call the *thermal potential*. (Throughout our previous work [27,28] the *nonrelativistic* intensities T^{-0} and μ^{-0} were used with the zero index neglected.) In terms of their relativistic counterparts the nonrelativistic intensities are defined as

$$\mu^{-0} \equiv \mu_*^{-0} - c^2 \quad (= \mu^0), \tag{24}$$

$$T^{-0} \equiv T_*^{-0} - \theta c^2 \quad (= T^0 + \theta \mu^0). \tag{25}$$

Comparison of Eqs. (22) and (24) shows that $\mu^{-0} = \mu^0$, i.e., the rest frame nonrelativistic canonical chemical potential coincides with the classical one, whereas $T^{-0} \equiv T^0 + \theta \mu^0$, i.e., the nonrelativistic thermal potential differs from the classical T due to the contribution of the nonrelativistic chemical potential. We stress that Eqs. (22)–(25) set the rest-frame inputs T^{-0} and μ^{-0} to their transformations into T^- and μ^- of moving systems. In moving frames μ and μ^- are not equal. [See Eqs. (30), (33), and (34) below for transformations of T^- and μ^- in moving systems.] Note that the classical densities ρ and ρ_s cease to be the natural variables of the energy density E in the sense of Callen [29] whenever the inertial effects are important.

However, not all thermodynamic properties change. The replacement of the rest temperature T^0 by the rest thermal potential T^{-0} leaves the densities of the internal energy and the total mass unchanged; hence the specific internal energy is an invariant. Moreover, the replacement of T^0 by T^{-0} does not change the pressure P ,

$$\begin{aligned}
P^{-0} &= T^{-0} \rho_s^{-0} + \mu^{-0} \rho_m^0 - \rho_e^{-0} \\
&= (T^0 + \theta \mu^0) \rho_s^0 + \mu^0 (\rho^0 - \theta \rho_s^0) - \rho_e^0 \\
&= T \rho_s^0 + \mu^0 \rho^0 - \rho_e^0 = P^0.
\end{aligned} \tag{26}$$

Of course, when $\theta=0$, T^{-0} , and T^0 are identical. As the simple calculation shows,

$$\begin{aligned}
h^{-0} &= (T^{-0} \rho_s^0 + \mu^{-0} \rho_m^0) / \rho^0 = [(T^0 + \theta \mu^0) \rho_s^0 + \mu^0 \rho_m^0] / \rho^0 \\
&= (T^0 \rho_s^0 + \mu^0 \rho^0) / \rho^0 = h^0,
\end{aligned} \tag{27}$$

the considered transformation does not change the specific enthalpy h^0 .

The pressure P is a scalar in all the theories. The density of the internal energy in the laboratory frame, $\rho_e = T^- \rho_s + \mu \rho_m - P$, is an invariant accurately up to the c^{-2} terms. If $k_B \theta c^2$ equals $m^0 c^2$, then the coefficient θ equals m^0/k_B , and the quantity $T + \theta c^2$ is the sum of the usual statistical temperature T and the (quantum, relativistic) de Broglie temperature of a particle, $T^r = m^0 c^2/k_B$. In our theory the partial derivative of the rest energy density E^0 with respect to the rest entropy density ρ_s^0 equals $T^{0*} = T^{0r} + T^0$. In the de Broglie theory [30] the statistical effects are ignored and the

derivative in question equals T^{0r} ; in the classical thermodynamics this derivative is, of course, T^0 . The absence of the usual statistical component T^0 in the de Broglie theory, identifying T^{0r} with an intrinsic temperature of a micro-object, made difficulty which has prevented a wider dissemination of his idea. Our relativistic approach incorporates the de Broglie microscopic thermodynamics and preserves the standard statistical meaning of the usual temperature. The Planck-Einstein formula for the relativistic temperature transformation, Eq. (6), pertains in our theory to the sum $T^{0*} = T^{0r} + T^0$, in the de Broglie theory to T^{0r} only, and in the classical relativistic theory to T^0 .

Our thermodynamics can now be applied to systems in motion. For this purpose we use the Lagrangian (19) with a finite inertial coefficient θ , the measure of the thermal mass per unit of the entropy, and the canonical variables, the densities of entropy and bare mass, ρ_s and ρ_m . The nonequilibrium intensities of the moving system follow immediately as

$$\begin{aligned} T_*^- &= -(\partial L / \partial \rho_s)_{\rho_s, u_s, u_m} = (\partial E^0 / \partial \rho_s^0)(\partial \rho_s^0 / \partial \rho_s) \\ &= T_*^{-0} \sqrt{1 - \mathbf{u}_s^2 / c^2} = [\theta c^2 + (\partial \rho^0 e^0 / \partial \rho_s^0)] \sqrt{1 - \mathbf{u}_s^2 / c^2} \\ &= [\theta c^2 + T^{-0}] \sqrt{1 - \mathbf{u}_s^2 / c^2}, \end{aligned} \quad (28)$$

and

$$\begin{aligned} \mu_*^- &= -(\partial L / \partial \rho_m)_{\rho_s, u_s, u_m} = (\partial E^0 / \partial \rho_m^0)(\partial \rho_m^0 / \partial \rho_m) \\ &= \mu_*^{-0} \sqrt{1 - \mathbf{u}_m^2 / c^2} = [c^2 + (\partial \rho^0 e^0 / \partial \rho_m^0)] \sqrt{1 - \mathbf{u}_m^2 / c^2} \\ &= [c^2 + \mu^{-0}] \sqrt{1 - \mathbf{u}_m^2 / c^2}, \end{aligned} \quad (29)$$

where the asterisk refers to the relativistic quantities containing the contribution of the rest mass. We stress that it is the *relativistic* canonical intensities, not their truncated nonrelativistic counterparts, that obey the Planck-Einstein formula for the relativistic temperature transformation. Equations (28) and (29) constitute the Einstein-Planck formulas for relativistic canonical intensities when the thermal inertia is described by the inertial coefficient θ . These intensities do not coincide with the classical ones because the L is differentiated with respect to the special densities, ρ_s and ρ_m . At disequilibrium our intensities incorporate two different velocities of the entropy and matter, contained in the transformations

$$T_*^- = T_*^{-0} (1 - \mathbf{u}_s^2 / c^2)^{1/2}, \quad \mu_*^- = \mu_*^{-0} (1 - \mathbf{u}_m^2 / c^2)^{1/2}. \quad (30)$$

Equation (30) is a shorthand expression of Eqs. (28) and (29). The low-velocity transformation rule for the canonical quantities is the c -independent transformation

$$T^- = T^{-0} - \theta \mathbf{u}_s^2 / 2 = T^0 + \theta \mu^0 - \theta \mathbf{u}_s^2 / 2, \quad \mu^- = \mu^{-0} - \mathbf{u}_m^2 / 2, \quad (31)$$

which applies for T^{-0} and μ^{-0} obeying Eqs. (24) and (25). When $\theta=0$, the velocity \mathbf{u}_m approaches the hydrodynamic velocity \mathbf{u} , $T=T^0$, and only chemical potential transforms; $\mu = \mu^0 - \mathbf{u}^2 / 2$. This result is well known in the theory of hydrodynamic fluctuations [31].

The pressure P preserves its scalar property even in a nonequilibrium situation

$$\begin{aligned} P^- &= T^- \rho_s^- + \mu^- \rho_m^- + \mathbf{p}_s^- \cdot \mathbf{u}_s^- + \mathbf{p}_m^- \cdot \mathbf{u}_m^- - E^- \\ &= (T + \theta \mu - \theta \mathbf{u}_s^2 / 2) \rho_s + (\mu - \mathbf{u}_m^2 / 2) \rho_m + \theta \rho_s \mathbf{u}_s^2 + \rho_m \mathbf{u}_m^2 \\ &\quad - (\rho_e + \theta \rho_s \mathbf{u}_s^2 / 2 + \rho_m \mathbf{u}_m^2 / 2) \\ &= T \rho_s + \theta \mu \rho_s + \mu \rho_m - \rho_e \\ &= T \rho_s + \mu \rho - \rho_e = P, \end{aligned} \quad (32)$$

where Eq. (90) of our previous work [27] was used to link E with P .

Intensities T^- and μ^- with tilde over their symbols include an effect of the external field; our previous work [12] shows such effective quantities to be useful. For example, in a Newtonian gravitational field, motions of the matter and entropy with the velocities \mathbf{u}_m and \mathbf{u}_s induce the transformations

$$\begin{aligned} \tilde{T}_*^-(\mathbf{u}_s) &= (T^0 + \theta \mu + \theta \psi + \theta c^2) \sqrt{1 - \mathbf{u}_s^2 / c^2} \\ &\cong T^0 + \theta \mu^0 - \theta \mathbf{u}_s^2 / 2 + \theta \psi + \theta c^2, \end{aligned} \quad (33)$$

$$\begin{aligned} \tilde{\mu}_*^-(\mathbf{u}_m) &= (\mu^0 + \psi + c^2) \sqrt{1 - \mathbf{u}_m^2 / c^2} \\ &\cong \mu^0 - \mathbf{u}_m^2 / 2 + \psi + c^2, \end{aligned} \quad (34)$$

where their second lines generalize Eq. (31). The transformations (33) and (34) can be used to generalize the kinetic potential L , Eq. (19), to cases with scalar external fields. The generalized L is the Legendre transform of the static pressure

$$\begin{aligned} &-L(\rho_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m) \\ &= \frac{\partial P}{\partial \tilde{T}_{*1}^-} \tilde{T}_{*1}^- + \frac{\partial P}{\partial \tilde{\mu}_*^-} \tilde{\mu}_*^- - P(\tilde{T}_{*1}^-, \tilde{\mu}_*^-, \mathbf{u}_s, \mathbf{u}_m) \\ &= \rho_s \tilde{T}_{*1}^-(\rho_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m) + \rho_m \tilde{\mu}_*^-(\rho_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m) \\ &\quad - P(\rho_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m) \\ &\cong T^0 \rho_s + (\mu^0 + \psi + c^2)(\rho_m + \theta \rho_s) \\ &\quad - P - \rho_m \mathbf{u}_m^2 / 2 - \theta \rho_s \mathbf{u}_s^2 / 2 \\ &\cong \rho_e + \rho(\psi + c^2) - \rho_m \mathbf{u}_m^2 / 2 - \theta \rho_s \mathbf{u}_s^2 / 2. \end{aligned} \quad (35)$$

The transformation theory of T^- and μ^- is also useful when the matter tensor G^{ik} is determined (Sec. IX) and a transition is made to express its components in terms of the traditional quantities. An example in our earlier work [28] shows that the components of the energy flux \mathbf{Q} , determined for a multicomponent counterpart of nonrelativistic L , Eq. (19), can be brought to the traditional form which contains all standard terms characterizing real fluids, with the heat flux, diffusion fluxes, and the work of the nonequilibrium stress per unit time, $\mathbf{\Pi} \cdot \mathbf{u}$.

VI. PARTIAL MOMENTA OF EXTENDED THERMODYNAMICS

In classical thermodynamics both the entropy and the heat behave like weightless continua, whereas in the extended approaches (Grad's theory, extended irreversible thermodynamics, de Broglie thermodynamics, and the thermal mass theory analyzed here) they have an inertia. For any model of a sufficiently small constant θ and the standard linear model of heat conduction, the heat pulse propagates in our theory with thermal speeds not exceeding $(T/\theta)^{1/2}$, in the classical relativistic hydrodynamics with the speed c^2 , and in the classical nonrelativistic hydrodynamics with infinite speed. A clarification of this issue will be given in Sec. VIII; here we aim to evaluate the momentum properties of the extended model.

With the kinetic potential (19) the thermal momentum and the momentum of the bare matter are

$$\begin{aligned}\mathbf{p}_s &= (\partial L / \partial \mathbf{u}_s)_{\rho_m, \rho_s, u_m} = -(\partial E^0 / \partial \rho_s^0) (\partial \rho_s^0 / \partial \mathbf{u}_s) \\ &= (\theta + c^{-2} T^{-0}) \rho_s \mathbf{u}_s / \sqrt{1 - \mathbf{u}_s^2 / c^2} \cong \theta \mathbf{J}_s + T^{-0} \mathbf{J}_s / c^2,\end{aligned}\quad (36)$$

$$\begin{aligned}\mathbf{p}_m &= (\partial L / \partial \mathbf{u}_m)_{\rho, \rho_s, u_s} = -(\partial E^0 / \partial \rho_m^0) (\partial \rho_m^0 / \partial \mathbf{u}_m) \\ &= (1 + c^{-2} \mu^{-0}) \rho_m \mathbf{u}_m / \sqrt{1 - \mathbf{u}_m^2 / c^2} \cong \mathbf{J}_m + \mu^{-0} \mathbf{J}_m / c^2.\end{aligned}\quad (37)$$

The total momentum density Γ or the sum of \mathbf{p}_m and \mathbf{p}_s can be written as

$$\begin{aligned}\Gamma = \mathbf{p}_s + \mathbf{p}_m &= (\theta + c^{-2} T^{-0}) \rho_s^0 \left(\frac{\mathbf{u}_s}{1 - \mathbf{u}_s^2 / c^2} - \frac{\mathbf{u}}{1 - \mathbf{u}^2 / c^2} \right) \\ &+ (1 + c^{-2} \mu^{-0}) \rho_s^0 \left(\frac{\mathbf{u}_m}{1 - \mathbf{u}_m^2 / c^2} - \frac{\mathbf{u}}{1 - \mathbf{u}^2 / c^2} \right) \\ &+ (1 + c^{-2} h^{-0}) \frac{\rho^0 \mathbf{u}}{1 - \mathbf{u}^2 / c^2}.\end{aligned}\quad (38)$$

We introduce into Eq. (38) the heat flux $\mathbf{q} = T^0 \mathbf{j}_s = \rho_s (\mathbf{u}_s - \mathbf{u})$ and the diffusion flux of bare mass $\mathbf{j}_m = \rho_m (\mathbf{u}_m - \mathbf{u})$, a sort of self-diffusion flux. The fluxes satisfy the standard equality resulting from the definition of the hydrodynamic velocity

$$\theta \mathbf{j}_s + \mathbf{j}_m = 0. \quad (39)$$

For small transfer velocities Eqs. (38) and (39) yield in terms of the fluxes

$$\begin{aligned}\Gamma &= \mathbf{p}_s + \mathbf{p}_m \\ &\cong (\theta + c^{-2} T^{-0}) \mathbf{j}_s + (1 + c^{-2} \mu^{-0}) \mathbf{j}_m + (1 + c^{-2} h^{-0}) \mathbf{J} \\ &= c^{-2} (T^{-0} - \theta \mu^{-0}) \mathbf{j}_s + (1 + c^{-2} h^{-0}) \mathbf{J} \\ &\cong c^{-2} T \mathbf{j}_s + (1 + c^{-2} h) \mathbf{J} \\ &= c^{-2} \mathbf{q} + (1 + c^{-2} h) \mathbf{J}.\end{aligned}\quad (40)$$

Thus the extended low-velocity expression for Γ coincides with that obtained from Eq. (17) in the previous case of the inertialess entropy, i.e., the classical and the extended models correspond with the same density of *total* momentum Γ . Yet, the extended model shows that $c^{-2} \mathbf{q}$ is merely the *resulting* or net momentum of heat. The $c^{-2} \mathbf{q}$ is residual, the result of incomplete compensation of $\mathbf{p}_s + \mathbf{p}_m$ in the frame where $\mathbf{J} = \mathbf{0}$; its definition rests on Eq. (39), which assures Eq. (17) or (40). In the limit of an infinite c the compensation of all internal momenta is complete.

On the other hand, Eq. (36) shows that for a finite θ the partial momentum associated with the entropy transfer can be considerable. In the nonrelativistic rest frame ($\mathbf{J} = \mathbf{0}$) this momentum equals simply $\theta \mathbf{q} / T$, where \mathbf{q} is the heat flux. As shown by the fourth line of Eq. (40), the low-velocity heat flux understood as the energy transferred in the fluid frame ($\mathbf{J} = \mathbf{0}$) equals $T \mathbf{j}_s$, with T being the ordinary temperature. This holds because $T^{-0} - \theta \mu^{-0} \cong T^{-0} - \theta \mu^{-0} = T^0 \cong T$, as the effect of difference $\theta \mathbf{u}_s^2 / 2 - \theta \mathbf{u}_m^2 / 2$ is negligible, and the ordinary T transforms as a scalar at low velocities. Thus the relativistic thermal momentum for the diffusive transfer of the entropy in the resting frame ($\mathbf{J} = \mathbf{0}$) equals $\mathbf{p}_s^0 = (c_0^{-2} + c^{-2}) T^{-0} \mathbf{j}_s \cong (T^{-0} / T) \mathbf{q} / c_0^2$, where $c_0 = (T^{-0} / \theta)^{1/2}$. The expression contains the ratio T^{-0} / T , the result of applying of the old definition $\mathbf{q} = T \mathbf{j}_s$ to a new situation where the definition $\mathbf{q}^{-} = T^{-0} \mathbf{j}_s$ would be not less relevant. With such a redefinition of heat the thermal momentum satisfies a mnemonic formula

$$\mathbf{p}_s^0 = (c_0^{-2} + c^{-2}) T^{-0} \mathbf{j}_s = (c_0^{-2} + c^{-2}) \mathbf{q}^{-}. \quad (41)$$

As \mathbf{p}_s^0 nearly compensates the diffusional momentum of matter, the resulting relativistic effect is always as small as \mathbf{q} / c^2 . But this practical compensation does not exclude large competitive currents and the role of their momenta. In any genuine extended thermodynamics the entropy flux carries a finite momentum in the nonrelativistic limit, as in Eq. (41), and this momentum cannot be ignored. Since the total momentum density Γ is constant, the large thermal momentum causes the compensating self-diffusion momentum of mass to preserve a definite mass flux \mathbf{J} .

The quantity \mathbf{p}_s is the (spatial component) density of the following relativistic four-vector of thermal momentum:

$$\begin{aligned}\mathbf{P}_s &= (\theta + c^{-2} T^{-0}) S \mathbf{u}_s / \sqrt{1 - \mathbf{u}_s^2 / c^2}, \\ P_s^4 &= (\theta + c^{-2} T^{-0}) S c^2 / \sqrt{1 - \mathbf{u}_s^2 / c^2},\end{aligned}\quad (42)$$

whose absolute value $|P_s| = [(P_s^4)^2 - \mathbf{P}_s^2]^{1/2}$ is $(\theta + c^{-2} T^{-0}) S c^2 \cong \theta S c^2$. Similarly for the momentum of the bare mass the following relations hold:

$$\begin{aligned}\mathbf{P}_m &= (1 + c^{-2} \mu^{-0}) M_m \mathbf{u}_m / \sqrt{1 - \mathbf{u}_m^2 / c^2}, \\ P_m^4 &= (1 + c^{-2} \mu^{-0}) M_m / \sqrt{1 - \mathbf{u}_m^2 / c^2},\end{aligned}\quad (43)$$

and $|\mathbf{P}_m| = (1 + c^{-2} \mu^{-0}) M_m c^2 \cong M_m c^2$. Thus the total four-momentum of an equilibrium volume is $|P| = (\theta S + M_m + c^{-2} h^{-0} M) c^2 = M (c^2 + h^0)$, where M is the total rest mass of the fluid and h^{-0} is the modified specific enthalpy. Since $h^{-0} = h^0$, the final result is the same as in the classical case

when $\theta=0$. Consequently, the classical result $|P|=M(c^2+h^0)$ still holds as exact. The surplus Mh^0 to the rest energy Mc^2 is due to the thermal agitation of molecules. This conforms to the standard theory which explains that the enthalpy, rather than the internal energy, plays the role of the time component of the energy-momentum vector when the system in motion is not a free system [19,23].

VII. ENERGY AND ENTROPY RELATIONS

In accord with the definition of the barycentric velocity, we apply the relations

$$\rho \mathbf{u} = \rho_m \mathbf{u}_m + \theta \rho_s \mathbf{u}_s, \quad (44)$$

$$\rho = \rho_m + \theta \rho_s, \quad (45)$$

to link the two representations in the laboratory frame. For the extended model the density of the total energy E (not E^0) is

$$\begin{aligned} E = \mathbf{p}_s \cdot \mathbf{u}_s + \mathbf{p}_m \cdot \mathbf{u}_m - L = c^{-2}(\theta c^2 + T^{-0}) \rho_s \mathbf{u}_s^2 / \sqrt{1 - \mathbf{u}_s^2 / c^2} \\ + c^{-2}(c^2 + \mu^{-0}) \rho_m \mathbf{u}_m^2 / \sqrt{1 - \mathbf{u}_m^2 / c^2} \\ + E^0(\rho_s \sqrt{1 - \mathbf{u}_s^2 / c^2}, \rho_m \sqrt{1 - \mathbf{u}_m^2 / c^2}). \end{aligned} \quad (46)$$

We transform this result to the rest densities, to compare it with the standard relativistic energy of a local-equilibrium fluid [2,7]. We obtain

$$\begin{aligned} E = (c^{-2}T^{-0} + \theta) \rho_s^0 \mathbf{u}_s^2 / (1 - \mathbf{u}_s^2 / c^2) \\ + (c^{-2}\mu^{-0} + 1) \rho_m^0 \mathbf{u}_m^2 / (1 - \mathbf{u}_m^2 / c^2) \\ + \theta \rho_s^0 c^2 + \rho_m^0 c^2 + T^{-0} \rho_s^0 + \mu^{-0} \rho_m^0 - P, \end{aligned} \quad (47)$$

where the second line represents the rest energy E^0 . A rearrangement yields

$$\begin{aligned} E = \rho_s^0 (T^{-0} + \theta c^2) / (1 - \mathbf{u}_s^2 / c^2) + \rho_m^0 (\mu^{-0} + c^2) / (1 - \mathbf{u}_m^2 / c^2) \\ - P. \end{aligned} \quad (48)$$

Equations (47) and (48) describe the nonequilibrium energy density of the extended fluid. At local equilibrium $\mathbf{u}_s = \mathbf{u}_m = \mathbf{u}$. In this case Eq. (48) simplifies to the well-known formula for the relativistic energy of a local equilibrium fluid,

$$E = (E^0 + P c^{-2} \mathbf{u}^2) / (1 - \mathbf{u}^2 / c^2) \quad (49)$$

[2,7]. In the low-velocity approximation, but still at disequilibrium,

$$\begin{aligned} E \cong \theta \rho_s^0 c^2 / (1 - \mathbf{u}_s^2 / c^2) \\ + \rho_m^0 c^2 / (1 - \mathbf{u}_m^2 / c^2) + T^{-0} \rho_s + \mu^{-0} \rho_m - P \\ \cong \frac{1}{2} \theta \rho_s \mathbf{u}_s^2 + \frac{1}{2} \rho_m \mathbf{u}_m^2 + (\theta \rho_s + \rho_m) [(c^2 + e(\rho_s, \theta \rho_s + \rho_m))], \end{aligned} \quad (50)$$

where a transition can be made from variables ρ_s and ρ_m to express the energy as the function of the traditional variables ρ_s and ρ .

Equation (48) generalizes the standard relativistic energy (49) for a finite θ . Equation (50) shows that for the same total entropy and mass the extended energies are higher than the corresponding energies of the local-equilibrium description [11–13]. Thus the nonequilibrium model increases the values of E above its equilibrium value. It is just this increase of the mechanical part of E which causes the larger (in comparison to classical) momenta of extended thermodynamics.

VIII. HEAT TRANSFER WITH FINITE PROPAGATION SPEED

We can now easily explain the large thermal momenta of Grad's theory [25]. We pursue the low- \mathbf{u} case as sufficient for practice in the frame of the present formalism. Using Eqs. (44) and (45) in the form of the constraint $\theta \mathbf{j}_s + \mathbf{j}_m = 0$ in the last line of Eq. (50) yields

$$\begin{aligned} L = \frac{1}{2} \rho \mathbf{u}^2 + \frac{1}{2} \theta^2 (\theta^{-1} \rho_s^{-1} + \rho_m^{-1}) \mathbf{j}_s^2 - \rho c^2 - \rho e \\ = \frac{1}{2} \rho \mathbf{u}^2 + \frac{1}{2} \rho_m^{-1} (\rho_m + \rho_s \theta) \rho_s \theta (\mathbf{u}_s - \mathbf{u})^2 - \rho c^2 - \rho e. \end{aligned} \quad (51)$$

Equation (51) may be transformed further, with conventional variables ρ_s , ρ , \mathbf{u}_s , and \mathbf{u} ,

$$\begin{aligned} L = \frac{1}{2} \rho \mathbf{u}^2 + \frac{1}{2} \frac{\rho \rho_s \theta}{\rho - \rho_s \theta} (\mathbf{u}_s - \mathbf{u})^2 - \rho c^2 - \rho e \\ = \frac{1}{2} \rho \mathbf{u}^2 + \frac{1}{2} \rho \theta s (1 - \theta s) \mathbf{v}_{sm}^2 - \rho c^2 - \rho e, \end{aligned} \quad (52)$$

where $\mathbf{v}_{sm} = \mathbf{u}_s - \mathbf{u}_m$ is the relative velocity. This shows that, as long as θ is finite, the partial momenta, $\partial L / \partial \mathbf{u}_s$, $\partial L / \partial \mathbf{v}_{sm}$, etc., appearing in various representations of extended dynamics may be many orders of magnitude larger than the residual (standard, relativistic) momentum of heat \mathbf{q} / c^2 . [The latter is actually ignored in L of Eq. (52) due to its approximate, low- \mathbf{u} , form.] These partial momenta do not vanish at the limit of infinite c as does the momentum \mathbf{q} / c^2 . Thus they can fit the inertia of Grad's solution of the Boltzmann equation or the experimental data of thermal conductivities by a pertinent choice of the coefficient θ [25,11,24].

For example, in the variables \mathbf{u} and \mathbf{v}_{sm} , the momentum density $\partial L / \partial \mathbf{v}_{sm} = \rho \theta s (1 - \theta s) \mathbf{v}_{sm}$, and, since $(1 - \theta s) \mathbf{v}_{sm}$ equals $\mathbf{u}_s - \mathbf{u} = \mathbf{q} / T \rho_s$, this density is \mathbf{q} / c_θ^2 , where $c_\theta \equiv (T / \theta)^{1/2}$. Hence, since θs are roughly $m / 3k_B$ [24], c_θ is close to the thermal speed. For Grad's model the density $\partial L / \partial \mathbf{v}_s$ is a suitable quantity; it equals $\rho g s^2 \mathbf{v}_s = g s \mathbf{q} / T = \mathbf{q} / c_g^2$, where $\mathbf{v}_s = \mathbf{u}_s - \mathbf{u}$ and $c_g = (T / g s)^{1/2}$, cf. Eq. (53) in Ref. [11]. This explains the origin of the momentum densities like \mathbf{q} / c_θ^2 in formulas for partial momenta and their squares in the nonequilibrium internal energy. Note that the coefficients c_θ and c_g need not describe the propagation speeds of heat (for the latter see below; they may be close to c_θ and c_g), although they still are convenient measures of inertial effects. Due to their speedlike nature, various internal momenta can be presented in simple forms.

The diffusive entropy flux $\mathbf{j}_s = \rho_s (\mathbf{u}_s - \mathbf{u})$ is a basic variable in our earlier work on nonrelativistic thermal inertia [11]; the corresponding L is given by Eq. (41) therein. How-

ever, another inertial coefficient, g , is used there, derived from Grad's [25] model, which is a function $g(\rho, s)$ rather than the constant θ used here. Grad's moment solution leads to evaluation of θ in terms of g as [24]

$$\theta(\rho, \rho_s) = \frac{\rho_s \rho g}{\rho^2 + g \rho_s^2} = \frac{g s}{1 + g s^2}. \quad (53)$$

For a constant g the θ in Eq. (53) depends solely on the ratio $s = \rho_s / \rho$. This θ does have a flat maximum with respect to s . Thus Eq. (53) approximates well a constant θ in the region of ρ_s and ρ surrounding the region of maximum θ , where $s = s^* = (\rho_s / \rho)^* = g^{-1/2} = (5/2)^{1/2} k_B / m$. In this sense the present constant- θ description is consistent with Grad's model and the related thermal conductivity data. For an ideal gas the corresponding maximum value of $\theta = (\frac{1}{2}) g^{1/2} = (\frac{1}{10})^{1/2} m / k_B$, i.e., θ equals about $(\frac{1}{3}) m / k_B$. This is the maximal value of the inertial coefficient θ obtained from Grad's model. In this respect, it should be stressed that Grad's model and the related relaxation time approximation produce data of thermal conductivity in quite good agreement with experiments. Thus our description uses g and θ actually confirmed by many experiments in heat conduction (see also the end of the Appendix). But, this being the case, the role of θ in Eqs. (19), (48), and (50) is substantial. The related thermal momenta understood in a broad sense as partial derivatives of L with respect to absolute or relative velocities, $\partial L / \partial \mathbf{u}_s$ or $\partial L / \partial \mathbf{v}_{sm}$, etc., can vary over orders of magnitude, and are by no means readily negligible quantities.

In the entropy representation, the kinetic energy surplus manifests itself as a flux-dependent or "kinetic" entropy. The dissipated entropy in the rest frame is obtained under the usual assumption that the conservation laws for energy and momentum are the same for both the reversible process and for an irreversible process with the entropy source σ_s . With Eq. (39) this entropy source can be expressed in terms of the ordinary temperature gradient and the entropy flux,

$$\begin{aligned} -T\sigma_s &= \mathbf{j}_s \cdot \left(\frac{\partial(\theta \mathbf{v}_s)}{\partial t} + \nabla T^- \right) + \mathbf{j}_m \cdot \left(\frac{\partial(\mathbf{v}_m)}{\partial t} + \nabla \mu^- \right) \\ &= \mathbf{j}_s \cdot \left(\frac{\partial \theta(\mathbf{v}_s - \mathbf{v}_m)}{\partial t} + \nabla T \right), \end{aligned} \quad (54)$$

where $\mathbf{v}_s = \mathbf{u}_s - \mathbf{u}$ and $\mathbf{v}_m = \mathbf{u}_m - \mathbf{u}$ are the relative velocities. Thus the gradient of the ordinary temperature persists as the driving force of heat transfer, and the inertial term appears with θ as the inertial coefficient. For practical purposes the difference $\mathbf{v}_s - \mathbf{v}_m$ should be expressed in terms of the heat flux $\mathbf{q} = T \mathbf{j}_s = T \rho_s (\mathbf{u}_s - \mathbf{u})$. From the definition of \mathbf{u} in terms of the \mathbf{u}_s and \mathbf{u}_m the difference $\mathbf{u}_s - \mathbf{u}$ equals $(\mathbf{u}_s - \mathbf{u}_m)(1 - \theta s)$ where $s = \rho_s / \rho$ is the specific entropy. Hence $\mathbf{q} = T \rho_s (\mathbf{u}_s - \mathbf{u}_m)(1 - \theta s)$ or, $\mathbf{v}_s - \mathbf{v}_m = \mathbf{u}_s - \mathbf{u}_m = [T \rho_s (1 - \theta s)]^{-1} \mathbf{q}$ and Eq. (54) implies the heat conduction in the Cattaneo form

$$\mathbf{q} = -\kappa \left\{ \frac{\partial}{\partial t} \left(\frac{\theta \mathbf{q}}{T \rho_s (1 - \theta s)} \right) + \nabla T \right\} = -\kappa \left\{ \frac{\partial}{\partial t} \left(\frac{g \mathbf{q}}{T \rho} \right) + \nabla T \right\}, \quad (55)$$

where g is the state function in Eq. (53). The last expression in Eq. (55) was introduced to make possible a comparison between the present model, based on the coefficient θ , and the model of our earlier work [11], derived from the constant ideal gas coefficient g . The present model is an alternative to the g -based model; they overlap in the range of state-independent θ and g . Equation (55) defines the relaxation time for heat flux, $\tau_q = \kappa g / (\rho T)$, and the propagation speed of the thermal wave $c_0 = (\kappa / \rho c_p \tau_q)^{1/2} = (T / c_p g)^{1/2}$. In terms of θ , in the plateau regime,

$$c_0 = \sqrt{T [c_p g(\theta)]^{-1}} = \sqrt{T s c_p^{-1} \theta^{-1} (1 - \theta s)}. \quad (56)$$

For an ideal gas, $g = (\frac{2}{5})(m/k_B)^2$ and $c_p = 5/2 k_B / m$. With these data Eq. (56) yields $c_0 = (k_B T / m)^{1/2}$. A discussion of the two models and of the variability of θ is available [24]. These models are more exact than the traditional Fourier model (implying $c_0 \rightarrow \infty$) in the sense that the infinite propagation speeds of the thermal disturbances are avoided. Yet they incorporate Fourier's thermal conductivities.

IX. EXTENSION OF RAY'S ANALYSIS BY INCLUDING EFFECT OF HEAT

We shall determine the matter tensor for a general relativistic fluid with thermal inertia. The Lagrangian density of the special theory

$$L = -E^0(\rho_s \sqrt{1 - \mathbf{u}_s^2 / c^2}, \rho_m \sqrt{1 - \mathbf{u}_m^2 / c^2}) \quad (19')$$

can be put in a manifestly invariant form by introducing the four-velocity vectors U_s^i and U_m^i and the corresponding fluxes, each obeying the same formulas as the four-vector of hydrodynamic velocity and the related mass flux

$$\begin{aligned} U^i &\equiv (U^\alpha, U^4) = \left(\frac{\mathbf{u}}{\sqrt{1 - \mathbf{u}^2 / c^2}}, -\frac{c}{\sqrt{1 - \mathbf{u}^2 / c^2}} \right), \\ U^i U^i &= \mathbf{U} \cdot \mathbf{U} - U^4 U^4 = -c^2, \end{aligned} \quad (57)$$

where $i = 1, \dots, 4$ and $\alpha = 1, \dots, 3$, so that any four-flux vector can be written as

$$\begin{aligned} (\mathbf{J}, J^4) &= (\rho \mathbf{u}, \rho c) = \left(\frac{\rho^0 \mathbf{u}}{\sqrt{1 - \mathbf{u}^2 / c^2}}, \frac{\rho^0 c}{\sqrt{1 - \mathbf{u}^2 / c^2}} \right) \\ &= (\rho^0 U^\alpha, \rho^0 U^4) = (\rho^0 U^i). \end{aligned} \quad (58)$$

These equations allow for an immediate inclusion of thermal inertia to general relativistic fluids. Our approach nicely extends the elegant variational formalism, formulated by Ray for perfect or adiabatic fluids [2]. Instead of searching for an extremum of an action A using the four variables (ρ, \mathbf{u}) Ray uses the five variables (ρ^0, U^i) and the general relativistic extension (59) of the constraint of Eq. (57) linking the four-velocity components. The introduction of the metric tensor contained in Eq. (59) is a step towards general relativity. Using such an approach, Ray achieves a variational treatment of standard perfect fluids and gravitational fields by incorporating into the field Lagrangian Λ constraints on mass conservation, specific entropy s , and fluid-particle identity.

The perfect fluid description is in terms of the rest density ρ^0 and the hydrodynamic four-velocity of matter U^i , obeying $J_i J^i = -\rho^0 c^2$, or the four-velocity constraint

$$g_{ik} U^k U^i + c^2 = 0, \quad (59)$$

which pertains to the case of gravitational fields with the metric g^{ik} . The signature convention is $(+++-)$. It is equivalent to the transformation of any rest density ρ_K^0 of a scalar quantity K from its rest frame to an arbitrary laboratory frame which moves with the four-velocity U^k .

Our extension of Ray's field Lagrangian allows free entropy flow, free bare mass flow, and the finite thermal inertia θ . Consequently, for a one-component fluid we have two different four-velocities, U_s^k and U_m^k , which describe, respectively, transport of the entropy and of the bare matter. The Ricci tensor is defined by $R_{ik} = R_{ilk}^l$. Inclusion of the entropic degrees of freedom yields the Lagrangian density

$$\begin{aligned} \Lambda = & \frac{c^3}{16\pi\kappa'} (-g)^{1/2} R - c^{-1} (-g)^{1/2} [(\theta\rho_s^0 + \rho_m^0)c^2 \\ & + \rho_e^0(\rho_s^0, \rho_m^0)] + (-g)^{1/2} \gamma_s (g_{ik} U_s^i U_s^k + c^2) \\ & + (-g)^{1/2} \gamma_m (g_{ik} U_m^i U_m^k + c^2) + (-g)^{1/2} \eta_s (\rho_s^0 U_s^i)_{;i} \\ & + (-g)^{1/2} \phi_m (\rho_m^0 U_m^i)_{;i} + (-g)^{1/2} \lambda_s X_{s,i} U_s^i \\ & + (-g)^{1/2} \lambda_m X_{m,i} U_m^i, \end{aligned} \quad (60)$$

where κ' is the gravitational constant, and the semicolon denotes covariant differentiation. R is the Riemannian curvature scalar which determines the Lagrangian of the gravitational field; for our purposes its implicit form is sufficient. The present $\Lambda = c^{-1} \Lambda^0$ (Λ^0 is an original Lagrangian), corresponding to the variable $ct = x^4$ used in the action integral.

As the calculations are similar to those made for the original model, we only briefly outline the derivation of the energy tensor G^{ik} , stressing the new results caused by the free entropy flow and referring the reader to Ray's original publications [2] for further details and generalizations for spinning fluids [32,33].

The energy-momentum tensor is obtained in a direct way by varying the action based on Eq. (60) with respect to the components of the metric tensor g_{ik} . Variations with respect to g_{ik} , ρ_s^0 , ρ_m^0 , U_s^i , U_m^i , γ_s , γ_m , ϕ , η , yield a set of equations of motion. When the extremum conditions with respect to the densities

$$\eta_{,i} U_s^i = -c^{-1} (\theta c^2 + T^{-0}), \quad (61)$$

$$\phi_{,i} U_m^i = -c^{-1} (c^2 + \mu^{-0}), \quad (62)$$

and those with respect to the velocity components

$$\gamma_s g_{ik} U_s^i = \rho_s^0 \eta_{,k} - \lambda_s X_{s,k}, \quad (63)$$

$$\gamma_m g_{ik} U_m^i = \rho_m^0 \phi_{,k} - \lambda_m X_{m,k}, \quad (64)$$

are combined with the velocity constraints for U_s and U_m , we obtain

$$\gamma_s = (2c^3)^{-1} \rho_s^0 (\theta c^2 + T^{-0}), \quad (65)$$

$$\gamma_m = (2c^3)^{-1} \rho_m^0 (c^2 + \mu^{-0}). \quad (66)$$

Thus the multipliers of four-velocities are additive components of the relativistic enthalpy density of the fluid. We recall that it is the total fluid enthalpy (not the total energy) which is the component of the four-vector at local equilibrium [23]. With our multipliers, Einstein's equations are contained in the extremum conditions of the action with respect to the components of the metric tensor

$$\frac{\partial \Lambda}{\partial g_{ik}} - \left(\frac{\partial \Lambda}{\partial g_{ik,r}} \right)_{,r} + \left(\frac{\partial \Lambda}{\partial g_{ik,r,s}} \right)_{,sr} = 0. \quad (67)$$

For the Lagrangian (60) these equations are obtained in the usual form

$$E^{ik} = (8\pi\kappa'/c^4) G^{ik}, \quad (68)$$

where E^{ik} is Einstein's tensor of the gravitational field. The energy-momentum tensor or the matter tensor G^{ik} is the source of this field. In the present theory, the matter tensor G^{ik} is affected by the heat flow, thermal inertia, and the nonequilibrium stress. Here it is obtained in the form

$$\begin{aligned} G^{ik} = & c^{-2} \rho_s^0 (\theta c^2 + T^{-0}) U_s^i U_s^k + c^{-2} \rho_m^0 (c^2 + \mu^{-0}) U_m^i U_m^k \\ & + g^{ik} [-(\theta \rho_s^0 + \rho_m^0) c^2 - \rho_e^0 + \rho_s^0 (\theta c^2 + T^{-0}) \\ & + \rho_m^0 (c^2 + \mu^{-0})], \end{aligned} \quad (69)$$

where the conditions (65) and (66) for the Lagrangian multipliers have been used. Since the expression in the last line is exactly the pressure scalar P , we find

$$\begin{aligned} G^{ik} = & c^{-2} \rho_s^0 (\theta c^2 + T^{-0}) U_s^i U_s^k + c^{-2} \rho_m^0 (c^2 + \mu^{-0}) U_m^i U_m^k \\ & + g^{ik} P. \end{aligned} \quad (70)$$

This equation takes into account effects of heat and nonequilibrium stress (the total viscous stress in the case of a purely dissipative fluid) through the relative four-velocities of the entropy and bare matter with respect to the hydrodynamic four-velocity. It allows one to investigate the effect of the dissipative phenomena on the solution of Einstein's equations. To prove this basic property it is essential that the rest mass (or relativistic energy *without* the statistical term) furnishes the definition of the hydrodynamic velocity U^i ,

$$\theta \rho_s^0 U_s^i + \rho_m^0 U_m^i - \rho^0 U^i = 0. \quad (71)$$

This is equivalent to Eqs. (44) and (45) which are the classical equations describing the additivity of laboratory-frame densities and fluxes for any fixed point in the space-time.

Now it is easy to prove that Eq. (70) can be cast into the traditional form of Eq. (4). An equivalent form of Eq. (4) uses the projection tensor $h^{ik} = g^{ik} + c^{-2} U^i U^k$,

$$G^{ik} = c^{-2} [(\rho^0 c^2 + \rho^0 e^0) U^i U^k + q^i U^k + q^k U^i] + \tau^{ik} + P h^{ik}, \quad (72)$$

where q^i is the four-vector of heat expressed as

$$q^i = \rho_s^0 (\theta c^2 + T^{-0}) U_s^i + \rho_m^0 (c^2 + \mu^{-0}) U_m^i - \rho^0 (c^2 + h^0) U^i, \quad (73)$$

and τ^{ik} is the four-tensor of nonequilibrium stresses expressed as

$$\begin{aligned}\tau^{ik} &= c^{-2}[\rho_s^0(\theta c^2 + T^{-0})v_s^i v_s^k + \rho_m^0(c^2 + \mu^{-0})v_m^i v_m^k] \\ &\cong \rho_s^0 \theta v_s^i v_s^k + \rho_m^0 v_m^i v_m^k.\end{aligned}\quad (74)$$

Here $v_s^i = U_s^i - U^i$ is the relative four-velocity. Note that the heat flux is defined as the difference between the actual energy flux and the energy flux of a corresponding perfect fluid. Equation (72) is known [5,8] but expressions (73) and (74) are new. Note that the relativistic heat flux in terms of the traditional temperature T^0 equals $q^i = T^0 \rho_s^0 (U_s^i - U^i) = T^0 \rho_s^0 v_s^i$, as in our earlier nonrelativistic definition of heat understood as the effect of entropy flow in the fluid frame [12]. Indeed,

$$\begin{aligned}q^i &= T^{-0} \rho_s^0 U_s^i + \mu^{-0} \rho_m^0 U_m^i - h^{-0} \rho^0 U^i \\ &= T^{-0} \rho_s^0 (U_s^i - U^i) + \mu^{-0} \rho_m^0 (U_m^i - U^i) \\ &= (T^{-0} - \theta \mu^{-0}) \rho_s^0 (U_s^i - U^i) = T^0 \rho_s^0 (U_s^i - U^i) = T^0 j_s^i.\end{aligned}\quad (75)$$

We stress that expressions (73) and (74) may be regarded as macroscopic definitions of the heat flux and nonequilibrium stress tensor. Such definitions should be distinguished from phenomenological equations which link fluxes with corresponding thermodynamic forces. The phenomenological equations cannot be obtained from our reversible Lagrangian, which works at the theoretical limit of a superconductor. Yet, unlike adiabatic fluid models, superconductor models preserve both heat and nonequilibrium stress, which is the substantial improvement. The phenomenological equations of dissipative fluids can be found from Israel's relativistic extension of Onsager's formalism involving the entropy production [8]. An example is our Eq. (54), consistent with this formalism.

For a special case when $U_s^i = U_m^i = U^i$, the entries of G^{ik} simplify to the form

$$G^{ik} = c^{-2} \rho^0 (c^2 + h^0) U^i U^k + g^{ik} P, \quad (76)$$

which describes an adiabatic relativistic fluid.

It is worth realizing that we have *derived*, not just assumed, relativistic definitions of heat and nonequilibrium stress, Eqs. (73) and (74), from the extended Hamilton principle allowing thermal degrees of freedom, represented macroscopically by the four-flux of the entropy. With these definitions, the energy-momentum tensor has the general form of Eq. (72). By furnishing these macroscopic definitions in the context of thermal inertia, the variational principle adds an important ingredient to the classical formulas (4) and (72).

X. DISCUSSION AND FINAL REMARKS

This work has shown that flows of matter and inertial entropy have similar effect on the matter tensor, and that the split of the total mass into the thermal mass and bare mass does not change observable effects at thermal equilibrium. Otherwise, nonequilibrium descriptions are benefited by the concept of the thermal mass, where both the heat flow \mathbf{q} and the nonequilibrium stress τ emerge as effects of the entropy

flow (thermal mass flow) in the fluid frame.

The virtue of the approach based on the Lagrangian of a superconducting fluid is that it does not truncate terms in the matter tensor; in fact, the obtained energy flux contains both the heat flux \mathbf{q} and the nonequilibrium flux of momentum, τ . (In the standard model of adiabatic fluid these terms are absent.) Thus general conservation laws are produced, applicable *even* for dissipative fluids. Yet, as shown by the equations of motion obtained from our Lagrangian, the superconducting model does not admit any dissipative mechanisms for fluxes. Indeed, the assumed zero entropy production admits that the fluxes can only be related to purely reversible effects, such as ‘‘ballistic’’ nondissipative heat transfer or purely elastic transport of momentum. This reversibility is typical of all classical action-type approaches. The problem of how to further modify Lagrangians in order to achieve equations of motion with dissipative terms and still preserve the same conservation laws (matter tensor) is unsolved to date. Nonetheless, some results obtained in Ref. [12] show that additive interaction Lagrangians might bring us closer to solving the problem. In the meantime, one must content himself by improving the conservation laws and the matter tensor, both derived from extended reversible models, such as our Eq. (60). This is, in fact, the only improvement necessary to properly describe gravitational metrics in general relativity, where the relativistic tensor of matter \mathbf{G} is the unique source of the gravitational field [cf. Einstein's Eq. (68)]. In other words, it is inessential whether the origins of the heat flux \mathbf{q} and of the nonequilibrium stress τ in \mathbf{G} are reversible or not. This statement should remove a common misunderstanding concerning the role of dissipative effects in the relativistic theory of gravitation. The effect of dissipation on gravitational fields is here shown to be *indirect at most*: as a *possible* phenomenon causing definite flows, \mathbf{q} and τ , which could otherwise be attributed to some reversible causes. This is similar to effects of electric currents which cause magnetic fields regardless of whether they are reversible (caused by the motion of the conductor) or irreversible (caused by the conductivity electrons).

The search for complete equations of evolution (with both reversible and irreversible terms) is recently quite intense. Grmela [34] has abandoned the idea of a single generalizing L and proposed a powerful two-bracket approach, in which the reversible terms are represented by the Poissonian bracket, and the irreversible terms by the so-called ‘‘dissipative bracket.’’ An approach of this sort was applied in the 1994 book by Beris and Edwards [35] to many complex thermodynamic systems (those with internal structure, rheological fluids, liquid crystals, etc.), where most of the results were derived by using a Hamiltonian as a sole generator. The idea of the two-bracket description was further generalized by Grmela and Öttinger to the so-called generic form [36] which uses two generators, the energy E and the entropy S . In the evolution equations, the functional derivatives of E and S are, respectively, multiplied by some antisymmetric and symmetric matrices (each satisfying a certain degeneracy condition). Again, the power of the method was shown in many applications [37], including relativistic systems [3]. It would seem from these results that reversible terms should be represented by an antisymmetric Poissonian bracket whereas irreversible ones should be represented by a symmet-

ric dissipative bracket. However, working in the context of the Onsager-like variational formulations, where a dissipative Lagrangian L_σ applies, Sieniutycz [38,39] has shown the (antisymmetric) Hamiltonian and Poisson-bracket structure for purely dissipative thermal fields with heat transfer and reaction-diffusion processes. The effect of the disequilibrium persisting in variational solutions [39] was exposed in a recent work [40] which treats lumped nonequilibrium systems which relax to equilibrium, subject to the conservation law constraints; the obtained dynamics was again Hamiltonian. Moreover, an information-theoretic variational formulation has recently been found for the minimum of *relative* Kullback entropy, still again showing the Hamiltonian structure of the irreversible evolution and leading to the Fokker-Planck equation [41]. All these results prove that the physical effects regarded as thermodynamically “irreversible” can certainly be described in terms of Hamiltonian or Lagrangian formalisms and associated Poissonian brackets. On the other hand, the results of the present paper show that fluxes usually regarded as “irreversible,” such as the heat flux \mathbf{q} and the nonequilibrium stress τ do naturally follow from Hamilton’s stationary action, in which the entropy flow is a varied extra variable. Thus the main difficulty in achieving a unifying Hamiltonian (Poissonian) formulation is caused by a composition of irreversible and reversible effects rather than by each of these effects taken separately.

Certainly, more experimental data are needed to achieve a quantitative conclusion on the role of thermal inertia in non-stationary fast-variable processes. Our theory should contribute well to organize systematic experiments in this area.

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APPENDIX: EXPERIMENTS CONFIRMING WAVE NATURE OF HEAT

Here we make reference to the literature which stresses experimental works confirming the wave nature of heat. Lin-

ear dependence of stationary heat flux on the temperature gradient ΔT rather than on its arbitrary power, $(\Delta T)^n$, is well confirmed by experiments [42]. Thermal waves were predicted in liquid ^2He by Peshkov [43]; the related experiments are described by Pellam [44]. Thermal inertia can be observed when the frequency of external oscillations is comparable to the reciprocal of the thermal relaxation time τ . In terms of this relaxation time, the thermal conductivity of fluids is $\kappa = T\rho\tau/g$, where g is the inertial coefficient discussed in the text. For an ideal gas, g is equal to $2m^2/5k_B^2$ in agreement with the thermal conductivity data and the relaxation approximation to the solution of Boltzmann’s equation. Data are available showing that such approximation frequently yields very good estimates of thermal conductivities [45]. The numerical values of the coefficient θ used in this work are evaluated on the basis of the thermal conductivity data or from Grad’s solution [25], as explained in Refs. [11, 24]. In other words, our choice of the coefficient θ is consistent with the experimental data of thermal conductivities in fluids, Sec. VIII.

As regards thermal inertia in other media, the reader is referred to pure dielectric crystals at low T (5 K), where heat pulses can propagate “ballistically” as longitudinal and transverse excitations. The fully developed second-sound pulse, predicted theoretically by Krumhansl and co-workers in a number of papers, see, for example [46], has been observed in experiments with solid ^4He and solid ^3He [47]. Helium crystals are highly anharmonic and hence nontypical. Therefore it is of interest that approach to the second-sound propagation and the short temperature pulses were detected in “more normal” solids, such as very pure samples of alkali-halide crystals. For experimental and computational studies, see [48,49]. For solids (for which there is no generally valid theory of thermal conductivity), a time-dependent relaxation model for the heat flux in metals was derived from the quantum mechanical form of the Boltzmann equation. The thermal and electrical relaxation times are not equal although they are estimated to be at the same order of magnitude, 10^{-14} sec, for the common monovalent metals [50]. For the relation of these evaluations to experiments, see Ref. [51].

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