Conservation laws in higher-order nonlinear Schrödinger equations

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Conservation laws of the nonlinear Schrödinger equation are studied in the presence of higher-order optical effects including the third-order dispersion and the self-steepening. In a context of group theory, we derive general expressions for infinitely many conserved currents and charges of a coupled higher-order nonlinear Schrödinger equation. The first few currents and associated charges are also presented explicitly. Due to the higher-order effects, the conservation laws of the nonlinear Schrödinger equation are violated in general. The differences between the types of the conserved currents for the Hirota and the Sasa-Satsuma equations imply that the higher-order terms determine the inherent types of conserved quantities for each integrable case of the higher-order nonlinear Schrödinger equation.
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In the ultrafast optical signal system, higher-order effects such as the third-order dispersion, the self-steepening, and the self-frequency shift become important if the pulses are shorter than $T_0 \leqslant 100$ fs [1]. When compared with the group velocity dispersion, the third-order dispersion is normally negligible but produces significant effects of asymmetrical temporal broadening for the ultrashort pulses [2,3]. The self-steepening effect, which is accompanied by an optical shock at the trailing edge, also leads to the asymmetrical spectral behavior of the pulses [4]. The self-frequency shift due to Raman gain stimulated to the long wavelength components costing the short wavelength components causes an increasing redshift to the propagating pulses [5,6]. These three types are in general the dominant higher-order effects to be considered for the propagation of femtosecond pulses in a monomode optical fiber. For a higher rate transmission of pulses, the wavelength division multiplexing [7] also can be taken into account. In this case, the use of optical pulses with multiple field components to accommodate degrees of freedom in distinct polarizations and/or frequencies requires the consideration of nonlinear cross-couplings between different modes of pulses.

For the description of the multimode transmission, extensions of the nonlinear Schrödinger equation (NSE) to include cross-coupling terms are required. The simplest case (vector NSE) in terms of two field components was first proposed and integrated by the method of inverse scattering transform [8]. A systematic generalization of the NSE was made only for the cross phase modulation terms using the structure of symmetric spaces [9] where the vector NSE is a special case. As mentioned above, the simultaneous inclusion of both the higher-order and the cross-coupling effects leads to the study on a coupled higher-order nonlinear Schrödinger equation (CHONSE) which is not in general integrable except for special cases of coupling constants. The CHONSEs described in [10] and [11,12] are the limited extensions of the Hirota [13] and the Sasa-Satsuma [14] equations, respectively. Recently, by making use of the matrix potential introduced in [15], we have proposed a general extension of the Hirota and the Sasa-Satsuma equations and clarified their relationships [16] in association with the formalism of Hermitian symmetric spaces [17].

It is well known that nonlinear equations which can be integrated by the method of inverse scattering transform possess an infinite number of conserved quantities. For example, the NSE has an infinite number of conserved charges in addition to the ones corresponding to the energy and the intensity-weighted mean frequency. However, the effect of the higher-order and the cross-coupling terms on the conservation laws has not been considered up to now. In this paper, utilizing the properties of the Hermitian symmetric space, we make a systematic study of the conservation laws in the presence of the higher-order and the cross-coupling terms. We first indicate that, except for the energy conservation, other conservation laws of the NSE such as the conservation of the intensity-weighted mean frequency do not hold any more due to the higher-order effects, unless the higher-order terms are of a unique type. In the case of the integrable CHONSE, we derive general expressions of an infinite number of conserved currents and charges from the Lax pair formulation. From the general expressions, explicit forms of the first few conserved currents and the associated charges of the Hirota and the Sasa-Satsuma equations are calculated in a consistent way of reduction. We then explain the correlations of conservation laws between the two integrable cases of the higher-order extension of the NSE.

In order to illustrate the issue, we first consider the NSE including the higher-order terms. In a monomode optical fiber, the propagation of an ultrashort pulse is governed by the higher-order NSE [18]

$$\bar{\partial}\psi = i(\gamma_1 \partial^2 \psi + \gamma_2 |\psi|^2 \psi) + \gamma_3 \partial^3 \psi + \gamma_4 \partial(|\psi|^2 \psi) + \gamma_5 \partial(|\psi|^2)\psi, \qquad (1)$$

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where $\bar{\partial}\equiv\partial/\partial\bar{z}$ and $\partial\equiv\partial/\partial z$ are derivatives in retarded time coordinates $(\bar{z}=x, z=t-x/v)$, and ψ is the slowly varying envelope function. The real coefficients γ_i $(i=1,2,3,4)$ in the first four terms on the right hand side of Eq. (1) specify in sequence the effects of the group velocity dispersion, the self-phase modulation, the third-order dispersion, and the self-steepening. With appropriate scalings of space, time, and field variables, one can readily normalize Eq. (1) so that $\gamma_1=1$, $\gamma_2=2$, $\gamma_3=1$ which we assume from now on. The remaining coefficient γ_5 in the last term is complex in general. The real and the imaginary parts of γ_5 are due to the effect of the frequency-dependent radius of fiber mode and the effect of the self-frequency shift by stimulated Raman scattering, respectively. It is well known that the above equation becomes integrable if $\gamma_4=-\gamma_5=6$ (Hirota case) [13] or $\gamma_4=-2\gamma_5=6$ (Sasa-Satsuma case) [14]. The physical conditions to observe the femtosecond soliton based on the measurable optical fiber parameters [19,20] and the existence even in a medium with an arbitrary dispersion law [21] are discussed with analytical solutions. Experimentally, the adiabatic compression and the redshift of the ultrashort pulses due to the delayed nonlinear response and the higher-order dispersion have been demonstrated [22]. Also some other models with similar types of the higher-order terms in Eq. (1) are proposed [23,24] with explicit soliton solutions.

In the absense of higher-order terms $(\gamma_3=\gamma_4=\gamma_5=0)$, Eq. (1) possesses an infinite number of conserved charges among which the first three charges [25] are

$$Q_1=\int_{-\infty}^{\infty}|\psi|^2 dt,$$

$$Q_2=\int_{-\infty}^{\infty} i(\psi^*\partial\psi-\partial\psi^*\psi)dt, \qquad (2)$$

$$Q_3=\int_{-\infty}^{\infty}(\partial\psi^*\partial\psi-|\psi|^4)dt,$$

where Q_1 represents conserved energy, and Q_2 the mean frequency weighted by the intensity of optical pulses. In the conventional NSE where the time and the space coordinates are interchanged, Q_1, Q_2, and Q_3, respectively, correspond to conserved mass, momentum, and energy. If we include higher-order terms, Q_i are not necessarily conserved but subject to the relations

$$\bar{\partial}Q_1=0,$$

$$\bar{\partial}Q_2=2i(\gamma_4+\gamma_5)\int_{-\infty}^{\infty}\partial|\psi|^2(\psi^*\partial\psi-\partial\psi^*\psi)dt, \qquad (3)$$

$$\bar{\partial}Q_3=(3\gamma_4+2\gamma_5-6)\int_{-\infty}^{\infty}\partial|\psi|^2\partial\psi^*\partial\psi dt.$$

The calculations indicate that the charge Q_1 which corresponds to energy is conserved for all values of γ_4, γ_5 while Q_2 and Q_3 are conserved provided $\gamma_4+\gamma_5=0$ and $3\gamma_4+2\gamma_5=6$, respectively. Note that Q_2 and Q_3 are conserved simultaneously only for the specific value $\gamma_4=-\gamma_5=6$ that is precisely the Hirota case. It is interesting to observe that integrability does not always imply the same types of conserved charges in the presence of higher-order terms. Another integrable case of the Sasa-Satsuma equation, where $\gamma_4=-2\gamma_5=6$, in consequence does not have Q_2 and Q_3 in Eq. (3) as the conserved charges. This result is rather remarkable in view of the fact that integrable equations possess an infinite number of conserved quantities. We will show, however, that the Sasa-Satsuma equation also possesses infinitely many conserved charges of different types other than the ones of the Hirota equation.

In the case where we include both the higher-order and the cross-coupling nonlinear effects, the propagating system is governed by a CHONSE. Without understanding physical settings, it would be meaningless to write down any general expression of the CHONSE. However, as explicitly derived in [16], there exists a group theoretic specification which admits a systematic classification of integrable cases of the CHONSE. In the following, we consider the group theoretic generalization of the NSE and define the CHONSE in association with a Hermitian symmetric space. By solving the linear Lax equations iteratively, we derive an infinite number of conserved currents and charges for the CHONSE. For later use, we briefly review the definition of Hermitian symmetric spaces [9,17] and the generalization of the NSE [16,26] according to the Hermitian symmetric spaces.

A symmetric space is a coset space G/K for Lie groups $G\supset K$ whose associated Lie algebras $\mathbf{g}$ and $\mathbf{k}$, with the decomposition $\mathbf{g}=\mathbf{k}\oplus\mathbf{m}$ satisfy the commutation relations

$$[\mathbf{k},\mathbf{k}]\subset\mathbf{k}, \quad [\mathbf{m},\mathbf{m}]\subset\mathbf{k}, \quad [\mathbf{k},\mathbf{m}]\subset\mathbf{m}. \qquad (4)$$

A Hermitian symmetric space is the symmetric space G/K equipped with a complex structure. One can always find an element T in the Cartan subalgebra of $\mathbf{g}$ whose adjoint action defines a complex structure and also the subalgebra $\mathbf{k}$ as a kernel, i.e., $\mathbf{k}=\{V\in\mathbf{g}: [T,V]=0\}$. That is, the adjoint action $J\equiv \mathrm{ad}\,T=[T,*]$ is a linear map $J:\mathbf{m}\to\mathbf{m}$ that satisfies the complex structure condition, $J^2=-I$, or $[T,[T,M]]=-M$ for $M\in\mathbf{m}$. Then, we define a CHONSE as

$$\bar{\partial}E=\partial^2\tilde{E}-2E^2\tilde{E}+\alpha(\partial^3E+\beta_1E^2\partial E+\beta_2\partial EE^2), \qquad (5)$$

where E and $\tilde{E}\equiv[T,E]$ are extended field variables belonging to $\mathbf{m}$. (We restrict to symmetric spaces $AIII=\mathrm{SU}(m+n)/[\mathrm{SU}(m)\mathrm{SU}(n)\mathrm{U}(1)]$, $CI=\mathrm{Sp}(n)/\mathrm{U}(n)$, and $DIII=\mathrm{SO}(2n)/\mathrm{U}(n)$ only so that the expression of CHONSE becomes simplified [16].) The arbitrary constant α may be normalized to 1 by an appropriate scaling but we keep it in order to exemplify the higher-order effects. Also the cross-coupling effects between different modes of polarizations or frequencies are accommodated in the matrix form of E which is determined by each Hermitian symmetric space. For example, in the case where $G/K=\mathrm{SU}(N+1)/\mathrm{U}(N)$, the matrices E and T are represented as

$$E=\begin{pmatrix} 0 & \psi_1 & \cdots & \psi_N \\ -\psi_1^* & 0 & & 0 \\ \vdots & & & \vdots \\ -\psi_N^* & 0 & \cdots & 0 \end{pmatrix},$$

$$T=\begin{pmatrix} \frac{i}{2} & 0 & \cdots & 0 \\ 0 & -\frac{i}{2} & & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & -\frac{i}{2} \end{pmatrix}, \qquad (6)$$

and the CHONSE becomes a higher-order vector NSE,

$$\bar{\partial}\psi_k=i\left[\partial^2\psi_k+2\left(\sum_{j=1}^{N}|\psi_j|^2\right)\psi_k\right]-\alpha\left[\beta_1\left(\sum_{j=1}^{N}|\psi_j|^2\right)\partial\psi_k+\beta_2\left(\sum_{j=1}^{N}\psi_j^*\partial\psi_j\right)\psi_k-\partial^3\psi_k\right],\quad k=1,2,\ldots,N. \qquad (7)$$

This equation is an obvious generalization of Eq. (1) to the multicomponent case. In a more general point of view, as can be seen from Eq. (7) the CHONSE does not include some other physically interesting equations, for example, the four-wave mixing. (A group theoretic treatment of the n-wave equation is also possible using reductive homogeneous space. See, for example, Ref. [9].) Anyway it is easy to see that Eq. (7) with $N=1$ and $\beta_1=\beta_2=-3$ is precisely the Hirota equation, which implies that the equation is an explicit N-coupled extension in itself. Remarkably, another N-coupled form of the Sasa-Satsuma equation also results from the same CHONSE in Eq. (5) through the consistent reduction [16].

As mentioned above, Eq. (5) is integrable if $\beta_1=\beta_2=-3$ because in such a case the CHONSE admits a Lax pair. That is, Eq. (5) with $\beta_1=\beta_2=-3$ arises from the compatibility condition ($[L_z,L_{\bar{z}}]=0$) of the associated linear equations,

$$L_z\Psi\equiv[\partial+E+\lambda T]\Psi=0,$$

$$L_{\bar{z}}\Psi\equiv[\bar{\partial}+U_K^0+U_M^0+\lambda(U_K^1+U_M^1)+\lambda^2(U_M^2+T)-\alpha\lambda^3T]\Psi=0, \qquad (8)$$

which holds for all values of the spectral parameter λ. The entities U_K^i and U_M^i in $L_{\bar{z}}$ are given by

$$U_K^0=-E\tilde{E}-\alpha[E,\partial E],\quad U_M^0=\partial\tilde{E}+\alpha(\partial^2E-2E^3),$$
$$U_K^1=\alpha E\tilde{E},\quad U_M^1=E-\alpha\partial\tilde{E},\quad U_M^2=-\alpha E. \qquad (9)$$

Here, the subscripts K and M signify that they belong to the subalgebra **k** and the remaining complement **m**, respectively. It is crucial that the Lax pair given in Eq. (8) covers the suggested types [10–12] in a generalized formulation. The algebraic decomposition can also be extended to a more general case including the matrix solution $\Psi=\Psi_K+\Psi_M$ with the properties that $[T,\Psi_K]=0$, $[T,\Psi_M]\in\mathbf{m}$, and the following multiplication properties:

$$[T,\Psi_K^1\Psi_K^2]=[T,\Psi_M^1\Psi_M^2]=0,\quad [T,\Psi_K^1\Psi_M^2]\in\mathbf{m}. \qquad (10)$$

The adjoint action of the element T in the Cartan subalgebra together with the complex structure condition, if applied to the decomposition, lead to a couple of general identities for any $M_1,M_2\in\mathbf{m}$;

$$[T,M_1M_2]=\tilde{M}_1M_2+M_1\tilde{M}_2=0,\quad \tilde{M}_1\tilde{M}_2=M_1M_2. \qquad (11)$$

These identities are useful for many calculations, for example, in deriving conserved currents or in verifying that the CHONSE in Eq. (5) is equivalent to the compatibility condition of the Lax pair in Eq. (8).

Having presented necessary ingredients, we are now ready to derive infinitely many conserved currents and charges of the integrable CHONSE by solving the associated linear equations in Eq. (8). In order to make use of the algebraic properties of Hermitian symmetric spaces, we make a change of the variable Ψ in Eq. (8) by

$$\Phi=\Psi\exp\{[\lambda z+(\lambda^2-\alpha\lambda^3)\bar{z}]T\}, \qquad (12)$$

which results in the change of the multiplicative term $T\Psi$ to the commutative term $[T,\Phi]$ in the linear equations. The adjoint action, $[T,\Phi]$, allows the splitting of the linear equations for Φ into the K and the M components as explained below. Let us first assume that the linear equations can be solved iteratively in terms of

$$\Phi(z,\bar{z},\lambda)\equiv\sum_{n=0}^{\infty}\frac{1}{\lambda^n}[\Phi_K^n(z,\bar{z})+\Phi_M^n(z,\bar{z})], \qquad (13)$$

where Φ_K^n and Φ_M^n denote the decomposition of a coefficient Φ^n satisfying the properties in Eq. (10). Then, the nth-order equation ($n\geqslant0$) separates into the K and the M components as

$$\partial\Phi_K^n+E\Phi_M^n=0, \qquad (14)$$

$$\partial\Phi_M^n+E\Phi_K^n+[T,\Phi_M^{n+1}]=0, \qquad (15)$$

while the $\bar{\partial}$ part of the linear equation becomes

$$\bar{\partial}\Phi_K^n+U_K^0\Phi_K^n+U_M^0\Phi_M^n+U_K^1\Phi_K^{n+1}+U_M^1\Phi_M^{n+1}+U_M^2\Phi_M^{n+2}=0, \qquad (16)$$

$$\bar{\partial}\Phi_M^n+U_K^0\Phi_M^n+U_M^0\Phi_K^n+U_K^1\Phi_M^{n+1}+U_M^1\Phi_K^{n+1}+U_M^2\Phi_K^{n+2}+[T,\Phi_M^{n+2}]-\alpha[T,\Phi_M^{n+3}]=0. \qquad (17)$$

In addition, there are equations arising from the positive powers of λ, which can be given by Eqs. (14)–(17) provided that $n=-1,-2,-3$ and $\Phi_K^{n<0}=\Phi_M^{n<0}=0$ are defined. These equations can be solved recursively for Φ_K^n, Φ_M^n ($n\geqslant0$) starting from a consistent set of initial conditions;

$$\Phi_M^0=0,\quad \Phi_K^0=-iI,\quad \Phi_M^1=-i\tilde{E}. \qquad (18)$$

Note that Eq. (15) can be solved for Φ_M^{n+1} by using the complex structure condition, that is, $\Phi_M^{n+1}=-[T,[T,\Phi_M^{n+1}]]=[T,\partial\Phi_M^n]+\tilde{E}\Phi_K^n$. Therefore Φ_M^{n+1} is obtained algebraically provided Φ_K^n and Φ_M^n are determined.

Contrary to Φ_M^{n+1}, the other solution Φ_K^{n+1} is calculated by a direct integration of Eq. (14) but overdetermined due to the additional equation in Eq. (16). Hence, in order for Φ_K^n in general to be integrable, the compatibility condition that $[\partial,\bar{\partial}]\Phi_K^n=0$ should be required for the solution Φ_K^n which is inherited from Eq. (8) generating the corresponding CHONSE. In this case the compatibility condition gives rise to infinitely many conserved currents labeled by integer n such that $\partial\overline{J_K^n}+\bar{\partial}J_K^n=0$;

$$J_K^n=-\partial\Phi_K^n=E\Phi_M^n, \tag{19}$$

$$\overline{J_K^n}=\bar{\partial}\Phi_K^n=-(\partial\tilde{E}+\alpha\partial^2E-3\alpha E^3)\Phi_M^n-\alpha E\partial^2\Phi_M^n \\ -(E-\alpha\partial\tilde{E})[T,\partial\Phi_M^n]. \tag{20}$$

In order to derive the local currents explicitly, we solve the recurrence relations in Eqs. (14)–(17) with the initial conditions as in Eq. (18). The first few conserved currents are listed below:

$$J_K^1=-iE\tilde{E}, \\ \overline{J_K^1}=-i[E,\partial E]+i\alpha([\partial^2E,\tilde{E}]+\partial\tilde{E}\partial E-3E^3\tilde{E}) \tag{21}$$

for $n=1$, and

$$J_K^2=-i\partial\Phi_K^1\Phi_K^1+iE\partial E,$$

$$\overline{J_K^2}=+i\bar{\partial}\Phi_K^1\Phi_K^1-i(E\partial^2\tilde{E}+\partial\tilde{E}\partial E-E^3\tilde{E})-i\alpha(E\partial^3E \\ +[\partial^2E,\partial E]+\partial EE^3-2E\partial EE^2-E^2\partial EE-4E^3\partial E), \\ J_K^3=-i(\partial\Phi_K^2\Phi_K^1+\partial\Phi_K^1\Phi_K^2-i\partial\Phi_K^1\Phi_K^1\Phi_K^1) \\ +i(E\partial^2\tilde{E}-E^3\tilde{E}), \tag{22}$$

$$\overline{J_K^3}=i(\bar{\partial}\Phi_K^1\Phi_K^2+\bar{\partial}\Phi_K^2\Phi_K^1-i\bar{\partial}\Phi_K^1\Phi_K^1\Phi_K^1)+i(E\partial^3E-\partial E\partial^2E \\ +\partial EE^3-2E\partial EE^2-E^2\partial EE-2E^3\partial E)-i\alpha(E\partial^4\tilde{E} \\ +\partial^2E\partial^2\tilde{E}+\partial\tilde{E}\partial^3E-5E^3\partial^2\tilde{E}-E^2\partial^2E\tilde{E}-3E\partial^2\tilde{E}E^2 \\ +\partial^2\tilde{E}E^3-2\partial\tilde{E}\partial EE^2-\partial\tilde{E}E\partial EE-2\partial\tilde{E}E^2\partial E \\ -3E\partial E\partial E\tilde{E}-5E\partial EE\partial\tilde{E}-3E^2\partial E\partial\tilde{E}+4E^5\tilde{E}) \tag{23}$$

for $n=2$ and $n=3$, respectively. Note that currents J_K^n and $\overline{J_K^n}$ for $n\geqslant 2$ contain nonlocal terms Φ_K^m with $m<n$. Fortunately, these nonlocal terms can be separated from the conservation law if we consider a scalar expression of the conserved current by taking an appropriate trace as follows:

$$S_K^n=\mathrm{Tr}(PJ_K^n), \quad \overline{S_K^n}=\mathrm{Tr}(P\overline{J_K^n}). \tag{24}$$

The parameter P is any matrix entity which commutes with matrices Φ_K^m, or we may choose $P=c_1I+c_2T$ for arbitrary constants c_1 and c_2. For instance, we have for $n=2,3$

$$S_K^2=-\partial\left[\mathrm{Tr}\,P\left(\frac{i}{2}(\Phi_K^1)^2\right)\right]+\mathrm{Tr}\,P(iE\partial E),$$

$$\overline{S_K^2}=\bar{\partial}\left[\mathrm{Tr}\,P\left(\frac{i}{2}(\Phi_K^1)^2\right)\right]+\mathrm{Tr}\,P\{-i(E\partial^2\tilde{E}+\partial\tilde{E}\partial E-E^3\tilde{E}) \\ -i\alpha(E\partial^3E+[\partial^2E,\partial E]-6E^3\partial E)\}, \tag{25}$$

$$S_K^3=-\partial(\mathrm{Tr}\,P\{i[\Phi_K^1\Phi_K^2-\tfrac{1}{3}(\Phi_K^1)^3]\}) \\ +\mathrm{Tr}\,P\{i(E\partial^2\tilde{E}-E^3\tilde{E})\},$$

$$\overline{S_K^3}=\bar{\partial}(\mathrm{Tr}\,P\{i[\Phi_K^1\Phi_K^2-\tfrac{1}{3}(\Phi_K^1)^3]\}),+\mathrm{Tr}\,P\{i(E\partial^3E-\partial E\partial^2E \\ -4E^3\partial E)-i\alpha(E\partial^4\tilde{E}+\partial^2E\partial^2\tilde{E}+\partial\tilde{E}\partial^3E-8E^3\partial^2\tilde{E} \\ +2\partial^2\tilde{E}E^3+E^2\partial\tilde{E}\partial E-\partial\tilde{E}E\partial EE+\partial\tilde{E}E^2\partial E \\ -5E\partial\tilde{E}E\partial E+4E^5\tilde{E})\}. \tag{26}$$

The derivations show that the nonlocal terms appear as total derivative terms thus they are conserved separately. Dropping the nonlocal terms and integrating over the time coordinate, we obtain an infinite number of global charges which are conserved in space, i.e., $\bar{\partial}Q^n=0$ where

$$Q^n\equiv\int_{-\infty}^{+\infty}dt\,S_K^n. \tag{27}$$

For the case of $G/K=\mathrm{SU}(N+1)/\mathrm{U}(N)$ as mentioned in Eq. (6), we work out explicitly and obtain the conserved charges for the Hirota case

$$Q_H^1=\int_{-\infty}^{+\infty}dt\sum_{k=1}^{N}\psi_k^*\psi_k \tag{28}$$

for $n=1$ and

$$Q_H^2=\int_{-\infty}^{+\infty}dt\sum_{k=1}^{N}i(\psi_k^*\partial\psi_k-\partial\psi_k^*\psi_k), \tag{29}$$

$$Q_H^3=\int_{-\infty}^{+\infty}dt\left[\sum_{k=1}^{N}\partial\psi_k^*\partial\psi_k-\left(\sum_{k=1}^{N}\psi_k^*\psi_k\right)^2\right] \tag{30}$$

for $n=2$ and $n=3$, respectively. Conserved charges for other cases of integrable CHONSE can be similarly obtained from the specification of E and T as classified in [16].

As noted in Eq. (3), the types of charges Q_2 and Q_3 are not conserved in the Sasa-Satsuma case. Nevertheless, the Sasa-Satsuma equation equivalently possesses infinitely many conserved charges of different types as well. These seemingly contradicting characteristics can be explained by the fact that the Sasa-Satsuma equation arises from the discrete Z_2 reduction of the SU(3)/U(2) CHONSE combined with a point transformation [16]. In this case, matrices E and T can be denoted as

$$E=\begin{pmatrix} 0 & \psi & \psi^* \\ -\psi^* & 0 & 0 \\ -\psi & 0 & 0 \end{pmatrix}, \quad T=\begin{pmatrix} \frac{i}{2} & 0 & 0 \\ 0 & -\frac{i}{2} & 0 \\ 0 & 0 & -\frac{i}{2} \end{pmatrix}. \tag{31}$$

Since the charge Q^n in Eq. (27) is invariant under the point transformation, we can also calculate the first few conserved charges of the Sasa-Satsuma equation using the expressions E and $\tilde{E}\equiv[T,E]$ given in Eq. (31). The resulting charges of the Sasa-Satsuma equation are

$$Q_S^1=\int_{-\infty}^{+\infty} dt\,\psi^*\psi,$$

$$Q_S^2=0, \tag{32}$$

$$Q_S^3=\int_{-\infty}^{+\infty} dt[3\partial\psi^*\partial\psi-6(\psi^*\psi)^2-i(\psi^*\partial\psi-\partial\psi^*\psi)].$$

If the charges in Eq. (32) are compared with those of the Hirota type in Eq. (2) [or equivalently Eqs. (28)–(30) for $N=1$], we note that the charge for $n=1$, which corresponds to energy, is the same but other charges are of different types. Remarkably, in Eq. (32) the charge for $n=2$ turns out to be trivial while the charge for $n=3$ is a new type that is seemingly a combination of charges for $n=2,3$ in Eq. (2). From Eq. (1) with normalized coefficients $\gamma_1=\gamma_2/2=\gamma_3=1$, one can readily confirm that the current $S_S^3=3\partial\psi^*\partial\psi-6(\psi^*\psi)^2-i(\psi^*\partial\psi-\partial\psi^*\psi)$ is conserved only if $\gamma_4+\gamma_5=3$ and $3\gamma_4+2\gamma_5=12$. Solving the equations results in $\gamma_4=-2\gamma_5=6$ that definitely leads to the Sasa-Satsuma case, to be compared with Eq. (3) for the Hirota case. Finally, we point out that the present formalism can be extended to other physically interesting cases, such as to the case where the self-steepening effect is dominant, or to the case of dark solitons which requires an appropriate renormalization of the conserved charges [27].

To summarize, using the properties of Hermitian symmetric space we have constructed the Lax pair formalism of a coupled higher-order nonlinear Schrödinger equation and derived general expressions of an infinite number of conservation laws. Remarkably, the conserved currents and charges for both the Hirota and the Sasa-Satsuma equations are calculated from the general expressions, accompanying the reduction procedure. We have shown that, except for the Hirota case, the current conservations of the nonlinear Schrödinger equation are in general broken by the higher-order effects. The types of conserved currents and charges for the Sasa-Satsuma case are different from the types for the Hirota case except for the energy conserved irrespective of all the higher-order effects. These differences may leave scope for more physical explanations and applications in the further study of higher-order effects including numerical analysis.

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