Hermite-Gaussian expansion for pulse propagation in strongly dispersion managed fibers

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We represent a pulse in the strongly dispersion managed fiber as a linear superposition of Hermite-Gaussian harmonics, with the zeroth harmonic being a chirped Gaussian with periodically varying width. We obtain the same conditions for the stationary pulse propagation as were obtained earlier by the variational method. Moreover, we find a simple approximate formula for the pulse shape, which accounts for the numerically observed transition of that shape from a hyperbolic secant to the Gaussian. Finally, using the same approach, we systematically derive the equations for the evolution of a pulse under a general perturbation. This systematic derivation justifies the validity of similar equations obtained earlier from the conservation laws.
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I. INTRODUCTION

Recent experimental [1–5], numerical [6–10], and analytical [11–16] studies have demonstrated that periodic dispersion compensation, or dispersion management (DM), can be used to significantly improve the performance of soliton transmission systems. Moreover, it has been shown in Ref. [17] that in a DM system, pulses in the soliton format can be transmitted, without being corrupted by certain pertubations, over longer distances than pulses in both the non-return-to-zero (NRZ) and the RZ formats at zero average dispersion. Here we have used the term ''soliton'' when referring to a pulse in a DM system, to indicate that stable propagation of such a pulse occurs via the balance of the nonlinearity and the small average dispersion in the fiber. To distinguish a stable pulse in a DM system from the conventional, nonlinear Schrödinger (NLS) soliton in a fiber with uniform dispersion, we will refer to the former as the DM soliton. The difference between the DM soliton and the RZ pulse in a DM system is that the RZ pulse is supposed to propagate at the zero average dispersion. Thus, the results reported in Ref. [17] indicate that the DM technique opens the way to upgrade the already installed telecommunication lines for the data transmission in the soliton regime.

The two main reasons that are behind the success of the DM technique for the soliton transmission are as follows. First, the periodic concatenation of segments of the fiber with opposite signs of dispersion allows one to make the average dispersion very small, which reduces the Gordon-Haus (GH) jitter for the soliton [18]. Moreover, the DM soliton has greater energy than its NLS counterpart, which further reduces the GH jitter [7,5,16]. Second, the high local dispersion in each of the segments of the dispersion map reduces the detrimental effect on the soliton by the four-wave-mixing fields, which arise in collisions of solitons in different wavelength channels [19,6,10]. (It is interesting to note that a reason similar to this last one was behind the original introduction of the DM for the NRZ transmission [20].)

Two significantly different regimes, those of weak and strong DM, have been considered in the literature. In both these regimes, the period L_{map} of the dispersion map is much less than both the nonlinear length L_{nonlin} and the *average* dispersion lengths L_{average}. Moreover, in the strong DM regime, the *local* dispersion length L_{local} is also much less than both L_{nonlin} and L_{average}. In the weak DM regime, all these three lengths are of the same order of magnitude. In other words, in the strong DM regime, the local dispersion is the dominant factor affecting the pulse evolution, whereas in the weak DM regime, the effects of the nonlinearity and dispersion are comparable in magnitude. Consequently, due to the existence of the *only* small parameter, say $L_{\mathrm{map}}/L_{\mathrm{nonlin}}$, in the weak DM regime, one can reduce the propagation equation to the leading-order NLS equation, corrections to which can be systematically computed. Thus, all properties of the DM soliton in the weak DM regime can be considered as known, at least in principle [21–23,11,12]. However, it is the strong DM regime that yields the stronger suppression of both the GH timing jitter and the jitter induced by collisions between solitons in different channels. At the same time, the existence of an additional *large* parameter, $L_{\mathrm{nonlin}}/L_{\mathrm{local}}$, in the strong DM regime, renders the results obtained for the weak DM formally invalid. Therefore, many numerical studies have been performed, which discovered that a number of properties of a soliton in the strong DM regime are in distinct contrast with properties of the NLS soliton. Here is a list of some of those properties:

(i) The DM soliton is strongly chirped. Moreover, in order to stably propagate in a fiber, it has to be launched at the beginning of the dispersion map with a particular value of the chirp [24,25,8].

(ii) For sufficiently strong periodic variations of the dispersion, the shape of the DM soliton can be closer to a Gaussian than to the convential hyperbolic secant (''sech''). This is manifested by the increase of the pulse time-bandwidth product from 0.32 for the sech up to about 0.44 for the Gaussian. For even stronger dispersion variations, the DM soliton has an even higher value of the time-bandwidth product (≈ 0.6) [26,8].

(iii) Moreover, the pulse changes its shape within one period of the dispersion map.

(iv) The energy of the DM soliton is considerably larger than the energy of the NLS soliton, with both solitons having the same average width and propagating at the same average dispersion [26].

(v) The DM soliton can propagate over very long distances even if the average dispersion in the system is zero or negative (i.e., normal) [27,28].

An explanation of those properties has been a subject of a large number of analytical studies. The first group of those used the variational method with a Gaussian trial function to obtain conditions for the amplitude, width, and initial chirp with which a DM soliton can propagate stationarily [29,30,16,31,32]. These conditions for stationary propagation explained properties (i), (iv), (v) listed above. However, the variational method with a Gaussian trial function could not possibly explain the observed shape of a DM soliton [i.e., properties (ii) and (iii)]. The second group of studies [33–35] used an averaging technique based on the Lie transformation to obtain a leading-order propagation equation of the form

$$iq_Z+\tfrac{1}{2}q_{TT}+q(|q|^2-CT^2)=0, \tag{1.1}$$

where q is some average pulse profile, Z and T are the evolution and spatial variables, respectively, and the constant C depends on the parameters of the dispersion map. By numerically finding a stationary solution of Eq. (1.1), the authors of Refs. [33,34] have been able to explain properties (i), (ii), (iv), and (v) above. However, since q was an average field, property (iii) still could not be explained. Also, since Eq. (1.1) is not integrable, then no *explicit* expressions for the stationary DM soliton could have been found within that approach.

The approach that we present here allows us to explain all of the properties (i)–(v). This approach is systematic, thus allowing one to rigorously account for the shape of the DM soliton. At the same time, it yields *explicit* expressions for the parameters of the stationary DM soliton, with those expressions being, in the leading order, the same ones obtained earlier by the variational method. The key step in our approach is an expansion of the DM soliton over a complete set of certain Hermite-Gaussian functions. We show that the DM soliton can be represented as an infinite sum of these functions (which we will call *harmonics*), with the dominant, zeroth, harmonic being the chirped Gaussian pulse. Taking into account the next nontrivial (see below) harmonic provides a very good approximation to the shape of the stationary DM soliton. We emphasize that the form in which we present our results is *explicit*. That is, all parameters of the DM soliton are given by explicit (and rather simple) expressions that depend only on the soliton's minimum width, provided that the parameters of the dispersion map are fixed and there is no losses and periodic amplification in the fiber. (In the case with losses and amplification, those expressions can be easily evaluated numerically by calculating a small number of certain definite integrals.)

As we mentioned above, the main advantage of using the DM soliton instead of the NLS soliton as an information carrier is that the DM soliton is less susceptible to perturbations than its NLS counterpart. Thus, we also present a perturbation theory for the DM soliton acted upon by an *arbitrary* perturbation. An example of a specific perturbation, which is produced by frequency filtering, is also considered.

The body of this work is organized as follows. In Sec. II, we present the expansion of the DM soliton over a complete set of Hermite-Gaussian harmonics. Using the equations for just the first two even harmonics, we recover the conditions for the stationary propagation of the DM soliton, which were earlier obtained by the variational method [16,31,32]. In Sec. III, we refine these conditions by taking into account the next even harmonic, and also obtain the correction to the shape of the DM soliton compared to the Gaussian. In Sec. IV, we develop the perturbation theory for the DM soliton. Section V contains the summary of this work. Secs. II and IV each have two subsections, with the first subsection containing the main results and the second subsection containing remarks. We also note that the main results of Sections II and III were announced in Ref. [36].

II. EXPANSION OF THE DM SOLITON OVER THE HERMITE–GAUSSIAN HARMONICS

A. Results

The propagation equation in the strong DM regime can be written in the following nondimensional form (see, e.g., [16]):

$$iu_z+\frac{1}{2}D(z)u_{\tau\tau}+\epsilon\left[\frac{1}{2}D_0u_{\tau\tau}+G(z)u|u|^2\right]=0. \tag{2.1}$$

Here $u(z,\tau)$ is proportional to the envelope of the electric field, z and τ are the distance along the fiber and the retarded time, $D(z)$ is the periodic (with period L_{map}) dispersion coefficient:

$$D(z)=\begin{cases}D_1, & 0<\mathrm{mod}(z,L_{\mathrm{map}})<L_1,\\ D_2, & L_1<\mathrm{mod}(z,L_{\mathrm{map}})<L_{\mathrm{map}},\end{cases} \tag{2.2}$$

such that the average of $D(z)$ over the map period L_{map} is zero:

$$D_1L_1+D_2L_2=0. \tag{2.3}$$

The variables in Eq. (2.1) are normalized [16] so as to have

$$L_{\mathrm{map}}=1,\quad |D_1L_1|=|D_2L_2|=1. \tag{2.4}$$

The small parameter ϵ in Eq. (2.1) is the ratio of the local dispersion length to the nonlinear length (for a pulse with unit amplitude and unit width); it characterizes the "strength" of the DM (see also a discussion in [16]). Finally, ϵD_0 is the average dispersion coefficient, and $G(z)$ is a periodic function, with period L_{amp}, which accounts for losses and periodic amplification (see Chap. 7 in [37]). For the idealized lossless fiber, $G(z)\equiv 1$.

To obtain a pulse solution of Eq. (2.1), we use the standard approach of any perturbation theory. Namely, we first exhibit the *general* solution of that equation with $\epsilon=0$, and then study how the presence of a small perturbation (i.e., when $0<\epsilon\ll 1$) will modify that solution. To exhibit the general solution of Eq. (2.1) with $\epsilon=0$, we first perform the following transformation of variables:

$$(z,\tau)\to\left(z,\xi=\frac{\tau-\tau_c(z)}{T_0\sqrt{1+\Delta^2/T_0^4}}\right), \tag{2.5}$$

where

$$\Delta\equiv\Delta(z)=\int_0^z D(z')dz'+\Delta_0, \tag{2.6}$$

$$\frac{d\tau_c(z)}{dz}=\omega_0 D(z), \tag{2.7}$$

and T_0, ω_0, and Δ_0 are constants. Next, we substitute into Eq. (2.1) with $\epsilon=0$ the solution of the following form:

$$u=c^{(0)}(z)f(\xi)\exp\left[i\left\{\frac{\xi^2\Delta}{2T_0^2}+\omega_0\xi T_0\sqrt{1+\frac{\Delta^2}{T_0^4}}+\phi(z)\right\}\right], \tag{2.8}$$

thus obtaining:

$$i\left(c_z^{(0)}+\frac{c^{(0)}D\Delta}{2T_0^4\left(1+\frac{\Delta^2}{T_0^4}\right)}\right)f+\frac{c^{(0)}D}{2T_0^2\left(1+\frac{\Delta^2}{T_0^4}\right)}(f_{\xi\xi}-\xi^2 f)=0, \tag{2.9}$$

provided that we take

$$\frac{d\phi}{dz}=\frac{\omega_0^2}{2}D. \tag{2.10}$$

Here and below we omit arguments of functions when they are obvious. The operator for f in Eq. (2.9) is easily recognized as the linear Schrödinger equation for the quantum harmonic oscillator, whose solutions are the Hermite-Gaussian functions. Hence the general solution of Eq. (2.1) with $\epsilon=0$ is

$$u^{(0)}=\sum_{n=0}^{\infty}a_n c_n^{(0)}(z)H_n(\xi)\exp\left[-\frac{\xi^2}{2}\left(1-\frac{i\Delta}{T_0^2}\right)+i\omega_0\xi T_0\sqrt{1+\frac{\Delta^2}{T_0^4}}+i\phi\right], \tag{2.11}$$

where

$$c_n^{(0)}(z)=\frac{1}{\sqrt{1+i\Delta/T_0^2}}\left(\frac{1-i\Delta/T_0^2}{1+i\Delta/T_0^2}\right)^{n/2}\equiv\frac{\exp[-in\arctan(\Delta/T_0^2)]}{\sqrt{1+i\Delta/T_0^2}}, \tag{2.12}$$

a_n are arbitrary constants, and $H_n(\xi)$ are the Hermite polynomials satisfying

$$H_n''-2\xi H_n'+2nH_n=0 \tag{2.13}$$

(here prime denotes differentiation). See also Remarks 1–4 about the solution (2.11) in Section II B below.

Next, we seek a solution of Eq. (2.1) with $0<\epsilon\ll 1$ using the method of multiple scales. We introduce the sequence of independent evolution variables

$$z_0=z,\quad z_1=\epsilon z,\quad \text{etc.} \tag{2.14a}$$

so that

$$\partial_z=\partial_{z_0}+\epsilon\partial_{z_1}+\cdots, \tag{2.14b}$$

and look for the solution in the form (2.11), where now we set

$$a_n=a_n^{(0)}(z_1,\ldots)+\epsilon a_n^{(1)}(z_0,z_1,\ldots)+\cdots. \tag{2.15}$$

In what follows, we will only consider the slow evolutions of the $a_n^{(0)}$ terms and ignore the $a_n^{(1)}$ terms on the grounds that they introduce corrections in the next order in the small parameter ϵ. We also note that those latter terms may be responsible for the radiation of small dispersive waves by the DM soliton.

Now we make the following two important assumptions about the perturbed solution (2.11). First, we use the fact that the "core" of the DM soliton is represented by the chirped Gaussian [i.e., the term with $n=0$ in the expansion (2.11)], with the higher Hermite-Gaussian harmonics providing corrections to the pulse shape. Thus, we require that the amplitude of the zeroth harmonic be dominant in the expansion (2.11):

$$|a_0^{(0)}|\gg|a_n^{(0)}|,\quad n=1,2,3,\ldots \tag{2.16}$$

As will be shown below, the ratios $|a_n^{(0)}/a_0^{(0)}|$ ($n=1,2,\ldots$) do *not* depend on the small parameter ϵ. [Note: one can easily show that a similar situation also occurs for a weakly nonlinear string satisfying the equations

$$u_{xx}-u_{tt}=\epsilon u^3,\quad u(0)=u(1)=0.]$$

Even with the *strong* inequality (2.16), one can quantitatively explain the numerically observed transition of the DM soliton's shape from the sech-like to that with a higher time-bandwidth product, as described in Sec. I. Also, this strong inequality guarantees that the shape of the DM soliton will not behave chaotically in z, at least for a sufficiently small ϵ. This statement is an analog of the well-known theorem due to Izrailev and Chirikov [38], according to which the motion of a weakly nonlinear chain of oscillators will never become chaotic provided that the energy is originally distributed among only a small number of the lowest harmonics (this is also known as the Fermi-Pasta-Ulam phenomenon).

Second, we will allow the pulse parameters T_0, Δ_0, ω_0, τ_c, and ϕ to be functions of the slow variable z_1:

$$T_0=T_0(z_1),\quad \Delta_0=\Delta_0(z_1),\quad \omega_0=\omega_0(z_1),\quad \tau_c=\tau_c(z_0,z_1),\quad \phi=\phi(z_0,z_1), \tag{2.17}$$

This is not really necessary since the set of functions $\{H_n(\xi)e^{-\xi^2/2}\}_{n=0}^{\infty}$, over which we expand the solution, is complete in the space of square-integrable functions. However, we will still use the superfluous degrees of freedom allowed by Eq. (2.17) because this significantly simplifies the following analysis and, in addition, will be required in Sec. IV where we develop the perturbation theory for the DM soliton.

We now substitute the expansion (2.11) into Eq. (2.1) and then collect the coefficients at each term $H_n(\xi)e^{-\xi^2/2}$. Since the set of the Hermite-Gaussian functions is complete, then the coefficient at each term must vanish individually and in all orders in ϵ. Obviously, all coefficients of order ϵ^0 identically vanish since the expansion (2.11) is the zeroth order solution of Eq. (2.1). Then in the order ϵ^1, the coefficient multiplying the nth term yields

$$
\begin{aligned}
&i\partial_{z_0}a_n^{(1)}+i\dot{a}_n^{(0)}+a_{n+2}^{(0)}\left[(n+2)(n+1)\left(\frac{D_0-\dot{\Delta}_0}{2T_0^2}-\frac{i\dot{T}_0}{T_0}\right)\right]+a_{n+1}^{(0)}\left\{(n+1)\left[-\dot{\omega}_0T_0\left(1-\frac{i\Delta}{T_0^2}\right)-\frac{i}{T_0}(\dot{\tau}_c-\omega_0D_0)\right]\right\}+a_n^{(0)}\\
&\times\left[\frac{(n+\frac{1}{2})}{2T_0^2}(\dot{\Delta}_0-D_0)+\frac{i\dot{T}_0}{2T_0}+\omega_0\left(\dot{\tau}_c-\frac{\omega_0D_0}{2}\right)-\dot{\phi}\right]+a_{n-1}^{(0)}\left\{\frac{1}{2}\left[-\dot{\omega}_0T_0\left(1+\frac{i\Delta}{T_0^2}\right)+\frac{i}{T_0}(\dot{\tau}_c-\omega_0D_0)\right]\right\}+a_{n-2}^{(0)}\\
&\times\left[\frac{1}{4}\left(\frac{D_0-\dot{\Delta}_0}{2T_0^2}+\frac{i\dot{T}_0}{T_0}\right)\right]+\frac{\sin[\pi(n+1)/2]e^{in\theta}a_0^{(0)}|a_0^{(0)}|^2G(z_0)}{2^{3n/2}\sqrt{2}(n/2)!\sqrt{1+\Delta^2/T_0^4}}\\
&+\sum_{\substack{m\geqslant 1,\\ (m+n)\ \text{is even}}}\frac{(-1)^{(m-n)/2}(m+n-1)!!G(z_0)}{2^n\sqrt{2}n!\sqrt{1+\Delta^2/T_0^4}}\{a_0^{(0)2}a_m^{(0)*}e^{i(m+n)\theta}+2|a_0^{(0)}|^2a_m^{(0)}e^{i(n-m)\theta}\}=0.
\end{aligned}
\tag{2.18}
$$

Here the overdot stands for ∂_{z_1}, $\theta=\arctan[\Delta(z_0)/T_0^2]$, and $(2k-1)!!\equiv 1\times 3\times 5\times\cdots(2k-1)$. Note that the term with $\sin\pi(n+1)/2$, which comes form the nonlinear term in Eq. (2.1), is present only for even n and vanishes for odd n. In the derivation of the next term, $\Sigma\{\cdots\}$, we used the strong inequality (2.16) and thus neglected terms like $a_0^{(0)}a_m^{(0)2}$ and $a_m^{(0)3}$. We also used, in addition to Eq. (2.13), the following relations satisfied by the Hermite polynomials (see, e.g., [39]):

$$H_n'(\xi)=2nH_{n-1}(\xi), \tag{2.19a}$$

$$H_{n+1}(\xi)-2\xi H_n(\xi)+2nH_{n-1}(\xi)=0, \tag{2.19b}$$

$$\int_{-\infty}^{\infty}d\xi H_n(\xi)H_m(\xi)e^{-\xi^2}=2^n n!\sqrt{\pi}\delta_{nm}, \tag{2.19c}$$

$$\int_{-\infty}^{\infty}d\xi H_n(\xi)H_m(\xi)e^{-2\xi^2}=\sqrt{\frac{\pi}{2}}(-1)^{(m-n)/2}(m+n-1)!!,\quad (m+n)\quad\text{is even.} \tag{2.19d}$$

The equation for $i\dot{a}_n^{(0)}$ follows from Eq. (2.18) after we impose the standard requirement that $a_n^{(1)}$ not grow linearly with z_0. Since $\Delta(z)$ is a periodic function with period $L_{\text{map}}\equiv 1$ and, moreover, since we assume that the amplification period L_{amp} is an integer factor of L_{map}, then that requirement amounts to averaging Eqs. (2.18) over one map period. Thus, taking into account that all $a_n^{(0)}$ are only functions of the slow variable z_1, and denoting

$$I_n=\int_0^1 dz\frac{G(z)}{\sqrt{1+\Delta^2/T_0^4}}e^{in\theta(z)}\equiv\int_0^1\frac{dzG(z)}{\sqrt{1+\Delta^2/T_0^4}}\left(\frac{1+i\Delta/T_0^2}{1-i\Delta/T_0^2}\right)^{n/2}, \tag{2.20}$$

we obtain

$$i\dot{a}_n^{(0)}+a_{n+2}^{(0)}\left[(n+2)(n+1)\left(\frac{D_0-\dot{\Delta}_0}{2T_0^2}-\frac{i\dot{T}_0}{T_0}\right)\right]+a_{n+1}^{(0)}\left[(n+1)\left(-\dot{\omega}_0T_0-\frac{i}{T_0}(\dot{\tau}_c-\omega_0D_0-\dot{\omega}_0[\tfrac{1}{2}\text{sgn}(D_1L_1)+\Delta_0)]\right)\right]+a_n^{(0)}$$

$$\times\left[\frac{(n+\frac{1}{2})}{2T_0^2}(\dot{\Delta}_0-D_0)+\frac{i\dot{T}_0}{2T_0}+\omega_0\left(\dot{\tau}_c-\frac{1}{2}\omega_0 D_0\right)-\dot{\phi}\right]+a_{n-1}^{(0)}\left[\frac{1}{2}\left(-\dot{\omega}_0 T_0+\frac{i}{T_0}(\dot{\tau}_c-\omega_0 D_0-\dot{\omega}_0[\tfrac{1}{2}\mathrm{sgn}(D_1L_1)+\Delta_0)]\right)\right]$$
$$+a_{n-2}^{(0)}\left[\frac{1}{4}\left(\frac{D_0-\dot{\Delta}_0}{2T_0^2}+\frac{i\dot{T}_0}{T_0}\right)\right]+\frac{\sin[\pi(n+1)/2]a_0^{(0)}|a_0^{(0)}|^2}{2^{3n/2}\sqrt{2}(n/2)!}I_n+\sum_{\substack{m\geq 1,\\ (m+n)\ \text{is even}}}\frac{(-1)^{(m-n)/2}(m+n-1)!!}{2^n\sqrt{2}n!}$$
$$\times\{a_0^{(0)2}a_m^{(0)*}I_{m+n}+2|a_0^{(0)}|^2a_m^{(0)}I_{n-m}\}=0. \tag{2.21}$$

From Eqs. (2.21), we immediately observe that the conditions for decoupling harmonics with even n from those with odd n are

$$\dot{\omega}_0=0,\quad \dot{\tau}_c=\omega_0 D_0. \tag{2.22}$$

Thus, if initially no harmonics with odd n are excited and conditions (2.22) hold, then such harmonics will never appear in the evolution. Note that the second of Eqs. (2.22) yields the same relation between the *average* velocity of the DM soliton [cf. Eq. (2.7)], its frequency ω_0, and the average dispersion D_0 that also holds in the case of uniform dispersion. This condition was earlier found in, e.g., Ref. [40] by different means.

Now, if we seek a *stationary* pulse solution of Eq. (2.1), we must set

$$\dot{\Delta}_0=\dot{T}_0=0. \tag{2.23}$$

Otherwise, if either of these conditions does not hold, then the pulse's width will increase without bound as z_1 increases [cf. Eq. (2.5)]. Under conditions (2.22) and (2.23), Eq. (2.21) is simplified considerably, although it still represents an infinite system of coupled equations. We begin analyzing it by restricting our attention to the first two harmonics, $n=0$ and $n=2$ [recall that all odd-numbered harmonics can be effectively eliminated by conditions (2.22)]. This simple first step will (i) give us the same conditions for the stationary propagation of the DM soliton as were obtained in, e.g., Refs. [16,31] by the variational method, and, more importantly, (ii) indicate how our analysis is to be extended to the case of an arbitrary number of harmonics.

Ignoring all amplitudes $a_n^{(0)}$ other than $a_0^{(0)}$ and $a_2^{(0)}$ and using Eqs. (2.22) and (2.23), we obtain from Eqs. (2.21) the following system:

$$i\dot{a}_0^{(0)}+a_0^{(0)}\left[-\frac{D_0}{4T_0^2}+\frac{|a_0^{(0)}|^2}{\sqrt{2}}I_0\right]+a_2^{(0)}\left[-\sqrt{2}|a_0^{(0)}|^2I_{-2}+\frac{D_0}{T_0^2}\right]$$
$$+a_2^{(0)*}\left[-\frac{a_0^{(0)2}}{\sqrt{2}}I_2\right]=0, \tag{2.24a}$$

$$i\dot{a}_2^{(0)}+a_0^{(0)}\left[\frac{D_0}{8T_0^2}-\frac{|a_0^{(0)}|^2}{8\sqrt{2}}I_2\right]+a_2^{(0)}\left[\frac{3}{4}\frac{|a_0^{(0)}|^2}{\sqrt{2}}I_0-\frac{5D_0}{4T_0^2}\right]$$
$$+a_2^{(0)*}\left[\frac{3a_0^{(0)2}}{8\sqrt{2}}I_4\right]=0. \tag{2.24b}$$

Now consider the following thought experiment. Let one launch a pure Gaussian pulse into a DM fiber, and, moreover, suppose that its evolution is governed by Eqs. (2.24). In general, the second harmonic, which was zero originally, i.e., $a_2^{(0)}(z_1=0)=0$, will be driven by the second term in Eq. (2.24b). If $a_2^{(0)}$ grows significantly, then the pulse will no longer have a simple bell-like shape with one maximum, and thus stable pulse propagation will not occur. Thus, the necessary condition for the pulse stationary propagation is that the coefficient multiplying $a_0^{(0)}$ in Eq. (2.24b) vanishes. This yields the following equation:

$$\frac{D_0}{T_0^2}=\frac{|a_0^{(0)}|^2}{\sqrt{2}}I_2, \tag{2.25}$$

which, in fact, implies two separate conditions for its imaginary and real parts:

$$\mathrm{Im}\, I_2=0, \tag{2.26a}$$

$$D_0=T_0^2\frac{|a_0^{(0)}|^2}{\sqrt{2}}\ \mathrm{Re}\ \ I_2. \tag{2.26b}$$

We now demonstrate that these are exactly the same conditions that were earlier obtained in Ref. [16] (see also references therein). Using Eq. (2.20), conditions (2.26) can be rewritten in explicit form:

$$\int_0^1\frac{G(z)\Delta(z)dz}{(T_0^4+\Delta^2(z))^{3/2}}=0, \tag{2.27a}$$

$$D_0=\frac{|a_0^{(0)}|^2}{\sqrt{2}}T_0^4\int_0^1\frac{G(z)[T_0^4-\Delta^2(z)]dz}{[T_0^4+\Delta^2(z)]^{3/2}}. \tag{2.27b}$$

Next, from Eqs. (2.2) and (2.6), one sees that $\Delta(z)$ is given by

$$\Delta(z)=\begin{cases}(\Delta_0+D_1z), & 0<\mathrm{mod}(z,1)<L_1,\\ [\Delta_0+D_1L_1+D_2(z-L_1)], & L_1<\mathrm{mod}(z,1)<1.\end{cases} \tag{2.28}$$

With Eq. (2.28), one performs the integration in Eq. (2.27) first over the interval $0<z<L_1$, using the substitution $z=L_1(s+\frac{1}{2})$, and then adds to the result the integral over $L_1<z<1$, in which one makes the substitution $(z-L_1)=L_2(\frac{1}{2}-s)$. Then, taking into account Eqs. (2.3) and (2.4), we transform conditions (2.27) into the form

$$\int_{-1/2}^{1/2} \frac{[s+\chi\Delta_0+\frac{1}{2}]g(s)ds}{[T_0^4+(s+\chi\Delta_0+\frac{1}{2})^2]^{3/2}}=0, \tag{2.29a}$$

$$D_0=\frac{|a_0^{(0)}|^2}{\sqrt{2}}T_0^4\int_{-1/2}^{1/2}\frac{[T_0^4-(s+\chi\Delta_0+\frac{1}{2})^2]g(s)ds}{[T_0^4+(s+\chi\Delta_0+\frac{1}{2})^2]^{3/2}}, \tag{2.29b}$$

where $\chi=\mathrm{sgn}(D_1L_1)$ and

$$g(s)=L_1G[L_1(s+\tfrac{1}{2})]+L_2G[L_1+L_2(\tfrac{1}{2}-s)]. \tag{2.30}$$

Upon noting the correspondence of notations:

$$(T_0^2)_{\text{this paper}}=\tfrac{1}{2}(\tau_0^2)_{\text{Ref. [16]}},$$

one sees that Eqs. (2.29) are the same as Eqs. (10) in [16], which were obtained there by the variational method. For a given minimum width T_0, the first of these conditions determines Δ_0 and hence the initial chirp; then the second condition determines the relation between the average dispersion D_0 and the amplitude $a_0^{(0)}$. In particular, these conditions predicted the possibility of stationary propagation of a DM soliton at normal average dispersion (i.e., at $D_0<0$) when $T_0\leqslant(T_0)_{\rm cr}\approx 0.39$. The validity of conditions (2.29) was confirmed in Ref. [16] by extensive numerical simulations.

B. Remarks

First, the transformation of variables (2.5) is the well-known ''lens transformation'' [41] that has also been used earlier in other analytical studies of the strong DM regime [34,33]. It exists since Eq. (2.1) with $\epsilon=0$ (i.e., the linear parabolic equation) has a certain symmetry group (denoted as G_6 in [42]).

Second, the fact that Eq. (2.11) yields the general solution to Eq. (2.1) with $\epsilon=0$ has recently been pointed out, in the context of linear optical pulse propagation, in Ref. [43]. Our Eq. (2.11) generalizes the corresponding equation in Ref. [43] in that it is written for the variable dispersion $D(z)$ and a nonzero frequency ω_0 of the pulse. It should also be noted that a *different* Hermite-Gaussian expansion for a pulse in a fiber laser has been used recently in Ref. [44].

Third, the solution (2.11) separates the variables z and ξ. Note that most classical partial differential equations are solved using a proper separation of variables.

Fourth, the first term of the solution (2.11) is just the familiar Gaussian pulse, which we rewrite here in the original variables z and τ:

$$u_0^{(0)}=\frac{a_0}{\sqrt{1+i\Delta/T_0^2}}\exp\left[-\frac{\tau^2}{2T_0^2(1+\Delta^2/T_0^4)}+i\frac{\tau^2\Delta}{2T_0^4(1+\Delta^2/T_0^4)}\right], \tag{2.31}$$

where we have put $\omega_0=\tau_c=0$ for simplicity. Thus the parameters T_0 and $\Delta_0/\sqrt{T_0^4+\Delta_0^2}$ have the meanings of the minimum pulsewidth and the initial chirp, respectively. Since Eq. (2.28) guarantees that $\Delta(z)$ periodically oscillates about some constant value, then the width, the amplitude, and the chirp of the pulse also oscillate about their respective constant values, rather than grow or decrease on average.

Fifth, we discuss the limit $T_0\to\infty$ of the DM soliton, with its amplitude being fixed at a finite value. It is easy to see from Eq. (2.1) that the size of the variable dispersion term decreases with the increase of T_0, whereas the magnitudes of both the average dispersion and nonlinear terms are independent of T_0 [cf. Eq. (2.26b)]. Thus, these two terms are dominant in the limit $T_0\to\infty$, and therefore this limit corresponds to the NLS equation with the uniform dispersion D_0. Therefore, we will always verify that our analysis gives correct results in the limit $T_0\to\infty$.

Sixth and last, it is fully expected that conditions (2.26), obtained here by considering the evolutions of the zeroth and the second Hermite-Gaussian harmonics, should also follow from the variational method. Indeed, the ''variations'' $\partial u^{(0)}/\partial T_0$ and $\partial u^{(0)}/\partial\Delta_0$, which are in fact used when finding the stationary solution of the variational equations in Ref. [16], both contain precisely those two harmonics, provided that the trial function was taken as the pure Gaussian pulse (2.31). This can be easily seen from the coefficients multiplying the terms $a_n^{(0)}$ and $a_{n+2}^{(0)}$ in Eq. (2.21).

III. SHAPE OF THE STATIONARY DM SOLITON

Let us now extend the previous analysis by taking into account the next even harmonic, i.e., that with $n=4$. As before, our main goal is to study the stationary DM soliton, and therefore we set $\dot{\Delta}_0=\dot{T}_0=0$ everywhere in this section. We also set

$$a_0^{(0)}=A[1+\hat{a}_0(z_1)]e^{i\Lambda z_1},\quad a_{2,4}^{(0)}=A\hat{a}_{2,4}(z_1)e^{i\Lambda z_1}, \tag{3.1}$$

where

$$\Lambda=\Lambda_0+\frac{A^2}{\sqrt{2}}\Lambda_1, \tag{3.2a}$$

$$\Lambda_0=\frac{A^2I_0}{\sqrt{2}}-\frac{D_0}{4T_0^2},\quad |\Lambda_1|\ll|\Lambda_0|. \tag{3.2b}$$

Λ_0 is the leading-order ''frequency'' of the slow oscillations of all harmonics [see the second term in Eq. (2.24a)], and the small correction Λ_1 is to be determined later. The condition (2.25) for the stationary propagation now needs to be taken in a modified form:

$$\frac{D_0}{T_0^2}=\frac{|a_0^{(0)}|^2}{\sqrt{2}}(I_2+\delta), \tag{3.3}$$

where the small correction δ is also to be found later. Finally, we assume that the correction $A\hat{a}_0$ to the amplitude of the zeroth harmonic, as well as the amplitudes $A\hat{a}_{2,4}$ of the higher harmonics, are small and of the same order of magnitude as Λ_1 and δ:

$$|A\hat{a}_0|\sim|A\hat{a}_2|\sim|A\hat{a}_4|\sim|\delta|\sim|\Lambda_1|\ll A. \tag{3.4}$$

All further calculations can be carried out for the general case when both losses and periodic amplification are present in the fiber. However, below we restrict our attention to the lossless case. Then $G(s)\equiv 1$ in Eq. (2.1), and using the same variable substitution as when obtaining Eqs. (2.29), all integrals I_n can be found explicitly as elementary functions of T_0. Moreover, the condition

$$\Delta_0 = -\tfrac{1}{2}\,\mathrm{sgn}(D_1 L_1), \tag{3.5}$$

which follows from Eq. (2.29a) for $G(s)\equiv 1$, guarantees that all these integrals are real valued:

$$\mathrm{Im}\, I_n = 0 \quad \text{for all even } n. \tag{3.6}$$

Substituting Eqs. (3.1)–(3.6) into the system (2.21), neglecting terms quadratic in δ (e.g., $\hat{a}_0\delta$, etc.), and considering only the harmonics with $n=0,2,4$, we obtain for the vector

$$\vec{Q}_4 = [\hat{a}_{0R}, \hat{a}_{0I}, \hat{a}_{2R}, \hat{a}_{2I}, \hat{a}_{4R}, \hat{a}_{4I}]^T \tag{3.7}$$

the following linear system:

$$\dot{\vec{Q}}_4 = \frac{A^2}{\sqrt{2}}[M_4\vec{Q}_4 - \vec{S}_4]. \tag{3.8}$$

Here $\hat{a}_{0R}=\mathrm{Re}\,\hat{a}_0$, $\hat{a}_{0I}=\mathrm{Im}\,\hat{a}_0$, etc., the matrix M_4 is given by

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -3I_4 \\ 2I_0 & 0 & -2I_2 & 0 & 9I_4 & 0 \\ 0 & 0 & 0 & \frac{1}{4}I_0 + I_2 + \frac{3}{8}I_4 & 0 & -\frac{9}{4}I_2 - \frac{15}{8}I_6 \\ -\frac{1}{4}I_2 & 0 & -\frac{1}{4}I_0 - I_2 + \frac{3}{8}I_4 & 0 & \frac{9}{4}I_2 - \frac{15}{8}I_6 & 0 \\ 0 & 0 & 0 & -\frac{3}{64}I_2 - \frac{5}{128}I_6 & 0 & \frac{29}{64}I_0 + 2I_2 + \frac{35}{128}I_8 \\ 0 & 0 & \frac{3}{64}I_2 - \frac{5}{128}I_6 & 0 & -\frac{29}{64}I_0 - 2I_2 + \frac{35}{128}I_8 & 0 \end{pmatrix}, \tag{3.9}$$

and the vector $\vec{S}_4$ is given by

$$\vec{S}_4 = \left[-\Lambda_1, 0, -\frac{\delta}{8}, 0, -\frac{I_4}{128}, 0 \right]^T. \tag{3.10}$$

The stationary solution is found by setting the left-hand side of Eq. (3.8) to zero, which immediately yields

$$\hat{a}_{0I} = \hat{a}_{2I} = \hat{a}_{4I} = 0. \tag{3.11}$$

Then $\hat{a}_{0R}$, $\hat{a}_{2R}$, and $\hat{a}_{4R}$ satisfy the following reduced linear system:

$$\begin{pmatrix} 2I_0 & -2I_2 & 9I_4 \\ -\frac{1}{4}I_2 & -\frac{1}{4}I_0 - I_2 + \frac{3}{8}I_4 & \frac{9}{4}I_2 - \frac{15}{8}I_6 \\ 0 & \frac{3}{64}I_2 - \frac{5}{128}I_6 & -\frac{29}{64}I_0 - 2I_2 + \frac{35}{128}I_8 \end{pmatrix} \begin{pmatrix} \hat{a}_{0R} \\ \hat{a}_{2R} \\ \hat{a}_{4R} \end{pmatrix} = \begin{pmatrix} -\Lambda_1 \\ -\frac{1}{8}\delta \\ -\frac{1}{128}I_4 \end{pmatrix}. \tag{3.12}$$

We now use the freedom in choosing Λ_1 and δ in order to set

$$\hat{a}_{0R} = 0, \quad \hat{a}_{2R} = 0. \tag{3.13}$$

The meaning of the first of these conditions is obvious: We simply require that the stationary amplitude of the zeroth harmonic identically equal A. The second condition in Eq. (3.13) amounts to the requirement that the width $W(z)$ of the stationary DM soliton, defined as

$$W^2(z) = \left(\int_{-\infty}^{\infty} \tau^2 |u|^2 d\tau \right) \Big/ \left(\int_{-\infty}^{\infty} |u|^2 d\tau \right), \tag{3.14}$$

be (approximately) equal to $T_0\sqrt{1+\Delta^2/T_0^4}$, i.e., the width of its zeroth harmonic. Indeed, from Eq. (2.11) one has

$$|u|^2 \approx |u^{(0)}|^2 \approx \left[\left| a_0^{(0)} c_0^{(0)} \right|^2 + \sum_{n\geq 2,\ n\ \text{even}} (a_0^{(0)} a_n^{(0)*} c_0^{(0)} c_n^{(0)*} + \text{c.c.}) H_n(\xi) \right] e^{-\xi^2}, \tag{3.15}$$

where we have used the strong inequality (2.16) and thus neglected terms quadratic in $a_n^{(0)}$ for $n \geq 2$. Then only the first term and the term with $n=2$ in the sum on the right-hand side of Eq. (3.15) contribute to the numerator in the definition (3.14), the other terms being orthogonal to τ^2 due to the orthogonality of the Hermite polynomials [cf. Eq. (2.19c)]. Thus, when $\hat{a}_{2R}=0$, then within the approximation considered, the width (3.14) of the DM soliton equals the width of its leading-order, Gaussian harmonic.

From Eqs. (3.13) and (3.12), one easily finds that

$$\hat{a}_{4R} = \frac{I_4}{128} \left[\frac{29}{64} I_0 + 2 I_2 - \frac{35}{128} I_4 \right]^{-1}, \tag{3.16}$$

and

$$\Lambda_1 = -9 I_4 \hat{a}_{4R}, \tag{3.17a}$$

$$\delta = (15 I_6 - 18 I_2) \hat{a}_{4R}. \tag{3.17b}$$

Note that $\hat{a}_{4R}$ is the function of the only parameter T_0. This function is plotted in Fig. 1 with a solid line.

Now, we have to answer the following important question: Will taking into account higher harmonics (with $n \geq 6$) significantly modify the results just obtained? This, in fact, involves two subquestions. First, is the expression (3.16) for $\hat{a}_{4R}$ significantly modified when harmonics with $n \geq 6$ are taken into consideration? Second, does the contribution of those higher harmonics to the pulse shape exceed, or become comparable with, the contribution of the harmonic with $n=4$? A rigorous answer to those questions requires estimation of the sizes of $a_n^{(0)}$ for *all* $n \geq 6$ and for all T_0. This we have been unable to do (see, however, [45]), since the complexity of the system (2.21), truncated at a certain *n*, very rapidly increases with the increase of *n*. Instead, we restricted our attention to just one next harmonic, i.e., that with $n=6$, and repeated the preceding analysis. We found that whenever $T_0 \gtrsim (T_0)_{\text{cr}} \approx 0.39$, the answers to both questions above are negative. To be more specific, in Fig. 1 we also plotted (dashed line) the quantity $\hat{a}_{4R}$ calculated when the harmonic with $n=6$ is taken into account. The two curves in Fig. 1 are seen to be sufficiently close to each other when $T_0 \gtrsim 0.4$. Also, we found that for $T_0 \gtrsim 0.4$, the ratio of the amplitudes $|a_6^{(0)}|/|a_4^{(0)}| < 0.01$ (except near the point where $a_4^{(0)}$ vanishes), and thus the ratio of the magnitudes of the 6th and 4th harmonics in the stationary soliton is less than 10%:

$$|a_6^{(0)} H_6(\xi) e^{-\xi^2/2}| / |a_4^{(0)} H_4(\xi) e^{-\xi^2/2}| < 0.1 \quad \text{for} |\xi| < 3.5.$$

Although harmonics with $n \geq 8$ also modify Eq. (3.16) and contribute to the shape of the DM soliton, their contributions are expected to be less than that of the harmonic with $n=6$ [43]. Thus, we conclude that when $T_0 \gtrsim (T_0)_{\text{cr}} \approx 0.39$, the shape of the stationary DM soliton in the interval $|\xi| < 3.5$ can be rather accurately approximated by just the two harmonics with $n=0$ and $n=4$.

In Fig. 2, we plotted the stationary profiles of the DM soliton at the point where its chirp vanishes [i.e., where $\Delta(z)=0$]. These profiles were calculated using the two-term truncation of the general solution (2.11) with $|a_0^{(0)}| \equiv A = 1$:

$$u^{(0)} \approx [1 + \hat{a}_{4R} H_4(\xi)] e^{-\xi^2/2} \tag{3.18}$$

for three different values of T_0. For a relatively large $T_0 = 2.5$, Eq. (3.16) yields $\hat{a}_{4R} = 0.0035$. This is very close to the value of the coefficient $s_4 \approx 0.0037$ in the Hermite-Gaussian expansion of the sech-shaped NLS soliton:

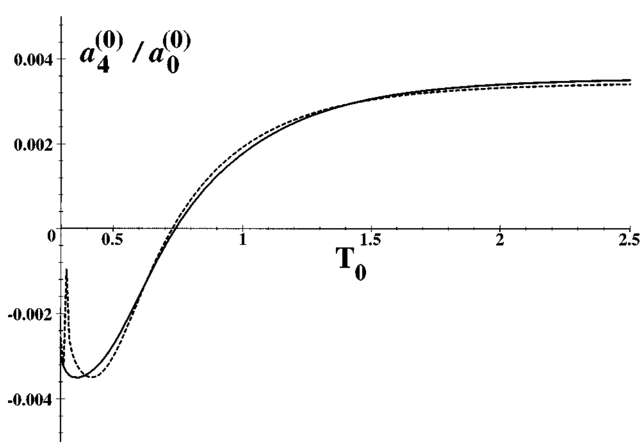

FIG. 1. Solid line: Magnitude of the fourth harmonic $\hat{a}_4$, as given by Eq. (3.16). Dashed line: Same quantity calculated when the harmonic with $n=6$ is taken into account.

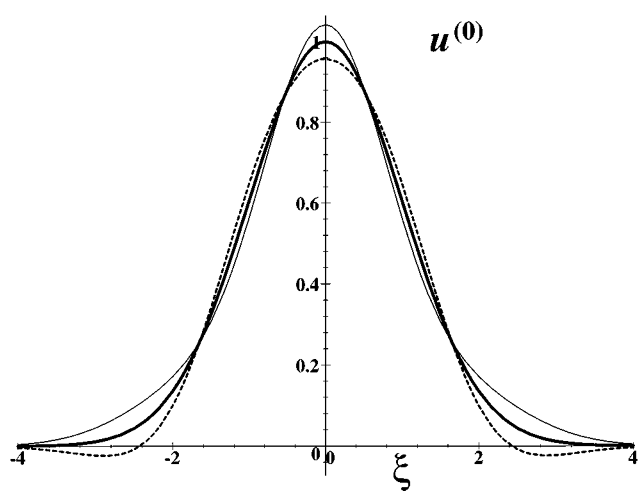

FIG. 2. Shapes of the stationary DM soliton at the point where $\Delta(z)=0$. Thin solid line: $T_0=2.5$, $\hat{a}_4=0.0035$. Thick solid line: $T_0=0.74$, $\hat{a}_4=0$. Dashed line: $T_0=0.39$, $\hat{a}_4=-0.0035$.

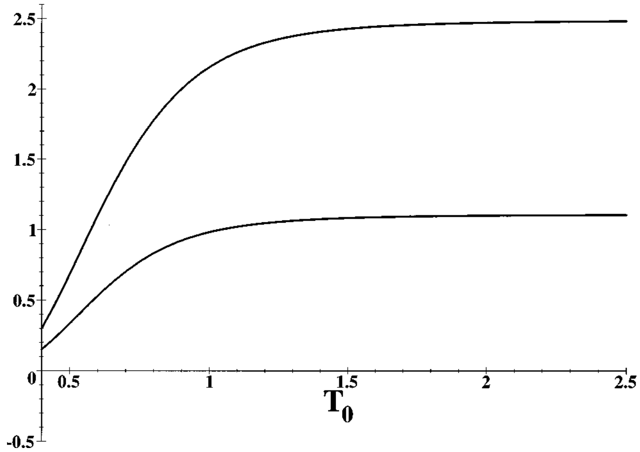

FIG. 3. Nonzero eigenfrequencies of small oscillations near the stationary solution of Eq. (3.8).

$$A_0 \operatorname{sech}(\xi/w_0) = \sum_{n=0}^{\infty} s_n H_n(\xi) e^{-\xi^2/2}, \tag{3.19}$$

where the constants $A_0 \approx 1.057$ and $w_0 = \sqrt{2/\pi} \approx 0.798$ are chosen so as to have $s_0 = 1$ and $s_2 = 0$. (One can also show that in that case, $s_6 = 0$.) Thus, for $T_0 \gg 1$ and for not too large $|\xi|$, the truncated expansion (3.18) yields an approximately sech-like profile (thin solid line in Fig. 2). This is consistent with Remark 5 in Sec. II B about the limit $T_0 \to \infty$. Next, as the minimum width T_0 decreases, the size of the fourth harmonic initially decreases, and the DM soliton becomes closer to a Gaussian. At $T_0 \approx 0.74$, the fourth harmonic vanishes, and sufficiently close to its center the pulse is very close to a Gaussian (thick solid line in Fig. 2). The contribution of the higher harmonics is expected to become more pronounced at the pulse ''tails'' far enough from its center. Finally, for $T_0 < 0.74$, $\hat{a}_{4R}$ becomes negative, and the pulse develops ''wings.'' Also note that since [39]

$$\int_{-\infty}^{\infty} e^{-i\omega\xi} H_n(\xi) e^{-\xi^2/2} d\xi = \frac{(i)^n}{\sqrt{2\pi}} H_n(\omega) e^{-\omega^2/2},$$

then Eq. (3.18) also yields an approximate spectrum of the DM soliton. This explains property (ii) stated in Sec. I. Namely, Eqs. (3.16) and (3.18) describe a smooth transition of the pulse shape from a hyperbolic secant to the Gaussian and then further to a shape with a higher time–bandwidth product. The latter part of the transition is manifested by the corresponding spectrum being broader at the top as T_0 decreases. Note that property (iii) is already explained by expansion (2.11), from which it is clear that the pulse shape will change depending on the value of $\Delta(z)$ inside the map.

We also analyzed small oscillations about the stationary solution of Eq. (3.8). The spectrum of these oscillations contains two zero eigenfrequencies and two pairs $(\pm\omega)$ of nonzero eigenfrequencies. The latter are real for all $T_0 \geqslant (T_0)_{\rm cr}$. In Fig. 3, we plotted the two different positive eigenfrequencies of the system (3.8). When one takes into account the next even harmonic $(n=6)$, then one more branch of oscillations, with a higher frequency, arises, while the frequencies of the two ''old'' branches may change by no more than 15% each. The two zero eigenfrequencies persist, as one can show, when *any* number of harmonics is taken into account. This fact will play an important part in Sec. IV.

Since a stationary DM soliton is an even function of ξ, then the stationary values of all its odd harmonics are zeros. Following the lines above, we can arrive at a linear system of the form (3.8) for the vector $\vec{Q}_{2k-1} = [\hat{a}_{1R}, \hat{a}_{1I}, \ldots, \hat{a}_{(2k-1)R}, \hat{a}_{(2k-1)I}]^T$, which describes small oscillations involving the first k odd harmonics. This system must always have two zero eigenmodes (i.e. the modes with zero eigenfrequencies). One of them, $\partial u^{(0)}/\partial\tau$, corresponds to a translation along τ, and the other, $\partial u^{(0)}/\partial\omega_0$, corresponds to a shift of the soliton's frequency. Let $\overline{a_n^{(0)}}$ be the coefficients in the expansion (2.11) of the stationary solution $u^{(0)}$. Then the above two zero eigenmodes can be shown to be as follows:

$$\frac{\partial u^{(0)}}{\partial\tau} = \sum_{n=1,\ n\ \text{odd}}^{\infty} \left[-\frac{1}{2}\overline{a_{n-1}^{(0)}} + (n+1)\overline{a_{n+1}^{(0)}} \right] \times c_n^{(0)}(z) H_n(\xi) e^{i\Psi}, \tag{3.20a}$$

$$\frac{\partial u^{(0)}}{\partial\omega_0} = iT_0 \sum_{n=1,\ n\ \text{odd}}^{\infty} \left[\frac{1}{2}\overline{a_{n-1}^{(0)}} + (n+1)\overline{a_{n+1}^{(0)}} \right] \times c_n^{(0)}(z) H_n(\xi) e^{i\Psi}, \tag{3.20b}$$

and the phase Ψ is the same as in the expansion (2.11).

The system

$$\dot{\vec{Q}}_1 = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix} \vec{Q}_1 \tag{3.21}$$

for just the first odd harmonic clearly does have those two zero eigenmodes. However, once we include higher harmonics (e.g., consider $\vec{Q}_5 = [\hat{a}_{1R}, \ldots, \hat{a}_{5I}]^T$), then the resulting truncated system no longer has modes with exactly zero eigenfrequencies. Instead, for $\vec{Q}_3$ and $\vec{Q}_5$ we verified, using MAPLE, that there are two eigenfrequencies whose magnitude is on the order of 0.1 and which may be complex for some T_0. The other eigenfrequencies are of order one and always real for all $T_0 \geqslant (T_0)_{\rm cr}$. The nonvanishing of the two eigenfrequencies in question can be attributed to our truncation of an infinite, coupled system for $\vec{Q}$ into a finite-dimensional system, and thus is a deficiency of our approach rather than an indication of any actual instability of the DM soliton. In fact, its long-term stability has been verified in many numerical simulations of Eq. (2.1).

IV. PERTURBATION THEORY FOR THE DM SOLITON

A. Results

As noted in Sec. I, the main advantage of the DM technique is that it significantly reduces the effect of perturbations on a soliton. Below we derive explicit expressions that allow one to calculate this effect for a z periodic (with period $L_{\rm map} \equiv 1$), but otherwise arbitrary perturbation $R(\tau, z)$. Consider a perturbed equation (2.1)

$$iu_z + \frac{1}{2}D(z)u_{\tau\tau} + \epsilon\left[\frac{1}{2}D_0 u_{\tau\tau} + G(z)u|u|^2\right] = \epsilon\mu R. \tag{4.1}$$

where $\mu \ll 1$ is a small parameter characterizing the size of the perturbation. The magnitudes of ϵ and μ are not supposed to be related, although for the consistency of our approach, we should require that $\mu \gg \epsilon$, in order to neglect terms of order ϵ^2. Following the approach of Sec. II, one obtains modified equations for the amplitudes $a_n^{(0)}$:

$$i\dot{a}_n^{(0)} + \{\text{as before}\} = \mu R_n, \tag{4.2}$$

where

$$R_n(z_1) = \int_0^1 dz \frac{[c_n^{(0)}(z)]^{-1}}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} d\xi R(\xi,z) H_n(\xi) e^{-\xi^2/2 - i\Psi}, \tag{4.3}$$

and the phase Ψ is the same as in the expansion (2.11) [see also Eqs. (3.20)]. The solution to system (4.2) is sought in the form

$$a_n^{(0)} = \overline{a_n^{(0)}} + \mu \tilde{a}_n^{(0)}, \tag{4.4}$$

where $\overline{a_n^{(0)}} = \overline{a_n^{(0)}}(T_0)$ are the expansion coefficients of the stationary unperturbed solution $u^{(0)}$. As before, conditions (2.22), which now need to hold only in the zeroth order in μ, guarantee that the equations for even- and odd-numbered $\tilde{a}_n^{(0)}$ are decoupled. However, we now allow the parameters Δ_0, T_0, ω_0, $\{\dot{\tau}_c - \omega_0 D_0 - \dot{\omega}_0[\frac{1}{2}\text{sgn}(D_1L_1) + \Delta_0]\}$, and $\varphi_0 = \phi - \omega_0^2 D_0 z/2$ [cf. the coefficients of $a_{n\pm1}^{(0)}$ and $a_n^{(0)}$ in Eq. (2.21)] to vary at a rate proportional to μ. Consequently, the amplitudes $\overline{a_n^{(0)}}(T_0)$ can also vary at the same rate. Note that the variations of Δ_0 and T_0 are related by Eq. (2.29a) and thus are not independent. In order to simplify the subsequent analysis and arrive at explicit expressions, below we will only consider the case of a lossless fiber, whereby one has $\Delta_0 = -\frac{1}{2}\text{sgn}(D_1L_1)$ for all T_0, and thus $\dot{\Delta}_0 = O(\mu^2)$.

Since our calculations in Sec. III have been carried out for the harmonics with $n \leq 4$, below we restrict the analysis only to those harmonics. For the vectors

$$\vec{\tilde{\mathcal{Q}}} = e^{-i\Lambda z_1}[\tilde{a}_{0R}^{(0)}, \tilde{a}_{0I}^{(0)}, \ldots]^T$$

and

$$\vec{\overline{\mathcal{Q}}} = e^{-i\Lambda z_1}[\overline{a_{0R}^{(0)}}, \overline{a_{0I}^{(0)}}, \ldots]^T,$$

where Λ is defined by Eqs. (3.2), one obtains the following linear system:

$$\dot{\vec{\tilde{\mathcal{Q}}}} = M\vec{\tilde{\mathcal{Q}}} + \vec{\mathcal{S}} - \dot{\vec{\overline{\mathcal{Q}}}} + \mu\vec{\mathcal{R}}. \tag{4.5}$$

First, we specify the form of various terms in Eq. (4.5) for even harmonics. In that case, matrix M is given by Eq. (3.9). The vector $\vec{\mathcal{S}}$ of the slow derivatives and the perturbation vector $\vec{\mathcal{R}}$ are given by the following formulas:

$$\vec{\mathcal{S}} = \left[-\frac{\dot{T}_0}{2T_0}\overline{a_0^{(0)}}, -\dot{\varphi}_0\overline{a_0^{(0)}}, -\frac{\dot{T}_0}{4T_0}(\overline{a_0^{(0)}} - 48\overline{a_4^{(0)}}), 0, \frac{\dot{T}_0}{2T_0}(-\overline{a_4^{(0)}} + 60\overline{a_6^{(0)}}), -\dot{\varphi}_0\overline{a_4^{(0)}} \right]^T, \tag{4.6}$$

$$\vec{\mathcal{R}} = [R_{0I}, -R_{0R}, R_{2I}, -R_{2R}, R_{4I}, -R_{4R}]^T, \tag{4.7}$$

where $R_{nR} = \text{Re}\, R_n, R_{nI} = \text{Im}\, R_n$. Note that $\overline{a_{nI}^{(0)}} = 0$ for all n and, in addition, $\overline{a_{nR}^{(0)}} = 0$ for all odd n. The stationary amplitude $|\overline{a_0^{(0)}}|$ of the zeroth harmonic was denoted as A in Sec. III.

Since Eq. (4.5) is linear, it can be analyzed in a great number of ways. Here, we chose to follow the method that emphasizes both similarities to and differences from the perturbation theory for the NLS (see, e.g., [46] and remark 1 below). Specifically, the solution for Eq. (4.5) is sought in the form

$$\vec{\tilde{\mathcal{Q}}} = k_0\vec{q}_0 + \left(k_0^A \vec{q}_0^A + \int_0^{z_1} k_0^A(s)ds\vec{q}_0 \right) + \sum_{n=1}^{4} k_n\vec{q}_n e^{\lambda_n z_1}. \tag{4.8}$$

Here $\vec{q}_n$ and λ_n are the eigenvectors and the eigenvalues of M_4:

$$M_4\vec{q}_n = \lambda_n\vec{q}_n, \quad n = 0, \ldots, 4 \tag{4.9}$$

with $\lambda_0 = 0$, and $\vec{q}_0^A$ is the associate eigenvector satisfying

$$M_4\vec{q}_0^A = \vec{q}_0. \tag{4.10}$$

The expansion coefficients $k_n (n=0, \ldots, 4)$ and k_0^A are functions of z_1. We will only require the forms of $\vec{q}_0$ and $\vec{q}_0^A$, which can be easily found:

$$\vec{q}_0 = [0,1,0,0,0,0]^T, \quad \vec{q}_0^A = [\hat{x}_{0R}, 0, \hat{x}_{2R}, 0, \hat{x}_{4R}, 0]^T, \tag{4.11}$$

where $\hat{x}_{0R}, \hat{x}_{2R}, \hat{x}_{4R}$ satisfy the modified system

$$M_{4R} \begin{pmatrix} \hat{x}_{0R} \\ \hat{x}_{2R} \\ \hat{x}_{4R} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{4.12}$$

and the matrix M_{4R} is that on the rhs of Eq. (3.12). The explicit form of the entries of $\vec{q}_0^A$, that involves the integrals $I_n(T_0)$, is rather cumbersome. However, from the last equation of the system (4.12) one can conclude that

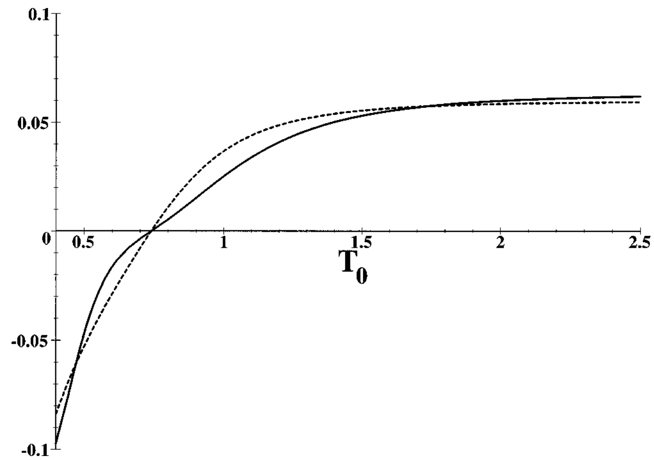

FIG. 4. Solutions of Eq. (4.16). Solid line: $\hat{y}_{2R}$. Dashed line: $\hat{y}_{4R}/20$.

$$\hat{x}_{4R} \approx \tfrac{1}{20} \hat{x}_{2R}, \tag{4.13}$$

which can indeed be confirmed by plotting these two quantities as functions of T_0 for $T_0 \geqslant (T_0)_{\rm cr} \approx 0.39$.

Next, one considers the solutions of the eigenvalue problem

$$-M_4^T \vec{p}_n = \lambda_n \vec{p}_n, \tag{4.14}$$

which is adjoint to the eigenvalue problem (4.9). Since the eigenvlaues of M_4 come in pairs as $\pm\lambda$, then the eigenvalues of Eq. (4.14) are the same as those of Eq. (4.9). In particular, $\lambda = 0$ is a double eigenvalue of Eq. (4.14), with the eigenvector $\vec{p}_0$ and the associate eigenvector $\vec{p}_0^A$ being given by

$$\vec{p}_0 = [1, 0, \hat{y}_{2R}, 0, \hat{y}_{4R}, 0]^T, \quad \vec{p}_0^A = [0, \hat{y}_{0I}, 0, \hat{y}_{2I}, 0, \hat{y}_{4I}]^T. \tag{4.15}$$

The entries in Eq. (4.15) satisfy the following systems:

$$\begin{pmatrix} \frac{1}{4}I_0 + I_2 + \frac{3}{8}I_4 & -\frac{3}{64}I_2 - \frac{5}{128}I_6 \\ -\frac{9}{4}I_2 - \frac{15}{8}I_6 & \frac{29}{64}I_0 + 2I_2 + \frac{35}{128}I_8 \end{pmatrix} \begin{pmatrix} \hat{y}_{2R} \\ \hat{y}_{4R} \end{pmatrix} = \begin{pmatrix} 0 \\ 3I_4 \end{pmatrix}, \tag{4.16}$$

$$M_{4R}^T \begin{pmatrix} \hat{y}_{0I} \\ \hat{y}_{2I} \\ \hat{y}_{4I} \end{pmatrix} = \begin{pmatrix} 1 \\ \hat{y}_{2R} \\ \hat{y}_{4R} \end{pmatrix}. \tag{4.17}$$

From the first line of the system (4.16) one can estimate that

$$\hat{y}_{2R} \approx \tfrac{1}{20} \hat{y}_{4R}, \tag{4.18}$$

which can be confirmed by plotting these quantities for $T_0 \geqslant 0.39$ (see Fig. 4). Similarly, the quantities $\hat{y}_{0I}, \hat{y}_{2I}, \hat{y}_{4I}$ can be shown to all have magnitudes of order one in the same range of T_0.

In the standard way, one can establish the following orthogonality relations between the vectors $\vec{p}$ and $\vec{q}$:

$$\vec{p}_k^T \vec{q}_n = 0 \quad \text{if} \quad \lambda_k \neq -\lambda_n, \tag{4.19}$$

$$(\vec{p}_0^A)^T \vec{q}_k = \vec{p}_k^T \vec{q}_0^A = 0 \qquad \text{for all} \quad \lambda_k \neq 0. \tag{4.20}$$

One can also directly verify that

$$\vec{p}_0^T \vec{q}_0 = (\vec{p}_0^A)^T \vec{q}_0^A = 0, \tag{4.21a}$$

$$\vec{p}_0^T \vec{q}_0^A = (\vec{p}_0^A)^T \vec{q}_0 \approx \hat{x}_{0R}, \tag{4.21b}$$

where the approximate equality in Eq. (4.21b) is obtained with the use of Eqs. (4.17), (4.13), and (4.18). Moreover, from Eqs. (4.12) and (4.13) one finds that

$$\hat{x}_{0R} \approx \frac{-\frac{1}{4}I_0 - I_2 + \frac{3}{8}I_4}{-\frac{1}{2}I_0^2 - 2I_0I_2 + \frac{3}{4}I_0I_4 - \frac{1}{2}I_2^2},$$

$$\hat{x}_{2R} \approx \frac{\frac{1}{4}I_2}{-\frac{1}{2}I_0^2 - 2I_0I_2 + \frac{3}{4}I_0I_4 - \frac{1}{2}I_2^2}, \tag{4.22}$$

where the expression for $\hat{x}_{2R}$ is given in view of its possible use in future work. By plotting the expression for $\hat{x}_{0R}$ for $T_0 \geqslant 0.39$, one can show that $\hat{x}_{0R} \neq 0$ in this range of T_0.

Finally, we substitute the expansion (4.8) into Eq. (4.5) and use the orthogonality relations (4.19)–(4.21) to obtain the following equations:

$$(\vec{p}_0^A)^T \vec{q}_0 \, \dot{k}_0 = (\vec{p}_0^A)^T (\mu \vec{\mathcal{R}} + \vec{\mathcal{S}} - \dot{\tilde{\vec{\mathcal{Q}}}}), \tag{4.23a}$$

$$\vec{p}_0^T \vec{q}_0^A \dot{k}_0^A = \vec{p}_0^T (\mu \vec{\mathcal{R}} + \vec{\mathcal{S}} - \dot{\tilde{\vec{\mathcal{Q}}}}). \tag{4.23b}$$

In order for the perturbation vector $\tilde{\vec{\mathcal{Q}}}$ not to grow secularly in z_1, one must require that $\dot{k}_0 = \dot{k}_0^A = 0$, which with the use of Eq. (4.6) yields

$$\begin{aligned} \frac{\dot{T}_0}{2T_0} \Bigg[& \left(\overline{a_0^{(0)}} + 2T_0 \frac{d\overline{a_0^{(0)}}}{dT_0} \right) + \frac{1}{2} (\overline{a_0^{(0)}} - 48 \overline{a_4^{(0)}}) \hat{y}_{2R} \\ & + \left(\overline{a_4^{(0)}} - 60 \overline{a_6^{(0)}} + 2T_0 \frac{d\overline{a_4^{(0)}}}{dT_0} \right) \hat{y}_{4R} \Bigg] \\ = {} & \vec{p}_0^T \mu \vec{\mathcal{R}}, \end{aligned} \tag{4.24}$$

$$\dot{\varphi}_0 (\overline{a_0^{(0)}} \hat{y}_{0I} + \overline{a_4^{(0)}} \hat{y}_{4I}) = (\vec{p}_0^A)^T \mu \vec{\mathcal{R}} \tag{4.25}$$

(recall that $\overline{a_{nI}^{(0)}} = 0$ for all n). In Eq. (4.24), the magnitude of the second term in the square brackets is no more than about 6% of the magnitude of the first one since $|\hat{y}_{2R}|$ is small (cf. Fig. 4). The magnitude of the third term is less than 1% of that of the first term since $\overline{a_4^{(0)}}$ and $\overline{a_6^{(0)}}$ are small too [cf. Eq. (3.16) and Fig. 1]. Moreover, since the signs of $\overline{a_4}$ in the second and the third terms are opposite, then the magnitude of these terms *combined* does not exceed 5% of the magnitude of the first term. Thus, with a 5% accuracy, one has

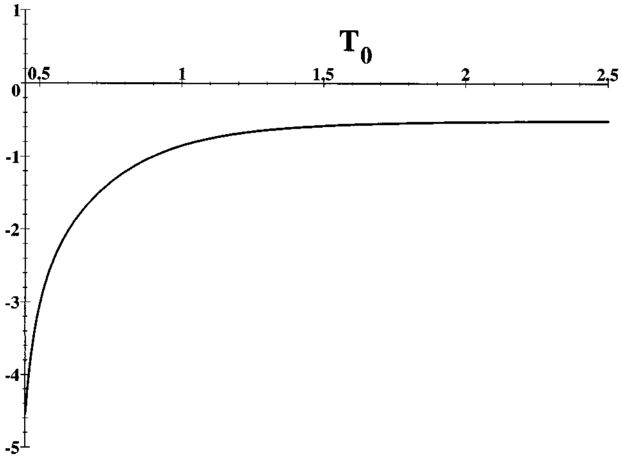

FIG. 5. The function in Eq. (4.30). Note that in this figure, the smallest value of $T_0=0.45>(T_0)_{\rm cr}$ in order to avoid the singularity at $T_0=(T_0)_{\rm cr}$.

$$\dot{T}_0\left(\frac{\overline{a_0^{(0)}}}{2T_0}+\frac{d\overline{a_0^{(0)}}}{dT_0}\right)\approx\vec{p}_0^{\,T}\mu\vec{\mathcal{R}}, \tag{4.26}$$

and similarly,

$$\dot{\varphi}_0\overline{a_0^{(0)}}\hat{y}_{0I}\approx(\vec{p}_0^{\,A})^T\mu\vec{\mathcal{R}}. \tag{4.27}$$

Note that if the first two entries of the perturbation vector $\vec{\mathcal{R}}$ have a much larger magnitude than its other entries, then Eqs. (4.26) and (4.27) can be simplified even further:

$$\frac{\dot{T}_0}{T_0}\left(\frac{\overline{a_0^{(0)}}}{2}+T_0\frac{d\overline{a_0^{(0)}}}{dT_0}\right)\approx\mu R_{0I}, \tag{4.28}$$

$$\dot{\varphi}_0\overline{a_0^{(0)}}\approx-\mu R_{0R}, \tag{4.29}$$

where R_{0I} and R_{0R} are defined after Eq. (4.7). Using Eq. (2.25), the coefficient of $(\dot{T}_0/T_0)$ in Eq. (4.28) is cast into the form

$$\left(\frac{\overline{a_0^{(0)}}}{2}+T_0\frac{d\overline{a_0^{(0)}}}{dT_0}\right)=-\frac{\overline{a_0^{(0)}}}{2I_2}\frac{d(I_2T_0)}{dT_0}, \tag{4.30}$$

which is plotted in Fig. 5 for $\overline{a_0^{(0)}}=1$. Note that for large T_0, when $I_2\to1$, this function tends to the NLS limit, which equals $-1/2$.

Now, as we turn to the analysis of Eq. (4.5) for the odd harmonics, we note the following. First, as mentioned in Sec. III, the analog of Eq. (3.9) that would take into account the 1st, 3rd, and 5th harmonics does *not* have zero eigenmodes, i.e., such eigenmodes whose eigenvalues are exactly zero. It is only Eq. (3.21) that has such eigenmodes. Second, the preceding analysis for the even harmonics has shown that the dominant contribution to the perturbation equations (4.26) and (4.27) comes from the lowest harmonic, i.e., $a_0^{(0)}$. Similarly, we can expect that it is the lowest odd harmonic, $a_1^{(0)}$, that should contribute the most to the perturbed evolution of the parameters ω_0 and τ_c. Thus, we consider Eq. (4.5), where now

$$M=\begin{pmatrix}0 & I_2\\ 0 & 0\end{pmatrix}, \tag{4.31}$$

$$\vec{\mathcal{S}}=\left[-\frac{\overline{a_0^{(0)}}}{2T_0}(\dot{\tau}_c-\omega_0D_0),-\frac{\overline{a_0^{(0)}}T_0}{2}\dot{\omega}_0\right]^T, \tag{4.32}$$

and

$$\vec{\mathcal{R}}=[R_{1I},-R_{1R}]^T. \tag{4.33}$$

Then, similarly to the even-harmonics case, we obtain

$$\frac{\overline{a_0^{(0)}}}{2T_0}(\dot{\tau}_c-\omega_0D_0)=\mu R_{1I}, \tag{4.34}$$

$$\frac{\overline{a_0^{(0)}}T_0}{2}\dot{\omega}_0=-\mu R_{1R}. \tag{4.35}$$

Finally, as a simple application of our perturbation theory, we consider the effect of fixed-frequency filters on a DM soliton. The equation in question is then

$$iu_z+\frac{1}{2}D(z)u_{\tau\tau}+\epsilon\left[\frac{1}{2}D_0u_{\tau\tau}+G(z)u|u|^2\right]=\epsilon\left(i\gamma u+i\frac{\beta}{2}u_{\tau\tau}\right), \tag{4.36}$$

where β is the filter strength and γ is the excess gain needed to compensate for the effective diffusion introduced by the filtering. Both β and γ are assumed to be small: $\beta,\gamma=O(\mu)$. The calculation of the γ term in the perturbation vector $\vec{\mathcal{R}}$ is straightforward, and that of the β term is also carried out easily upon replacing D_0 with $(D_0-i\beta)$ in Eq. (2.21). The relevant components of $\vec{\mathcal{R}}$ are as follows

$$R_0=i\left[\gamma-\frac{\beta}{2}\left(\frac{1}{2T_0^2}+\omega_0^2\right)\right]\overline{a_0^{(0)}}, \tag{4.37a}$$

$$R_1=\frac{\omega_0\beta}{T_0}\overline{a_0^{(0)}}, \tag{4.37b}$$

$$R_2=i\frac{\beta}{8T_0^2}(\overline{a_0^{(0)}}+48\overline{a_4^{(0)}}), \tag{4.37c}$$

$$R_4=i\left[\gamma-\frac{\beta}{2}\left(\frac{9}{4T_0^2}+\omega_0^2\right)\right]\overline{a_4^{(0)}}+i\frac{15\beta}{T_0^2}\overline{a_6^{(0)}}. \tag{4.37d}$$

Following the argument presented after Eq. (4.25), we observe that the perturbation equations (4.26) can be reasonably well approximated by Eq. (4.28). Thus, one finds

$$\dot{T}_0\frac{d(\overline{a_0^{(0)2}}T_0)/dT_0}{(\overline{a_0^{(0)2}}T_0)}\approx2\gamma-\beta\left(\omega_0^2+\frac{1}{2T_0^2}\right), \tag{4.38a}$$

$$\dot{\omega}_0\approx-\frac{2\beta}{T_0^2}\omega_0. \tag{4.38b}$$

Equations that are equivalent to these were earlier obtained by Matsumoto [14] (see also [13]) from the conservation laws for Eq. (4.36); cf. remark 3 below.

B. Remarks

First, we point out the difference of the above perturbation theory from that for the NLS soliton. In the latter case, the zero eigenmodes of the linearized NLS correspond to the infinitesimal shifts of the four soliton parameters: the amplitude (same as the inverse width), the overall phase, the velocity (same as the frequency), and the center coordinate (see, e.g., [46]). In the case of the perturbed DM soliton, the last two shift modes are easily recognized in the expressions in Eqs. (3.20). However, as we mentioned in Sec. III, these eigenmodes are indeed the zero eigenmodes for the corresponding truncated system of equations *only* when the truncation is done at the lowest odd harmonic (i.e., at that with $n=1$). As for the shifts corresponding to the amplitude and the overall phase, those are not the zero modes of their corresponding truncated system for two different reasons. First, that system's coefficients depend on T_0, and hence $\partial u_0^{(0)}/\partial T_0$ is not a zero mode. Second, the stationary value of the coefficient $a_4^{(0)}$ is found from the equation with a nonzero rhs [Eq. (3.12)], hence $\partial u_0^{(0)}/\partial \varphi_0$ is not a zero mode.

Second, the form of the function in Eq. (4.30) and that of matrix M in Eq. (4.31) both present additional evidence that the results obtained in this section can be rigorously justified only for $T_0>(T_0)_{\rm cr}\approx 0.39$, since $I_2((T_0)_{\rm cr})=0$.

Third, the perturbation equations (4.28), (4.29), (4.34), and (4.35) appear to be the same as the analoguous equations that are obtained using either conservation laws or the variational method for Eq. (4.1), with the form of the unperturbed solution being taken as the Gaussian [14,13,16]. This confirms the validity of the results obtained in Refs. [14,13]. It also explains the very good agreement, found in [5], between the experimentally measured GH jitter of the DM soliton and its theoretical estimate, which was obtained in Ref. [5] by simply dividing the corresponding formula for the NLS soliton by the energy enhancement factor (cf. the discussion in [16]). We emphasize that even if one were to use the *exact* profile of the DM soliton to obtain the perturbation equations from the conservation laws or the variational method, those equations would still hold only approximately (although with good accuracy, as explained above). The reason for that, which involves the orthogonality of the radiation modes to the soliton, was discussed in [16].

V. CONCLUSIONS

In this work, we have used the expansion (2.11) over the appropriate set of Hermite-Gaussian functions to represent a pulse propagating in the strong DM regime. We found (Sec. II) that with just the first two lowest even harmonics, one obtains the same conditions for the pulse stationary propagation as were earlier obtained by the variational method. Taking into account the next even harmonic, we found the correction [given by Eqs. (3.3) and (3.17b)] to the above conditions of stationary propagation. Note that the amplitude of the DM soliton can be found from Eqs. (3.18) and (3.16), whereas the width, given by Eq. (3.14), is the same as the width of its zeroth (purely Gaussian) harmonic, i.e., it equals $T_0\sqrt{1+\Delta^2/T_0^4}$. Moreover, Eqs. (3.18) and (3.16) provide an accurate approximation for the soliton's shape not too far from its center ($|\xi|<3.5$) and when the minimum width T_0 of the soliton exceeds a certain threshold value. That threshold value, $(T_0)_{\rm cr}\approx 0.39$, appears to be the same as (or, at least, very close to) the threshold where the average dispersion in the system turns negative in order to support (quasi-)stationary propagation of a DM soliton. For narrower pulses with $T_0<(T_0)_{\rm cr}$, higher harmonics (with $n\geqslant 6$) significantly contribute to the pulse shape.

We emphasize that all the results obtained in this work are explicit functions of the only parameter T_0, provided that the parameters of the dispersion map are fixed and there is assumed to be no losses in the fiber. [When the periodically compensated losses are included into consideration, the above results can still be obtained by evaluating a small number of certain definite integrals; cf. Eqs. (2.29) and (2.20).] Note that instead of varying the minimum pulse-width T_0 and keeping the dispersion map parameters fixed, one can equivalently vary the dispersion strength, $(D_1L_1-D_2L_2)$, while keeping the pulsewidth fixed. It is this later convention that was adopted in most studies of the DM systems. The stronger dispersion maps in that convention correspond to shorter pulses (smaller values of T_0) in our notations. An invariant parameter, which can be proposed to facilitate the comparison of results obtained by different researchers, is the stretching factor of the DM soliton. This stretching factor can be defined [16] as the ratio of the maximum and minimum widths of the pulse:

$$S=\sqrt{1+\frac{\Delta_{\max}^2}{T_0^4}}. \tag{5.1}$$

For the lossless fiber, $\Delta_{\max}^2=1/4$; cf. Eqs. (2.28) and (2.4). Thus, the stretching factor corresponding to the threshold width $(T_0)_{\rm cr}$ equals

$$S_{\rm cr}\approx\sqrt{1+\frac{1/4}{0.39^4}}\approx 3.43. \tag{5.2}$$

Similarly, that factor corresponding to the minimum width $T_0=0.74$ where the pulse is closest to the Gaussian (cf. Fig. 2) is

$$S_{\rm Gauss}\approx\sqrt{1+\frac{1/4}{0.74^4}}\approx 1.35. \tag{5.3}$$

Note that this is reasonably close to the value $S\lesssim 1.5$, which was reported in Ref. [8] for a numerically found DM soliton with an almost Gaussian spectrum.

We also used our Hermite-Gaussian expansion to derive (Sec. IV) the equations for a perturbed evolution of the DM soliton under a general perturbations. We showed that with an accuracy of about 5%, these equations can be also obtained by considering the conservation laws for the corresponding evolution equation, where the form of the unperturbed soliton is taken as a chirped Gaussian pulse. This justifies the results of earlier studies [5,15,13,14], where the perturbed evolution of the DM soliton was studied using either the conservation laws or the variational method. As was explained in Ref. [16], without that justification, the validity of the approach based on the conservation laws or the variational method was far from obvious.

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$$|\hat{a}_{nR}| \sim \frac{1}{(n/2)(n/2)!2^{3n/2}}.$$

With this estimate, one can show that the shape of the DM soliton in this limit can be adequately described, for not too large $|\xi|$, by just the two harmonics with $n=0$ and $n=4$.
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