Modulational instability of dust-acoustic and dust-ion-acoustic waves

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Using the standard reductive perturbation technique, a nonlinear Schrödinger equation is derived to study the modulational instability of finite amplitude dust-acoustic (DA) and dust-ion-acoustic (DIA) waves against oblique perturbations (with respect to the propagation direction of the carrier waves) in an unmagnetized dusty plasma. It is shown that both the DA and DIA waves are modulationally unstable. Possible stationary states of the wave packets can appear as envelope solitons. [S1063-651X(98)06111-X]

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I. INTRODUCTION

In recent years, nonlinear wave propagation in plasmas has become one of the most important subjects of plasma physics [1–6]. In particular, it is now well established that the slow modulation of a monochromatic plane wave can be described by the so-called nonlinear Schrödinger equation (NLSE), which is familiar as the resultant equation of the reductive perturbation theory (RPT) developed by Taniuti and his collaborators [3,4]. For a medium with a positive coefficient of the cubic nonlinearity term in the NLSE, an instability that takes place in the transverse directions is usually called self-focusing, while that in the longitudinal direction is referred to as the modulational instability. Even when a modulation of the wave mode takes place in a direction oblique to the direction of the carrier wave propagation, the instability can still be called a modulational instability [6].

Most of the theoretical efforts to investigate the modulational instability of ion-acoustic wave (IAW) are focused on a two-component plasma whose constituents are singly charged positive ions and electrons. Earlier, Kako and Hasegawa [6] studied the modulational instability of an IAW when the modulation takes place oblique to the propagation direction of the carrier wave. The modulational instability of ion-acoustic waves has also been studied by Chhabra and Sharma [7] in a plasma consisting of two-ion species with singly ionized positive ions. Mishra *et al.* [8] have recently studied the modulational instability of obliquely modulated ion-acoustic waves in a collisionless plasma consisting of two-ion species with different masses, concentrations, and charge states. In that analysis, the ion species were taken as cold.

However, when an electron-ion plasma contains extremely massive, micrometer-size charged dust grains, there appears the possibility of new normal modes [9,10]. The latter include the dust-acoustic (DA) and the dust-ion-acoustic (DIA) waves. The dust-acoustic wave (DAW) [9] is an extremely low phase velocity (in comparison with the electron and ion thermal velocities) normal mode of a three-component dusty plasma comprising electrons, ions and extremely massive micrometer-size charged dust grains. In the DAW potential, both the electrons and ions are Boltzmann distributed, whereas the charged dust particles are inertial. Thus, in the DAW, the pressures of the electrons and ions provide the restoring force, whereas the inertia comes from the dust mass. The phase velocity of the DAW scales as $\omega_{pd}\lambda_D \equiv c_d$, where ω_{pd} is the dust-plasma frequency, λ_D is the effective Debye length of the dusty plasma, $c_d = (k_B T_e/m_d)^{1/2}$ is the DA velocity, k_B is the Boltzmann constant, T_e is the electron temperature, and m_d is the mass of a dust particle. The DA waves have been observed in several laboratory experiments [11–13]. On the other hand, in the dust-ion-acoustic wave (DIAW), which is an extension of the usual IAW, the pressure of the inertialess electrons provides the restoring force, whereas the inertia mainly comes from the ion mass. The phase velocity of the DIAW is proportional to $\omega_{pi}\lambda_{De} \equiv (Z_i n_{i0}/n_{e0})^{1/2} c_s$, where ω_{pi} is the ion plasma frequency, λ_{De} is the electron Debye length, n_{j0} is the unperturbed number density of the particle species j (j equals e for the electrons and i for the ions), Z_i is the charge state of ions, $c_s = (k_B T_e/m_i)^{1/2}$ is the ion-acoustic velocity, and m_i is the mass of an ion. The DIAW, whose phase velocity is somewhat larger than that of the usual IAW in an electron-ion plasma when the dust grains are absent, is also observed in a laboratory experiment [14]. A critical review of wave phenomena in dusty plasmas is presented recently by Verheest [15].

In the present paper, we present an investigation of the modulational instability of the DAW [9] and the DIAW [10], when the modulation on the amplitude of the carrier wave takes place oblique to the direction of the pump carrier wave propagation.

The manuscript is organized in the following fashion. In Sec. II, we present the relevant nonlinear equations for DA and DIA waves. By introducing the two-time and space scales and the reductive perturbation method, we then derive the NLSE governing the dynamics of obliquely modulated DA and DIA waves in Sec. III. In Sec. IV, we derive a nonlinear dispersion relation for the amplitude modulation and we discuss the possible two types of solutions of the resulting NLSE. For the oblique direction with respect to the pump carrier wave propagation, it is found that the DA and DIA waves are modulationally unstable. Section V contains a brief summary of our investigation.

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II. MATHEMATICAL FORMALISM

To study the oblique modulation of DA and DIA waves in a dusty plasma, we consider a two-dimensional (2D) geometry. For simplicity, the plasma is assumed to be unmagnetized and the wave is assumed to propagate in the xy plane with wave vector $\boldsymbol{k}\equiv(k\cos\theta, k\sin\theta)$, where k is the magnitude of the wave vector and θ is the angle made by the wave vector $\boldsymbol{k}$ with the positive x direction. We assume that the wave modulation takes place along the x direction. In a dusty plasma, the dust grains are very much heavier ($m_d/m_p \sim 10^{12}-10^{15}$) compared to the ordinary ions and electrons. The dusty plasma is composed of three species, namely, the cold dust particles, usually considered to be negatively charged, and the electrons and positive ions. We consider two slightly different models for the two types of wave modes, namely, the DA and DIA waves.

CASE I: Dust-acoustic wave

Here, we assume that the dust component of the dusty plasma is cold and inertial, whereas both the positive ions and the electrons are hot, isothermal, and Boltzmann distributed. For the propagation of a DA wave mode $(\omega,\boldsymbol{k})$ in the dusty plasma, we consider the following set of equations:

$$\frac{\partial n_d}{\partial t}+\nabla\cdot(n_d\boldsymbol{v}_d)=0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}_d}{\partial t}+\boldsymbol{v}_d\cdot\nabla\boldsymbol{v}_d=\frac{Z_d e}{m_d}\nabla\phi, \tag{2}$$

$$n_e=n_{e0}\exp(e\phi/k_BT_e), \tag{3}$$

$$n_i=n_{i0}\exp(-Z_ie\phi/k_BT_i), \tag{4}$$

$$\nabla^2\phi=4\pi e(n_e+Z_dn_d-Z_in_i), \tag{5}$$

and in equilibrium, the following charge neutrality condition

$$n_{e0}+Z_dn_{d0}=Z_in_{i0}, \tag{6}$$

is fulfilled; where n_j $(j=e,i,d)$ is the number density of different plasma species with n_{j0} being their equilibrium values when there is no any plasma perturbation, $\boldsymbol{v}_d$ is the dust-fluid velocity, ϕ is the plasma perturbation potential; $Z_i(Z_d)$ is the charge state of ions (dust particles), e is the magnitude of the electronic charge, $T_e(T_i)$ is the electron (ion) temperature, and m_d is the mass of a dust particle in the plasma.

It is convenient to normalize different quantities in Eqs. (1)–(6). We introduce the following normalizations: $t=(\lambda_D/c_d)T$, $\boldsymbol{x}=\lambda_D\boldsymbol{X}$, $\boldsymbol{v}_d=c_d\boldsymbol{V}$, $n_d=n_{d0}N$, $\phi=(k_BT_e/Z_de)\Phi$, where

$$\lambda_D^{-2}=\lambda_{De}^{-2}+\lambda_{Di}^{-2}, \tag{7}$$

and

$$\lambda_{Dj}=(k_BT_j/4\pi q_j^2n_{j0})^{1/2}, \tag{8}$$

$q_e=e$ for electrons and $q_i=Z_ie$ for ions; λ_{Dj} is the Debye length for species j. After this normalization, Eqs. (1)–(6) take the following forms:

$$\frac{\partial N}{\partial T}+\nabla_X\cdot(N\boldsymbol{V})=0, \tag{9}$$

$$\frac{\partial \boldsymbol{V}}{\partial T}+\boldsymbol{V}\cdot\nabla_X\boldsymbol{V}=\nabla_X\Phi, \tag{10}$$

$$(\nabla_X^2-1)\Phi+\alpha\Phi^2=\beta(N-1), \tag{11}$$

where the parameters α and β are given by

$$\alpha=\left(\frac{1}{2Z_d}\right)\left[Z_i^3\left(\frac{T_e}{T_i}\right)^2\frac{n_{i0}}{n_{e0}}-1\right]\left[Z_i^2\frac{T_e}{T_i}\frac{n_{i0}}{n_{e0}}+1\right]^{-1}\simeq\frac{1}{2}\frac{Z_iT_e}{Z_dT_i}, \tag{12}$$

$$\beta=\frac{\omega_{pd}^2\lambda_D^2}{c_d^2}\simeq\left(\frac{Z_d}{Z_i}\right)^2\left(\frac{n_{d0}T_i}{n_{i0}T_e}\right), \tag{13}$$

and $\omega_{pd}=(4\pi Z_d^2e^2n_{d0}/m_d)^{1/2}$ is the dust plasma frequency. In deriving expressions (12) and (13), we have assumed $T_e>T_i$ and $n_{i0}>n_{e0}$ for the usual dusty plasma parameters. We have also expanded the exponentials in Eqs. (3) and (4) under the assumption that $e\phi/k_BT_e$, $Z_ie\phi/k_BT_i\ll 1$, and have retained terms up to Φ^2 so as to include the second harmonic effect.

CASE II: Dust-ion-acoustic wave

For the DIAW, we assume that the ion and dust components are cold inertial fluid, whereas the electrons are inertialess hot isothermal fluid and are Boltzmann distributed. Thus, the model equations in this case are

$$\frac{\partial n_i}{\partial t}+\nabla\cdot(n_i\boldsymbol{v}_i)=0, \tag{14}$$

$$\frac{\partial \boldsymbol{v}_i}{\partial t}+\boldsymbol{v}_i\cdot\nabla\boldsymbol{v}_i=-\frac{Z_i e}{m_i}\nabla\phi, \tag{15}$$

$$\frac{\partial n_d}{\partial t}+\nabla\cdot(n_d\boldsymbol{v}_d)=0, \tag{16}$$

$$\frac{\partial \boldsymbol{v}_d}{\partial t}+\boldsymbol{v}_d\cdot\nabla\boldsymbol{v}_d=\frac{Z_d e}{m_d}\nabla\phi, \tag{17}$$

$$n_e=n_{e0}\exp(e\phi/k_BT_e), \tag{18}$$

and

$$\nabla^2\phi=4\pi e(n_e+Z_dn_d-Z_in_i). \tag{19}$$

However, as has been shown earlier [10], the effects of including the equations of motion of the massive dust component of the dusty plasma introduce a very small correction to the physics of the dispersion relation of the DIAW, we can then safely assume that the dust particles are immobile. The overall charge neutrality condition $n_{e0}+Z_dn_{d0}=Z_in_{i0}$ is also fulfilled here.

We now normalize the equations here in a slightly different way. We use the normalizations $t=(\lambda_{De}/c_s)T$, $\boldsymbol{x}$

$=\lambda_{De}\boldsymbol{X}$, $\boldsymbol{v}_i=c_s\boldsymbol{V}$, $n_i=n_{i0}N$, $\phi=(k_BT_e/Z_ie)\Phi$. After these normalizations, Eqs. (14)–(19) take the following forms:

$$\frac{\partial N}{\partial T}+\nabla_X\cdot(N\boldsymbol{V})=0, \tag{20}$$

$$\frac{\partial \boldsymbol{V}}{\partial T}+\boldsymbol{V}\cdot\nabla_X\boldsymbol{V}=-\nabla_X\Phi, \tag{21}$$

$$(\nabla_X^2-1)\Phi-\alpha\Phi^2=\beta(1-N), \tag{22}$$

where now the parameters α and β are different from the previous case and are given by

$$\alpha=1/Z_i, \tag{23}$$

and

$$\beta=Z_i^2n_{i0}/n_{e0}. \tag{24}$$

III. DERIVATION OF THE NONLINEAR SCHRÖDINGER EQUATION

CASE I: Dust-acoustic wave

In order to investigate the modulational instability of oblique modulations on DA and DIA waves in the dusty plasma, we assume that the perturbed quantities of all orders depend on X and T through the wave amplitude and the phase factor of the monochromatic plane wave, and on Y through the phase factor only [6]. To derive the nonlinear Schrödinger equation, we employ the standard reductive perturbation technique [4]. We introduce the following stretched variables ζ and τ such that

$$\zeta=\epsilon(X-\lambda T), \tag{25}$$

$$\tau=\epsilon^2T, \tag{26}$$

where ϵ is a small parameter and λ is the group velocity of the wave along the X direction. We then expand the variables N, $\boldsymbol{V}$, and Φ in terms of the expansion parameter ϵ as

$$N(\boldsymbol{X},T)=1+\sum_{n=1}^{\infty}\epsilon^n\sum_{l=-\infty}^{\infty}N_l^{(n)}(\zeta,\tau)\exp[il(\boldsymbol{k}\cdot\boldsymbol{X}-\omega T)], \tag{27}$$

$$[\boldsymbol{V}(\boldsymbol{X},T),\Phi(\boldsymbol{X},T)]=\sum_{n=1}^{\infty}\epsilon^n\sum_{l=-\infty}^{\infty}[\boldsymbol{V}_l^{(n)}(\zeta,\tau),\Phi_l^{(n)}(\zeta,\tau)]\times\exp[il(\boldsymbol{k}\cdot\boldsymbol{X}-\omega T)], \tag{28}$$

where N, $\boldsymbol{V}$ and Φ satisfy the reality condition $A_{-l}^{(n)}\equiv A_l^{(n)*}$ and the asterisk denotes complex conjugate. The operators $\partial/\partial t$, ∇_X, and ∇_X^2 then take the following forms to account for the slow variation of the wave amplitude:

$$\frac{\partial}{\partial T}\rightarrow\frac{\partial}{\partial T}-\epsilon\lambda\frac{\partial}{\partial\zeta}+\epsilon^2\frac{\partial}{\partial\tau}, \tag{29}$$

$$\nabla_X\rightarrow\nabla_X+\hat{x}\epsilon\frac{\partial}{\partial\zeta}, \tag{30}$$

and

$$\nabla_X^2\rightarrow\nabla_X^2+2\epsilon\frac{\partial^2}{\partial X\partial\zeta}+\epsilon^2\frac{\partial^2}{\partial\zeta^2}. \tag{31}$$

Substituting the expressions (25)–(31) into Eqs. (9)–(11), we obtain the nth-order reduced equations as follows:

$$-il\omega N_l^{(n)}+il\boldsymbol{k}\cdot\boldsymbol{V}_l^{(n)}-\lambda\frac{\partial N_l^{(n-1)}}{\partial\zeta}+\frac{\partial V_{lX}^{(n-1)}}{\partial\zeta}+\frac{\partial N_l^{(n-2)}}{\partial\tau}+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}il\boldsymbol{k}\cdot\boldsymbol{V}_{l-l'}^{(n-n')}N_{l'}^{(n')}+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\frac{\partial}{\partial\zeta}(V_{(l-l')X}^{(n-n'-1)}N_{l'}^{(n')})=0, \tag{32}$$

$$-il\omega\boldsymbol{V}_l^{(n)}-il\boldsymbol{k}\Phi_l^{(n)}-\lambda\frac{\partial\boldsymbol{V}_l^{(n-1)}}{\partial\zeta}-\hat{x}\frac{\partial\Phi_l^{(n-1)}}{\partial\zeta}+\frac{\partial\boldsymbol{V}_l^{(n-2)}}{\partial\tau}+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}il'\boldsymbol{k}\cdot\boldsymbol{V}_{l-l'}^{(n-n')}\boldsymbol{V}_{l'}^{(n')}+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}V_{(l-l')X}^{(n-n'-1)}\frac{\partial\boldsymbol{V}_{l'}^{(n')}}{\partial\zeta}=0, \tag{33}$$

and

$$-(1+l^2k^2)\Phi_l^{(n)}-\beta N_l^{(n)}+2ilk_X\frac{\partial\Phi_l^{(n-1)}}{\partial\zeta}+\frac{\partial^2\Phi_l^{(n-2)}}{\partial\zeta^2}+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\alpha\Phi_{l-l'}^{(n-n')}\Phi_{l'}^{(n')}=0. \tag{34}$$

For the first order $(n=1)$ equations, we obtain

$$-il\omega N_l^{(1)}+il\boldsymbol{k}\cdot\boldsymbol{V}_l^{(1)}=0, \tag{35}$$

$$-il\omega\boldsymbol{V}_l^{(1)}-il\boldsymbol{k}\Phi_l^{(1)}=0, \tag{36}$$

and

$$-(1+l^2k^2)\Phi_l^{(1)}-\beta N_l^{(1)}=0. \tag{37}$$

The equations for $l=1$ give rise to the following dispersion relation for the DAW:

$$\omega^2=\frac{\beta k^2}{1+k^2}, \tag{38}$$

which agrees well with the standard dispersion relation $\omega^2=\omega_{pd}^2\lambda_D^2k^2/(1+k^2\lambda_D^2)$ of the DAW in an unmagnetized dusty plasma studied earlier by Rao, Shukla, and Yu [9], if we go back to the dimensional form by replacing $\omega\rightarrow(\lambda_D/c_d)\omega$ and $\boldsymbol{k}\rightarrow\lambda_D\boldsymbol{k}$ and substituting the value of β from Eq. (13) into Eq. (38).

From Eqs. (35)–(37), we can express the first-order quantities in terms of $N_1^{(1)}$ as

$$\boldsymbol{k}\cdot\boldsymbol{V}_1^{(1)}=\omega N_1^{(1)}, \tag{39}$$

$$V_{1X}^{(1)}=\frac{\omega}{k}\cos\theta N_1^{(1)}, \tag{40}$$

$$V_{1Y}^{(1)}=\frac{\omega}{k}\sin\theta N_1^{(1)}, \tag{41}$$

$$\Phi_1^{(1)}=-\left(\frac{\omega}{k}\right)^2 N_1^{(1)}. \tag{42}$$

To the second order, one obtains the second-order correction to the quantities in Eqs. (39)–(42) in terms of a function $N_1^{(2)}(\zeta,\tau)$ and $\partial N_1^{(1)}/\partial\zeta$. For $l=1$ in the second-order equations, the following compatibility condition is obtained:

$$\lambda=\frac{1}{\beta}\left(\frac{\omega}{k}\right)^3\cos\theta=\frac{\partial\omega}{\partial k}. \tag{43}$$

It is to be noted here that as Taniuti [4] has shown, the first-order quantities with zeroth harmonic can be taken to be zero.

The second harmonic mode of the carrier wave is also obtained in terms of $N_1^{(1)}N_1^{(1)}$. This comes from nonlinear self-interaction. The component of $l=2$ for the second-order $(n=2)$ equations determine the second harmonic quantities

$$N_2^{(2)}=\left[2+\frac{1}{2k^2}-\frac{1}{3}\frac{\alpha}{k^2}\left(\frac{\omega}{k}\right)^2\right]N_1^{(1)}N_1^{(1)}, \tag{44}$$

$$\boldsymbol{k}\cdot\boldsymbol{V}_2^{(2)}=\omega\left[1+\frac{1}{2k^2}-\frac{1}{3}\frac{\alpha}{k^2}\left(\frac{\omega}{k}\right)^2\right]N_1^{(1)}N_1^{(1)}, \tag{45}$$

$$\Phi_2^{(2)}=\left[-\frac{1}{2}\frac{\beta}{k^2}+\frac{1}{3}\frac{\alpha}{k^2}\left(\frac{\omega}{k}\right)^4\right]N_1^{(1)}N_1^{(1)}. \tag{46}$$

The zeroth-harmonic mode also appears due to the self-interaction of the modulated carrier wave. Its expression cannot be determined completely within the second order and we will have to consider the third-order equations. Thus, the $l=0$ components of the third-order part of the reduced equations determine the following second-order quantities in the zeroth harmonic as follows:

$$N_0^{(2)}=\frac{-1}{\beta-\lambda^2}\frac{1}{\beta}\left(\frac{\omega}{k}\right)^4[1-2\alpha\beta+k^2+2\cos^2\theta]|N_1^{(1)}|^2, \tag{47}$$

$$\boldsymbol{k}\cdot\boldsymbol{V}_0^{(2)}=\frac{-1}{\beta-\lambda^2}\frac{1}{\beta}(\omega\cos^2\theta)\left[2\beta^2+\frac{1}{\beta}\left(\frac{\omega}{k}\right)^6\times(1-2\alpha\beta+k^2)\right]|N_1^{(1)}|^2, \tag{48}$$

$$\Phi_0^{(2)}=\frac{1}{\beta-\lambda^2}\left(\frac{\omega}{k}\right)^4[1-2\alpha\lambda^2+k^2+2\cos^2\theta]|N_1^{(1)}|^2. \tag{49}$$

Finally, substituting the above derived expressions into the $l=1$ component of the third-order part of the reduced equations, we obtain the following nonlinear Schrödinger equation:

$$i\frac{\partial a}{\partial\tau}+P\frac{\partial^2 a}{\partial\zeta^2}+Q|a|^2a=0, \tag{50}$$

for the slow evolution of the first-order amplitude of the plasma perturbation potential $\Phi_1^{(1)}\equiv a$. In the above equation, The coefficients P and Q are given by

$$P=\frac{1}{\beta}\left(\frac{1}{2\omega}\right)\left(\frac{\omega}{k}\right)^4\left[1-\left(1+\frac{3}{\beta}\omega^2\right)\cos^2\theta\right] \tag{51}$$

and

$$Q=Q_0+Q_2, \tag{52}$$

where

$$Q_0=\frac{\omega}{2\beta}\left(\frac{k}{\omega}\right)^4(\beta-\lambda^2)^{-1}\left[\beta\left(\frac{\omega}{k}\right)^2\left\{1-4\alpha\left(\frac{\omega}{k}\right)^2\right\}+4\cos^2\theta\left\{\beta^2+\left(\frac{\omega}{k}\right)^4-2\alpha\left(\frac{\omega}{k}\right)^6+\left(\frac{\alpha}{\beta}\right)^2\left(\frac{\omega}{k}\right)^{12}\right\}\right], \tag{53}$$

and

$$Q_2=-\left(\frac{1}{2\omega}\right)\left(\frac{k}{\omega}\right)^4\left[4\omega^2+\frac{3}{2}\left(\frac{\omega}{k}\right)^2-2\alpha\left(\frac{\omega}{k}\right)^4+\frac{2}{3}\frac{\alpha^2}{\beta}\left(\frac{\omega}{k}\right)^8\right]. \tag{54}$$

The expressions for the parameters α and β appearing in Eqs. (51)–(54) are given respectively by $\alpha=Z_iT_e/2Z_dT_i$ and $\beta=Z_d^2n_{d0}T_i/Z_i^2n_{i0}T_e$ and θ is the angle between the direction of the pump carrier wave propagation and the direction of the propagation of the amplitude modulation. In Eq. (52), Q_0 is the contribution to Q due to the zeroth harmonic and Q_2 is the contribution due to the second harmonic wave, respectively.

CASE II: Dust-ion-acoustic wave

We can proceed in the similar way as we have just done for case I for the DAW and arrive finally at the NLSE, Eq. (50), with expressions for the coefficients P and Q, which have exactly the same forms as to those given by Eqs. (51) and (52). However, the constants α and β in this case are different than for case I and are given by $\alpha=1/Z_i$ and $\beta=Z_i^2n_{i0}/n_{e0}$.

IV. STABILITY OF DUST-ACOUSTIC AND DUST-ION-ACOUSTIC WAVES WITH OBLIQUE MODULATION

In this section, we study the stability of DA and DIA waves when modulation on the wave amplitude (packet) takes place in a direction, which is oblique to the direction of the pump carrier wave propagation. Instead of a stationary solution, here we consider the dynamic solution of the NLSE, derived in the previous section. Accordingly, we separate the amplitude a into two parts:

$$a=[a_0+\delta a(\xi)]\exp(-i\Delta\tau), \tag{55}$$

where $\xi=K\zeta-\Omega\tau$ is the modulation phase with $K(\ll k)$ and Ω $(\ll\omega)$ are respectively the wave number and the frequency of the modulation; a_0 is the constant (real) amplitude of the pump carrier wave, $\delta a(\ll a_0)$ is the small amplitude perturbation, and Δ is a nonlinear frequency shift. After substituting Eq. (55) into Eq. (50) and collecting terms of the same order, we obtain

$$\Delta=-Q|a_0|^2 \tag{56}$$

and

$$i\frac{\partial\delta a}{\partial\tau}+P\frac{\partial^2\delta a}{\partial\zeta^2}+Q|a_0|^2(\delta a+\delta a^*)=0, \tag{57}$$

where δa^* is the complex conjugate of δa. Introducing $\delta a=U+iV$ in Eq. (57), and separating the real and imaginary parts, one obtains the following two coupled equations:

$$\frac{\partial V}{\partial\tau}=P\frac{\partial^2 U}{\partial\zeta^2}+2Q|a_0|^2U \tag{58}$$

and

$$\frac{\partial U}{\partial\tau}=-P\frac{\partial^2 V}{\partial\zeta^2}. \tag{59}$$

Let us now assume that the amplitude perturbation δa varies as $\sim\exp[i(K\zeta-\Omega\tau)]$, we then obtain from Eqs. (58) and (59), the following nonlinear dispersion relation for the amplitude modulation of the DA or DIA wave modes:

$$\Omega^2=PK^2(PK^2-2Q|a_0|^2), \tag{60}$$

where P and Q are respectively the coefficients of the dispersive and the nonlinear terms in the NLSE, Eq. (50), and are given respectively by Eqs. (51) and (52). Since $\alpha=Z_iT_e/2Z_dT_i\ll 1$ is a very small quantity and also from Eq. (38), $\omega^2=\beta-(\omega/k)^2\ll\beta$, it can be shown from Eqs. (53) and (54) that $Q_0>0$ and $Q_2<0$. It is also interesting to note that one can find a wide domain of (θ,ω) where $Q<0$. Similarly, from the expression of P, Eq. (51), we observe that P changes sign from negative to positive when θ exceeds a value $\cos^{-1}[1+3\omega^2/\beta]^{1/2}$. We note that this value of θ is $0°$ when $\omega=0$ and $60°$ when $\omega=\sqrt{\beta}$. Thus, we see that there is a wide domain where both P and Q take the negative sign. On the other hand, we see from the nonlinear dispersion relation, Eq. (60), that when both P and Q are negative, $\Omega^2<0$ for a small value of the modulation wave number K. That is, the wave is modulationally unstable for long wavelength perturbations with wave number K satisfying $K^2<2|Q/P||a_0|^2$, and the maximum growth rate is obtained for $K=\sqrt{|Q/P|}|a_0|$ with maximum growth rate $\gamma_{\max}=\mathrm{Im}(\Omega)_{\max}=|Q||a_0|^2$. It is interesting to note from Eq. (56) that the magnitude of the nonlinear frequency shift is equal to the maximum growth rate $|\Delta|=\gamma_{\max}$. Instability sets in for perturbation wavelength $\lambda>\lambda_{\mathrm{cr}}$ where the critical threshold wavelength corresponding to the critical wave number $K_{\mathrm{cr}}=\sqrt{2|P/Q|}|a_0|$ is $\lambda_{\mathrm{cr}}=2\pi/K_{cr}=\sqrt{2|P/Q|}(\pi/|a_0|)$.

We note that for nearly parallel direction of modulation $(\theta\approx 0)$ with respect to the pump carrier wave propagation and with $\omega\neq 0$, P is negative, however, when ω is not very large, Q is positive. This then indicates from Eq. (60) that $\Omega^2>0$, that is, the wave packet is now modulationally stable. This happens due to the nonlinear self-interaction originating from the zeroth harmonic mode (sometimes called slow mode), which is referred to as the so-called ponderomotive force. Additionally, when the modulation takes place along a direction that is nearly perpendicular ($\approx 90°$) to the pump carrier wave propagation direction, P becomes positive, while Q is always negative here, indicating from Eq. (60) that $\Omega^2>0$, that is, in this case also, the wave packet is modulationally stable. This happens due to the second harmonic self-interaction. On the other hand, for a modulational instability, second harmonic mode is essential to enforce Q to take a negative value, since in the instability domain $P<0$.

To this end, we now briefly discuss the possible localized stationary solutions of the NLSE, Eq. (50). This also applies to both DA and DIA waves. First we note that, since the wave packet can be unstable as well as stable in different conditions for θ and ω, P and Q can both be negative or they can have different signs, there are accordingly two types of stationary solutions of the NLSE. For the unstable wave packet (P and Q have same sign) it can be shown that the DAW or DIAW propagates as an envelope soliton. On the other hand, for stable wave packet (P and Q have different signs), the wave can propagate in the form of an envelope hole or an envelope shock [16]. To obtain the profiles in both the cases, we let $a(\zeta,\tau)=\rho(\zeta,\tau)\exp[i\sigma(\zeta,\tau)]$, where ρ and σ are two real variables. Substituting this in the NLSE, Eq. (50), and separating the real and imaginary parts, we solve for ρ and σ in a straightforward manner [16]. When both P and Q have the same sign (modulationally unstable wave), we obtain the following envelope soliton:

$$\rho(\zeta,\tau)=\rho_m\,\mathrm{sech}\left(\sqrt{\frac{1}{2}\left|\frac{Q}{P}\right|}\rho_m\zeta\right), \tag{61}$$

where ρ_m is a constant and represents the nonlinear maximum amplitude. On the other hand, when P and Q have the opposite signs (modulationally stable wave), we obtain

$$\rho(\zeta,\tau)=\rho_1\left[1-b^2\mathrm{sech}^2\left(\sqrt{\frac{\rho_1}{2}\left|\frac{Q}{P}\right|}b\zeta\right)\right]^{1/2}, \tag{62}$$

where $b^2=(\rho_1^2-\rho_m^2)/\rho_1^2\leqslant 1$, ρ_1 is a constant. The solution, Eq. (62), is referred to as an envelope hole sometimes called a dark soliton [16], i.e., a soliton for the absent region of the distribution $|a|^2$. In other words, this solution corresponds formally to the accumulation of density in a region where the wave intensity is very low. The parameter b in Eq. (62) determines the depth of the modulation. For $b=1$, we have

$$\rho(\zeta,\tau)=\rho_1 \tanh\left(\sqrt{\frac{\rho_1}{2}\left|\frac{Q}{P}\right|}\, b\,\zeta\right), \tag{63}$$

which is known as an envelope shock [16].

It is important here to note that, besides the NLSE described above, the nonlinear dynamics of DA and DIA waves can also be described by an another nonlinear equation, called the KdV equation [19,20], the solution of which is usually called the KdV soliton (sometimes called topological soliton). The following points make clear the physical differences between the KdV soliton and the envelope soliton (solution of the NLSE):

(1) The KdV equation describes the evolution of nonmodulated DAW and DIAW in which the harmonic generation nonlinearities involving the divergence of fluxes, fluid convection, as well as fully nonlinear electron and ion number densities are in balance with the dispersion of the waves. The resulting stationary solutions of the KdV equation do not have an envelope of waves; rather we obtain a bare pulse (either compressive or rarefactive) with no fast oscillations inside the packet.

(2) The NLSE governs the dynamics of a modulated wave packet. Here, the DAW and DIAW are modulated by nonresonant disturbances whose frequency is much smaller than the carrier frequencies. Here, the two time and space scales are necessary and stretchings are different from those used to derive the KdV equation. The modulated amplitude varies slowly (on the time and space scales of the nonresonant modulations). The essential nonlinearities come from the ponderomotive force that arises from the beating of the carrier wave and the sidebands that are created on account of the nonlinear interactions between the carrier and the quasistationary density modulations. The latter, in turn, are reinforced by the ponderomotive force of the DAW and DIAW envelopes. In the NLSE, the nonlinearities are in balance with the wave group dispersion and the resulting stationary solutions of the NLSE have an envelope structure consisting of the modulated DAW and DIAW (with rapidly varying phasors and slowly varying envelopes) packets and large scale quasistationary density perturbations that are created by the wave envelopes.

Accordingly, the stationary solutions of the KdV equation and the NLSE are entirely different and the physics of the two processes differ significantly.

V. DISCUSSIONS

We have derived a nonlinear Schrödinger equation for the two-dimensional nonlinear propagation of DA and DIA waves in a homogeneous unmagnetized dusty plasma by applying the standard reductive perturbation theory [3,4]. To study the modulational instability of the DA and DIA waves, we have considered amplitude modulations that propagate oblique to the direction of the pump carrier wave propagation. We have then derived a nonlinear dispersion relation from the viewpoint of the dynamic solution of the NLSE. We have found that a wide domain in (θ,ω) exists where both the coefficients P and Q of the dispersive and the nonlinear terms of the NLSE, take the negative sign. The nonlinear dispersion relation then predicts that the modulations of DA and DIA waves are unstable when they propagate oblique to the direction of the pump carrier wave propagation. It thus confirms the earlier analysis of Kako and Hasegawa [6] that modulational instability is a general property of a wave in a nonlinear and dispersive medium when the modulation on wave amplitude is allowed to take place in a direction that is oblique to the direction of the pump carrier wave propagation.

The above discussion about the nonlinear evolution of the wave amplitude applies equally to both DA as well as DIA waves. However, since the coefficients P and Q of the dispersive and the nonlinear terms of the NLSE are slightly different (although the forms of the expressions are exactly the same) due to the difference in the parameters α and β in the two cases, the growth rates may not be exactly the same. Additionally, the domain of the modulational instability in the (θ,ω) space may also be changed accordingly. Furthermore, we observe from Eq. (61) that the width of the envelope soliton in the two cases may also be changed due to the possible difference in $|Q/P|$ ratio.

We have found that the stationary state of the modulationally unstable DAW and DIAW can appear as an envelope soliton [16–18]. On the other hand, modulationally stable DAW and DIAW can propagate either as an envelope hole sometimes called a dark soliton or an envelope shock [16].

It is to be noted here that besides the NLSE to describe the slow modulation of the wave envelopes, the nonlinear dynamics of the acoustic waves in a dusty plasma can also be described by a KdV equation [19,20]. However, there are some differences between these two descriptions. One important difference is that in the case of the KdV soliton, the velocity depends on the wave amplitude, leading to faster speeds for larger amplitudes, but envelope solitons all move at the same group velocity and therefore do not overtake each other.

In the present investigation, we have assumed constant dust charge. The inclusion of the dust charge perturbation [21,22] in our analysis would give rise to a damped NLSE. The latter can still have an envelope soliton with a tail [18]. Finally, it is stressed that the results of our investigation should be useful in understanding the features of modulated DA and DIA wave packets in a weakly coupled dusty plasma.

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