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Enhancement in the dynamic response of a viscoelastic fluid flowing in a tube

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In order to gain insight into the effects of elasticity on the dynamics of fluids in porous media, we have analyzed the flow of a Maxwell fluid in a tube. The entire problem is transformed to the frequency domain through a time Fourier transform and this allows for the derivation of a dynamic generalization of Darcy's law. Analytical results are provided showing a dramatic enhancement at certain frequencies. We present a parametric analysis of the dissipative and elastic behavior of the system, and we obtain an analytic expression for the maximum value of the permeability and the frequency at which this maximum occurs in terms of the physical properties of the fluid and the radius of the tube. With these results, we show that the human pulse occurs at a frequency that produces a permeability in the neighborhood of the maximum permeability.
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I. INTRODUCTION

Many problems associated with the petroleum, plastic, and chemical industries involve the response of a fluid in a porous medium to a frequency dependent pressure drop. The study of the frequency dependence of sound propagation can be used to determine the permeability in some cases of saturated porous media, and the same approach may provide the parameters that characterize the geometry of the media [1]. A useful way to describe such frequency-dependent processes is through a dynamic permeability function [2]. Expressions for the *dynamic permeability* of compressible Newtonian fluids have been obtained for rigid, isotropic porous media [3] and for deformable porous media [4]. In addition, dimensional analysis has also been used in connection with frequency-dependent processes [5,6]. Many studies, based on such an analysis, have attempted to describe viscoelastic flow through porous media in terms of the friction factor [5]. It must be pointed out that most of those studies were for either steady or quasisteady situations [7]. The study of transient flow of non-Newtonian fluids has been used to investigate the normal stress for flow in a tube [8], and in particular to describe the role of an increasing pressure gradient on the velocity profile. Nevertheless, in all of these studies the frequency dependence of the permeability has not been considered.

In a similar vein, during the past few years attention has been given to the study of the dynamic response of non-Newtonian fluids in tubes, focusing on the problem of transport of inelastic fluids through pipes [9]. In those studies, an enhancement of the mean flow rate was found, and the percentage increase in terms of the magnitude of the mean pressure gradient was given. In addition, other experimental evidence of similar enhancement for viscoelastic fluids was later provided [10]. In this last case, the observed phenomena could not be described in terms of a non-Newtonian, *inelastic* fluid and elastic behavior had to be incorporated in order to explain the experimental observations. The general aim of these studies was directed at the relation between the magnitude of the mean pressure and the increasing mean flow rate [11]. Biorheology is another field in which this type of investigation is of particular interest since blood or other biofluids are forced through capillaries by a periodic pressure gradient [12]. When blood is modeled as a Maxwell fluid with several relaxation times, good agreement is found [13] with the experimental measurements.

Electrorheological and magnetic phenomena in viscoelastic fluids, where frequency-dependent properties may play an important role, have also been the subject of recent interest. In the steady state, cluster formation in electrorheological fluids has been found to produce a nonlinear dependence of the viscosity on the shear rate [14]. Bounds for the physical properties involved in the electrorheological flow have been given [15] . The dynamic orientational response of a dilute suspension of single-domain magnetic particles immersed in a Maxwell fluid has been described as a complex comblike structure [16]. Experimental studies of the pressure drop and the flow rate of dilute solutions of polyethylene oxide flowing through beds of packed beads have shown that the pressure drop is greater than that for a Newtonian fluid, except for some solutions in a small range of flow rates where there was an interval in which the pressure drop decreased with increasing velocity [17]. In this range of velocities, referred as the *velocity gap*, it was not possible to obtain steady conditions [17]. This velocity gap was not fully explored, although it is of interest from both the fundamental and the applied points of view.

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Recently, we studied the flow of a Maxwell fluid in porous media [18], and found that in the linearized case it is possible to get an enormous enhancement of the dynamic permeability in comparison with the static one. The purpose of this paper is to present additional results for the case of the flow of a linearized Maxwell fluid in a tube [19] subjected to an oscillating pressure gradient. We obtain an analytic result that exhibits an enhancement of the mean frequency-dependent velocity. Our analysis is centered on the frequency dependence of the permeability and on the competition between dissipative and elastic effects. We are able to identify the values where it is possible to obtain a resonance and therefore an enhancement in the flow rate. Also, we give two simple formulas, one for the value of the maximum permeability at a given value of the Deborah number and another for the frequency where this maximum occurs. We use these results to show that the pulse of the human heart is in the most efficient range to produce a maximum flow through arteries and veins.

In the next section we present an analytic solution for a Maxwell fluid flowing in a tube. In the derivation, we will neglect the nonlinear effects to obtain a dynamic permeability for the mean flow. This dynamic permeability exhibits a resonantlike behavior that depends only on the Deborah number. In the third section we present numerical calculations and develop expressions for the first maximum of the dynamic response and the specific frequency at which it occurs as functions of the Deborah number. Finally, we close the paper in Sec. IV with some concluding remarks.

II. FREQUENCY DEPENDENT PERMEABILITY

In this section we present the analysis of a Maxwell fluid flowing in a cylindrical tube. We solve analytically the equation of motion in the frequency domain and integrate it in order to obtain an expression for the mean flow velocity. With this expression we can define a dynamic permeability which shows an enhancement with respect to its value at steady state.

The physical process that we wish to analyze is described by a continuity equation for incompressible flow,

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

and the linearized momentum equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p - \nabla \cdot \boldsymbol{\tau}, \tag{2}$$

in the absence of any nonuniform external field. Here p represents pressure that may contain the effect of gravity, $\mathbf{v}$ is the fluid velocity, ρ is the mass density of the fluid, and $\boldsymbol{\tau}$ represents the viscous stress tensor.

To study a viscoelastic fluid we consider the linear form of the Maxwell model

$$t_m \frac{\partial \boldsymbol{\tau}}{\partial t} = -\eta \nabla \mathbf{v} - \boldsymbol{\tau}, \tag{3}$$

where η is the viscosity and t_m the relaxation time. We now begin our analysis of the momentum equation by substituting Eq. (3) into Eq. (2) and constructing the Fourier transform to obtain

$$\rho(t_m \omega^2 + i\omega)\mathbf{V} + \eta \nabla^2 \mathbf{V} = (1 - i\omega t_m)\nabla P. \tag{4}$$

Here ω is the frequency, and in order to minimize complications with the nomenclature, we have used upper case letters to represent the time Fourier transform. In cylindrical coordinates, the solution of Eq. (4) is given by

$$V(r) = -\frac{(1 - i\omega t_m)}{\beta^2}\left[1 + \frac{J_0(\beta r)}{J_0(\beta a)}\right]\frac{\partial P}{\partial z}, \tag{5}$$

in which the no-slip condition has been imposed at the radius of the cylinder, $V(a) = 0$. In Eq. (5) we have used $V(r)$ to represent the Fourier transform of the z component of the velocity and we have used β to represent the parameter defined by

$$\beta = \left(\frac{\rho}{\eta t_m}[(t_m\omega)^2 + i\omega t_m]\right)^{1/2}.$$

To gain some insight concerning the mean flow in the tube, we form the area average denoted by $\langle V \rangle$ to obtain

$$\langle V \rangle = -K(\omega)\frac{\partial P}{\partial z}. \tag{6}$$

Here the dynamic permeability is given by

$$K(\omega) = -\frac{a^2(1 - i\omega t_m)}{\alpha \varpi}\left[1 - \frac{2}{\sqrt{\alpha\varpi}\, J_0(\sqrt{\alpha\varpi})} J_1(\sqrt{\alpha\varpi})\right], \tag{7}$$

in which α^{-1} is the Deborah number, $\alpha = \rho a^2/\eta t_m$, and the parameter ϖ is defined by

$$\varpi(\omega) = (\omega^*)^2 + i\omega^*$$

with the dimensionless frequency ω^* given by

$$\omega^* = t_m \omega.$$

This expression is similar to the one found using the method of volume averaging [18], and in the limit $t_m \to 0$ we recover the result of Zhou and Sheng. We observe that the inverse of a Deborah number α, can be expressed as

$$\alpha = \frac{t_v}{t_m},$$

where $t_v = \rho a^2/\eta$ is a characteristic time for purely viscous effects. Thus α is the governing parameter that determines whether the behavior is elastic or viscous. The critical value of α may be obtained by analyzing the imaginary part of the permeability. In order to do this, it is convenient to use the dimensionless permeability defined by

$$K^*(\omega) = -\frac{8K(\omega)}{a^2},$$

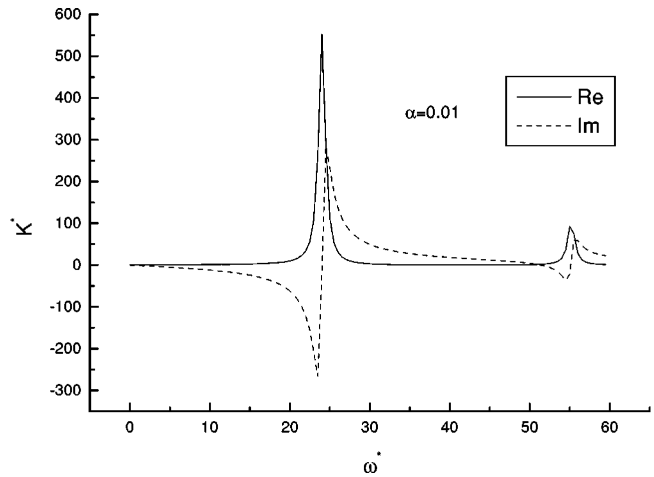

FIG. 1. The dimensionless permeability is plotted against the dimensionless frequency for the case of a Deborah number of $\alpha = 0.01$. The solid and dashed lines correspond to the real and imaginary parts, respectively.

so, that Eq. (7) takes the form

$$K^*(\omega) = \frac{8(1-it_m)}{\alpha\varpi}\left[1 - \frac{2}{\sqrt{\alpha\varpi}J_0(\sqrt{\alpha\varpi})}J_1(\sqrt{\alpha\varpi})\right]. \tag{8}$$

This expression gives the dynamic behavior of a Maxwell fluid flowing in a tube. It is important to note that it does not take into account the nonlinear aspects of viscoelasticity; , Eq. (8) does, however, contain the intrinsic elastic behavior. To gain some insight into the importance of this formula, we present in the following section some numerical calculations.

III. DETERMINATION OF THE MAXIMUM VELOCITY

It is clear that the maximum value of the permeability corresponds to the maximum value for the mean velocity for a given pressure gradient. In order to illustrate the results that we have obtained, we now consider some particular values. For example, using a relaxation time on the order of seconds, a mass density and viscosity on the order of those of water, and a tube radius on the order of centimeters, we find $\alpha \approx 0.01$. The behavior of $K^*(\omega)$ for this value of α can be seen in Fig. 1. It is important to emphasize that the maximum of the permeability is not at the phase point, because the dissipation shifts this maximum to slightly higher frequencies.

In the case of Newtonian fluids a dissipative behavior is always found in transient problems, while in Fig. 1 we see an elastic resonance at a specific frequency. This resonance implies that the dynamic response could be several orders of magnitude higher than the steady one. The critical value for the parameter α which determines whether a dissipative behavior prevails or a resonance at a given frequency appears is [18]

$$\alpha_c = 11.64.$$

If $\alpha > \alpha_c$, the behavior of the system is dissipative, while resonant frequencies can be found if $\alpha < \alpha_c$. Now, two questions immediately arise that will serve to assess the importance of such resonances. The first one concerns the maximum value of $\mathrm{Re}[K^*(\omega)]$ for a given value of α, while the second has to do with the value of the frequency corresponding to such a maximum. In order to present useful results, we have plotted in Figs. 2 and 3 the numerically determined values of $K^*_{\max}$ and $\omega^*_{\max}$ as functions of α, respectively. These results can be fitted rather well for several orders of magnitude of the parameter α by means of

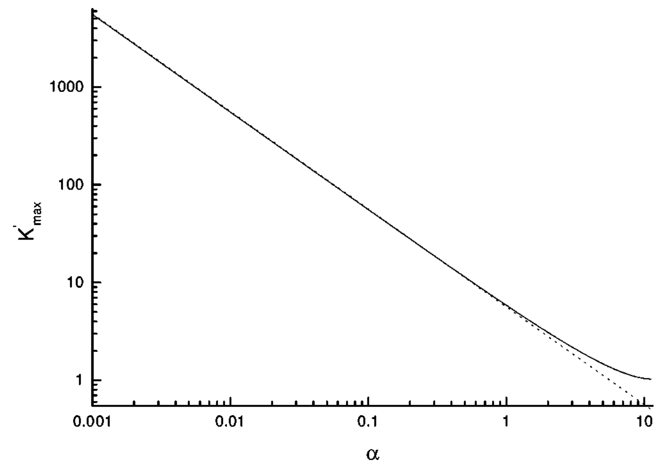

FIG. 2. This graph shows $\mathrm{Re}(K^*_{\max}) - \alpha$ pairs (solid line) and the adjusting formula Eq. (9) (dashed line).

$$K^*_{\max} = \frac{10^{3/4}}{\alpha}, \tag{9}$$

and

$$\omega^*_{\max} = \frac{10^{2/5}}{\sqrt{\alpha}}. \tag{10}$$

Combination of these two results leads to

$$K^*_{\max} = 10^{-1/5}(\omega^*)^2.$$

This expression indicates that, in the elastic regime, the maximum value for the velocity grows more rapidly than the frequency of the oscillating pressure. Up to this point, we

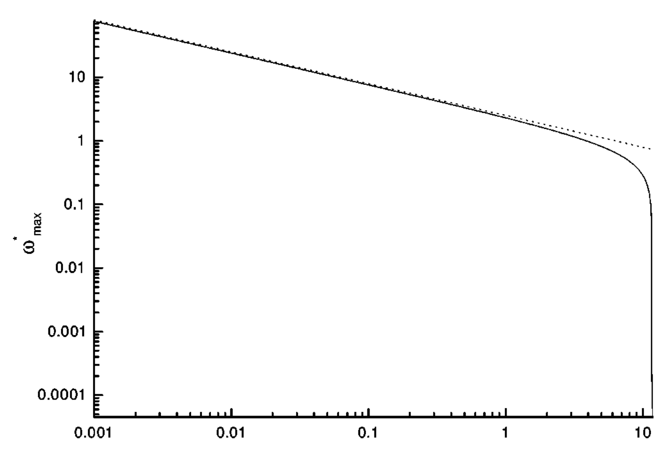

FIG. 3. This graph shows the comparison between the adjusting formula [dashed line Eq. (10)] and $\omega_{\max}$-α pairs (solid line).

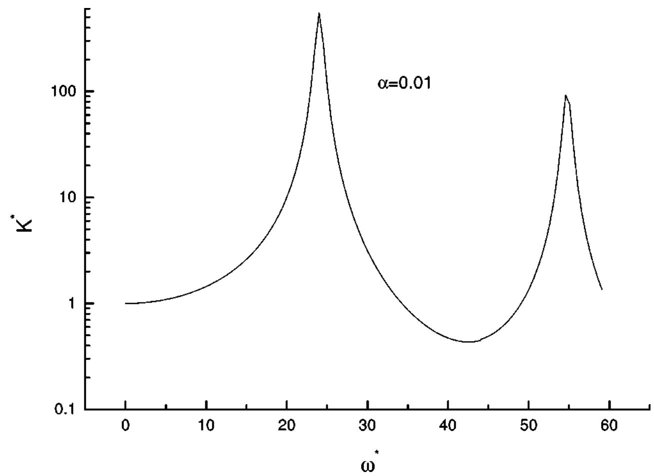

FIG. 4. In this figure the real part of the dimensionless permeability is shown. Here it is clear that the second maximun also can be important.

have analyzed only the main maximum but, as can be seen in Fig. 4, there are other local maxima. The second maximum shown in Fig. 4 may also be important because it can be orders of magnitude larger than the permeability for steady flow, and the (reduced) permeability values between the first and the second maximum are less than 1. This illustrates the filtering characteristic of a viscoelastic fluid flowing in a tube when oscillations are present.

As an application of our analysis of the linearized Maxwell fluid flowing in a tube, we consider the case of human blood flowing in the arteries. In biorheology it is considered that a value for α less than one is representative of elastic behavior [20,21]. For blood cells the value of the relaxation time is on the order of 0.06 s [22], the radius of arteries go from 0.02 to 0.35 cm [20] and the density of the blood is approximately 1.05 g/cm^3 [20,23]. Assuming linear viscoelasticity,

$$\eta^* = \eta' - i\eta''$$

it is found that

$$\eta' = a\exp(bP),$$

where $a = 1.055$, $b = 0.035$, and P ($P \in [0.25, 0.60]$) is the packed cell volume fraction [24], the viscosity units being in cp. The viscosity of blood depends on the shear rate, ranging from 5 cp at shear rates on the order of 500 s^{-1} up to 20 cp at low shear rates [25] . Here we consider two representative values for the viscosity, 5 cp and 20 cp, in the region of 100 to 500 s^{-1} shear rate, with hematocrit 31% at 18 °C [25].

From these data we can find that in blood the parameter α takes on values between a lower limit given by

$$\alpha_l = \frac{1.05(0.02)^2}{20(0.06)} = 3.5\times 10^{-4}$$

and an upper limit given by

$$\alpha_h = \frac{1.05(0.35)^2}{5(0.06)} = 0.42875.$$

Thus the optimum pumping frequency range for these values of α lies between $\omega_{\max l} = 10^{2/5}/\sqrt{0.000\,35} = 134$ rad/s and $\omega_{\max h} = 10^{2/5}/\sqrt{0.42875} = 3.8$ rad/sec, or equivalently,

$$\nu_l = \frac{134}{2\pi} \approx 20 \text{ Hz}$$

and

$$\nu_h = \frac{3.8}{2\pi} \approx 0.6 \text{ Hz}.$$

With these calculations, we may ask whether the heart pumps our blood in the most efficient manner according to the viscoelastic properties of the blood and the radius of the arteries. The range of our estimations is in agreement with the experimental results in oscillatory flow in rigid blood-filled tubes [20]. But before addressing the answer, it is important to clarify some points involved in such calculations. There are other times associated with the viscoelastic response of the blood cell ensemble on the order of seconds [13]. Moreover, the higher value of α is probably correct, indicating that the most efficient frequency for blood flow in arteries with 0.35 cm, is of the order of 1 Hz. The lower value we found for α is open to question, because the blood flowing in very small capillaries has a lower viscosity than that of blood flowing in thick arteries, due to a smaller packed cell volume fraction in the former [22]. However, our simple linear calculation gives a correct order of magnitude without considering complications such as the fact that the arteries and veins are elastic and that there are other nonlinearities in the viscoelastic characteristics of the blood. In this sense, our estimates in this example serve as an indication of the order of magnitude for a resonantlike response in the oscillating flow of viscoelastic fluids in a tube. Of course, we are fully aware of the fact that in the circulatory system the elasticity of blood vessels is often important. Nevertheless, the results for a Newtonian fluid in a deformable porous medium [4] indicate that the elasticity of the medium has no effect on the expression for the dynamic permeability and one would expect that the same applied also in this case, but clearly this must be verified. Therefore, an obvious extension of the present study, which we may address in the future, would be to consider elastic tubes and include a more realistic model for the complex structure of the blood.

IV. CONCLUSION

The developments presented in the previous sections deserve a few further comments. While it is clear that the Maxwell fluid is a drastic simplification of a viscoelastic fluid, we are persuaded that the results we have derived for the dynamic permeability capture the essential physics of the problem, albeit in an approximate manner.

In transport phenomena in porous media, there has already been some work dealing with the flow of non-Newtonian fluids through porous media using several techniques [7,5]; however, the dynamic generalization of Darcy's law for such fluids that we derived in Ref. [18] had not been attempted previously to our knowledge. From the results of this paper we are able to give a criterion to select the range of frequency to obtain an enhancement in the oscillating flow rate. There is no simple scaling relation, such as the one found by Zhou and Sheng [4] for the Newtonian fluid. However, it is clear that many different combinations of fluid properties and tube diameters lead to the same value of α and thus some kind of ''universality'' is implied by Eq. (7). Therefore, we would expect that in a real system one could obtain enhanced transport by identifying the parameter corresponding to our α that combined the characteristics of the fluid (density, viscosity, and stress tensor relaxation time) and the radius of the tube. The existence of a velocity gap [17] in the dilute solution flows through porous media may be qualitatively understood as a resonance phenomena close to the critical value of α. It should thus be clear that the performance of experimental studies to assess the value of our predictions would be called for. The same comments can be applied to biofluids flowing in capillaries for which the results described in the third section are particularly suggestive.

Finally we would hope our findings encourage further experimental and theoretical studies for (a) the possible use of porous media in organic filter applications using an oscillatory pressure gradient, (b) the consequences of the enhancement in the dynamic permeability for oil recovery techniques, and (c) the nonlinear behavior in blood or organic flow in unsteady situations.

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