Critical slowing down in synchronizing nonlinear oscillators

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We investigate the transient properties of two van der Pol oscillators that are interacting with various types of couplings. As the coupling constant varies, the transient dynamics changes qualitatively and new intermediate or asymptotic attractors may appear. This can be considered as a kind of dynamic phase transition in nonequilibrium systems. It is interesting to find that two nonlinear oscillators could be phase locked and synchronized with appropriate couplings, and that critical slowing down might occur near the boundaries of the synchronization domain. Besides the genuine asymptotic synchronization, we also observe the transient synchronization that occurs only momentarily. For both classes of synchronization, the relevant exponent describing the slowing down dynamics is found to be equal to a mean field value of unity. [S1063-651X(98)11111-X]

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I. INTRODUCTION

Considerable attention has been given recently to the dynamics of coupled systems, especially the synchronization of chaotic motions [1–4]. It is indeed an astonishing fact that two chaotic motions that are practically random and sensitive to initial conditions could be synchronized by a simple coupling. While more and more systems are found to exhibit chaotic synchronization with various types of coupling, transient dynamics leading to the final synchronization deserve more attention. In this study, we investigate the synchronization dynamics of two limit cycle oscillators since the limit cycle is a more elementary attractor and is much easier to monitor.

The van der Pol model of self-sustained oscillation can be described by an autonomous differential equation,

$$\ddot{x}=f(x,\dot{x})=-\alpha x-\beta\dot{x}(x^2-1). \tag{1}$$

This model and its variations, which were initially designed to describe oscillating circuits, have many applications in science and engineering [5]. It is an important model in nonlinear dynamics since it is a paradigm for the limit cycle. Furthermore, this limit cycle can easily bifurcate to a fixed point by adding a constant bias term, and to a chaotic motion by modulating with a simple periodic function of time.

In nonequilibrium systems, the phase transition is usually associated with a bifurcation between attractors, which could be fixed points, limit cycles, and the like [6]. When a control parameter is adjusted toward its critical value, the dynamic system loses its stability and the existing phase yields to a new one. Within a close vicinity of the instability, the relaxation process from an initial state to the expected attractor slows down [7]. According to the linear stability analysis, the slow transient process leading to a fixed point is characterized by a relaxation time that diverges to infinity following a simple power-law relationship,

$$T_{\rm rel}\approx|K-K_c|^{-\gamma}, \quad \gamma=1.0, \tag{2}$$

where K_c is the critical control parameter, and the *critical exponent* γ assumes a mean field value of unity [7–12].

This phenomenon of *critical slowing down* resembles that which occurs in equilibrium systems. While it is a natural consequence in the deterministic transition between fixed points, it is also observed in stochastic multistable systems in which noise intensity is treated as a control parameter [9]. In recent years, this unique behavior has also been detected in a noisy Hopf bifurcation in which a fixed point yields to a limit cycle [10]. The same phenomenon was also found in period-doubling bifurcation of discrete maps [11] and in deterministic Hopf bifurcation [12].

In this study, we investigate the transient process before synchronization of two limit cycles is finally achieved. We observe the same phenomenon of *critical slowing down* near the boundaries of synchronization domains. We could interpret the phenomena of synchronization and desynchronization as two *nonequilibrium phases*, and the transition between them as a result of stability loss. Therefore, the *nonequilibrium phases* are now extended to include those dynamic aspects that are not necessarily the traditional attractors. It is striking to find that two limit cycle oscillators could also reach transient synchronization that exists momentarily, and that this unique process exhibits critical slowing down as well.

In order to study the synchronization processes systematically, we examine several types of interactions. Both one- and two-way couplings will be considered. Dynamic effects caused by feedback coupling [2–4,13] will be compared with those caused by driving with differences in locations [14–16].

II. ONE-WAY DRIVING WITH DIFFERENCE IN LOCATIONS

When two identical van der Pol oscillators are started with different initial conditions, their trajectories will circulate along a common limit cycle with different phases. Since both circulations are nonuniform and have the same frequency, it is reasonable to expect that an oscillator driven by a difference in locations could manage to phase lock to the

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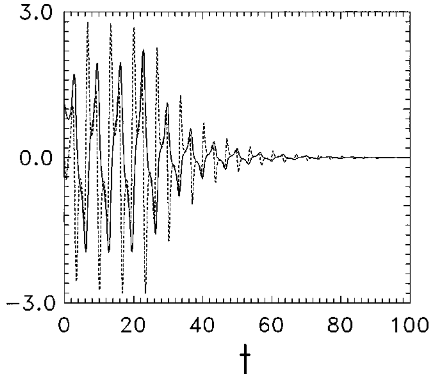

FIG. 1. Synchronization is demonstrated in a coupled system described by Eq. (5) with driving constant $K=0.3$. Solid and dashed curves represent $x-u$ and $y-v$, respectively.

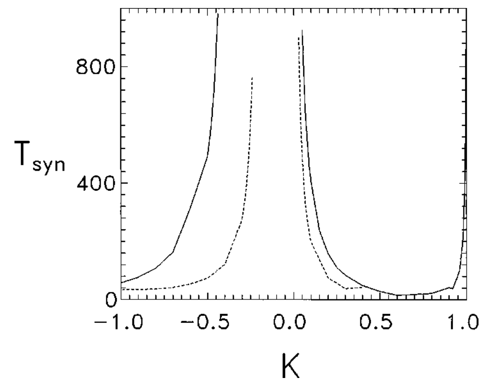

FIG. 2. Synchronization time is plotted against coupling constant. The solid curve stands for one-way drive defined by Eq. (3), and dashed curve stands for two-way drive defined by Eq. (8).

other oscillator. In this section, we study the one-way driving system described by

$$\ddot{x}=f(x,\dot{x}),$$
$$\ddot{u}=f(u,\dot{u})+K(u-x)H(t-T_0), \tag{3}$$

where K is the driving constant and T_0 is the onset time of the driving. In the above, the function $f(x,\dot{x})$ is given by Eq. (1), and $H(x)$ is the Heaviside function defined by

$$H(x)=\begin{cases}0, & x<0\\ 1, & x\geqslant 0.\end{cases} \tag{4}$$

The onset of driving is designed to guarantee that both oscillators have already relaxed to the limit cycle from their initial states. For most cases, $T_0=20$ is used.

Numerical solution of Eqs. (3) is carried out by rewriting them as a set of four coupled differential equations,

$$\dot{x}=y,$$
$$\dot{y}=f(x,y),$$
$$\dot{u}=v, \tag{5}$$
$$\dot{v}=f(u,v)+K(u-x)H(t-T_0),$$

which can be solved with the double precision Runge-Kutta scheme. Since a van der Pol oscillator with $\alpha=\beta=1$ takes a period $T\approx 6.66$, a time step of $dt=0.01$ is fine enough for most cases. Extensive studies include all possible ranges of driving constant K, both positive and negative. Computational time has been extended to $t=10^8$ or more in order to scan for possible synchronization. For most cases, initial conditions $(x_0,y_0)=(2,2)$ and $(u_0,v_0)=(1,1)$ are used to generate two limit cycle circulations with a phase lag, which is time dependent since the limit cycle oscillation is nonuniform. From time to time, different initial states are used to check the qualitative properties of the results.

Figure 1 shows that with properly chosen K values, the slave system (u,v) could be phase locked to the master system (x,y) shortly after the driving is turned on. We define *the synchronization time* as

$$T_{syn}=t_{syn}-T_0, \tag{6}$$

where t_{syn} is the time instant at which the two trajectories are close enough to be considered as synchronized. A practical criterion of synchronization could be defined as

$$\delta=\sqrt{(x-u)^2+(y-v)^2}<10^{-3}. \tag{7}$$

For higher accuracy with smaller δ, T_{syn} will assume a larger value. However, qualitative features discussed below remain unchanged.

The result of T_{syn} is plotted against the driving constant K in Fig. 2. The *synchronization domain* is defined as the range of K over which T_{syn} is finite. As shown in Fig. 2, it consists of two parts: $0.0<K<1.0$ and $K<-0.3904$. It can be seen that T_{syn} diverges near the domain boundaries $K_c=0$ and $K_c=1$. The critical slowing-down behavior with exponent $\gamma=1.0$ is demonstrated in Fig. 3.

After the coupling initiates, the driven trajectory first detours from the force-free limit cycle, and then turns back to have a quick synchronization with the driver system (x,y). Slowing down of the synchronization process reflects the loss of stability for asymptotic synchronization, and the loss of attraction to the original limit cycle. This argument applies to both critical values of $K_c=0$ and $K_c=1$, since both cases result in an instability for the synchronization process.

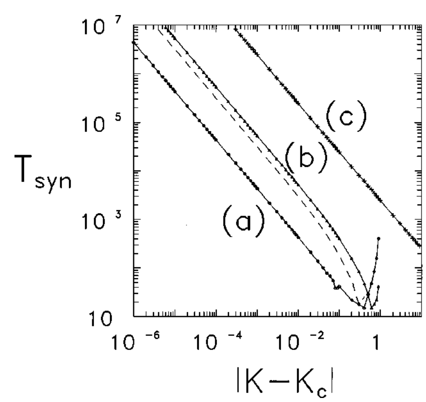

FIG. 3. *Critical slowing down* near the boundary of a synchronization domain. The solid curves stand for one-way coupling with critical coupling constant, $K_c=1.0$ (a) and $K_c=0.0$ (b). The dashed curve stands for two-way coupling with $K_c=0.0$. Curve (c) represents an overlapping of two identical plots of $K>0$ and $K<0$, (with $K_c=0$), for a modified two-way coupling defined by Eq. (9).

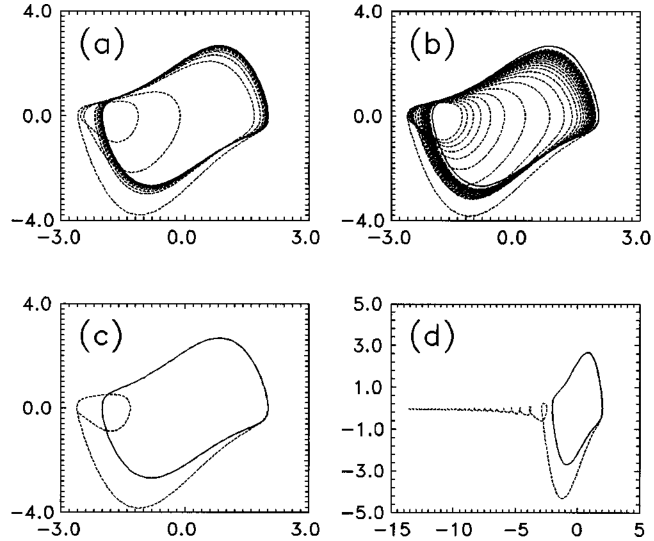

FIG. 4. Transient processes of one-way coupling defined by Eq. (3). Solid and dashed curves stand for the master system (x,y) and slave system (u,v), respectively. In (a), synchronization is achieved quickly with $K=0.92$. In (b), only part of the synchronization process is shown (up to $t=200$) for $K=0.96$. In (c), for $K=K_c=1.0,(u,v)$ is attracted to a deformed and displaced limit cycle. In (d), for $K=1.5,(u,v)$ is drifting with time and no synchronization with (x,y) is possible. In all four graphs, velocity ($y=\dot{x}$ or $v=\dot{u}$) is plotted against the position (x or u).

This unique phenomenon of critical slowing down resembles that observed in equilibrium transitions, and in nonequilibrium systems undergoing Hopf bifurcation from a fixed point to a limit cycle [9–12]. From the transitional point of view, it is reasonable to suggest that both synchronization and desynchronization modes of coupled systems can be considered as *nonequilibrium phases*. Closer to the domain boundary of $K_c=1.0$, the driven trajectory takes a longer time in spiralling about a deformed and displaced cycle before being attracted back to the original cycle for a final synchronization. At the threshold value of $K_c=1$, trajectory (u,v) is attracted to the new limit cycle, as seen in Fig. 4(c). For $K>1,(u,v)$ is continuously drifting away from the original cycle, as shown in Fig. 4(d). No synchronization with (x,y) is possible for $K\geqslant 1$. This could be explained qualitatively in the following. When K is positive, the restoring force for the (u,v) oscillation is weakened, and is equal to zero as $K=1$. For $K>1$, the restoring force turns out to be repelling and the original cycle loses its attraction to the disturbed trajectory. This also results in the asymmetry of synchronization domains for $K>0$ and $K<0$.

III. MUTUAL DRIVING WITH DIFFERENCES

The limit cycle is known to be a fairly strong attractor since it attracts all trajectories except the one initiated from the trivial fixed point $(x_0,y_0)=(0,0)$. In the previous section, we find that the driving with difference in locations would help the driven system to adjust its pace for a quick synchronization with the driver system on the limit cycle. It was reported that both one- and two-way driving could result in synchronization in chaotic systems [14–16]. In this section, we investigate the synchronization dynamics of two oscillators with mutual coupling,

$$\ddot{x}=f(x,\dot{x})+K(x-u)H(t-T_0),$$
$$\ddot{u}=f(u,\dot{u})+K(u-x)H(t-T_0). \tag{8}$$

Each of the two oscillators is adjusting its pace, according to the difference of its own position with respect to that of the other.

With this symmetric coupling, both trajectories (x,y) and (u,v) detour and then turn back to the original limit cycle for a phase locking. As a result, synchronization is more effective with mutual coupling. The corresponding T_{syn} is smaller than that of one-way driving. Results are shown in Fig. 2 and Fig. 3 for a comparison. However, the synchronization domain does not expand accordingly. Again, it consists of two parts: $0<K<0.4145$ and $K<-0.2169$.

Critical slowing down is also observed near the domain boundary with $K_c=0.0$, as shown in Fig. 3. For $K>0.4145$, the deformed trajectories are attracted to a new attractor of period three, and so there is no synchronization. For larger K, the trajectory is drifting continuously away from its original cycle, in a manner similar to that shown in Fig. 4(d).

Synchronization mechanism depends on the form of driving terms. We study another model of two-way coupling by modifying Eq. (8) slightly as

$$\ddot{x}=f(x,\dot{x})+K(u-x)H(t-T_0),$$
$$\ddot{u}=f(u,\dot{u})+K(u-x)H(t-T_0). \tag{9}$$

It is found that for this antisymmetric coupling, T_{syn} becomes much larger while the synchronization domain expands to include all values of K except the uncoupled case with $K=0$. It is also found that the T_{syn} K curve is symmetric with respect to $K=0$. These results are shown in Fig. 3, in which the curve (c) represents the two identical plots of $K>0$ and $K<0$.

IV. ONE-WAY FEEDBACK INTERACTION

When a van der Pol oscillator is driven by a simple sinusoidal function of time, the limit cycle can drift to other attractors, like the deformed limit cycle, torus, or chaotic motion. The driving frequency and amplitude are treated as control parameters for these transitions [1]. It was reported that [2–4] feedback between two chaotic attractors could induce synchronization. In this section, we study the driving effect due to a nonuniform oscillation, which is represented by the feedback of another oscillator,

$$\ddot{x}=f(x,\dot{x})$$
$$\ddot{u}=f(u,\dot{u})+KxH(t-T_0). \tag{10}$$

For the uncoupled case with $K=0$, the two independent trajectories circulate along a common limit cycle with time-dependent phase lag. For $K\neq 0$, the slave oscillator (u,v) is modulated by a nonsinusoidal periodic function $x(t)$. The trajectory (u,v) will be deformed since it is periodically kicked by another one with the same frequency. As long as the master oscillator remains on the original cycle, synchro-

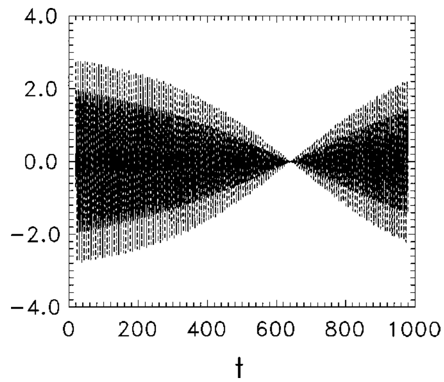

FIG. 5. Instantaneous synchronization in a one-way feedback system defined by Eq. (10). Solid and dashed curves stand for $x-u$ and $y-v$, respectively.

nization between them seems very unlikely. The same reason applies to the case when the slave system is kicked in the reverse direction with negative K.

Numerical results show that there is no genuine synchronization indeed. Careful studies reveal that there exists a very narrow domain, $0<K<0.0039$, over which, trajectory (u,v) manages to stay on the original limit cycle without deformation. It is striking to find that, for small driving, (u,v) can reach a transient synchronization with the undisturbed (x,y) as shown in Fig. 5. If we adapt the same definition of T_{syn}, as defined by Eq. (6), for the transient synchronization, we observe the same critical slowing down near $K_c=0$. The relevant critical exponent γ is also found to be one, as shown in Fig. 6.

Since this phenomenon occurs only once and, momentarily, an appropriate choice of synchronization criterion δ and time step dt is required. Very detailed and careful investigations are carried out to confirm this unique transient characteristic. For such a small driving, the (u,v) trajectory does remain on its original limit cycle all the time. In Fig. 7(a), we put together three sections of trajectories; each consists of at least three circulations (with period $T\approx 6.66$). The first one is the undisturbed (x,y) limit cycle recorded between $20\leqslant t\leqslant 40$, the second one is (x,y) with $10^6\leqslant t\leqslant 10^6+100$, and the third one is (u,v) with $10^6\leqslant t\leqslant 10^6+100$. The maximum time used to investigate this unique behavior is 10^8. Figure 7(b) shows a typical case of a (u,v) trajectory, which is deformed by larger driving. No synchronization is possible for feedback with $K\geqslant 0.0039$, and for reverse driving with negative K.

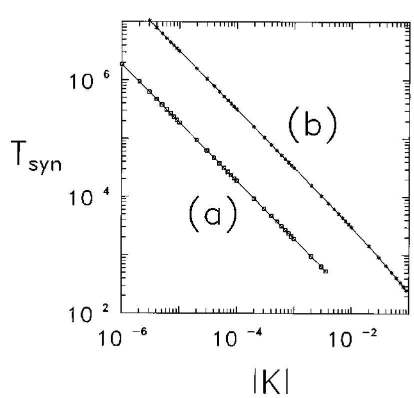

FIG. 6. *Critical slowing down* is demonstrated for (*a*) transient synchronization in a one-way feedback system with $K>K_c=0$, and (b) asymptotic synchronization in a two-way feedback system with $K<K_c=0$.

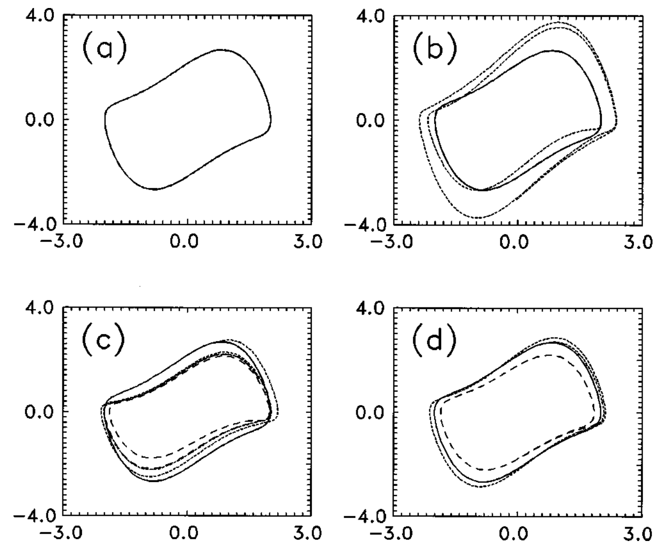

FIG. 7. Transient dynamics of coupled oscillators. (a) For one-way feedback with $K=0.003$, both (u,v) and (x,y) remain on the van der Pol limit cycle all the time, and reach an instantaneous synchronization there. (b) For one-way feedback with $K=0.5$, no synchronization is achieved. (c) For two-way feedback with $K=0.5$, two oscillators synchronize on a deformed limit cycle. (d) For two-way feedback with $K=0.15$, there is no synchronization since two trajectories [long dashed for (x,y) and short dashed for (u,v)] detour to different limit cycles. The solid curves are the original van der Pol limit cycle. In all four graphs, velocity ($y=\dot{x}$ or $v=\dot{u}$) is plotted against the position (x or u).

V. TWO-WAY PERIODIC DRIVING

Since periodic driving with sinusoidal or nonsinusoidal functions will cause deformation on the driven trajectory, there is no hope for a genuine synchronization. In this section, we look for the possibility of synchronization of the two deformed trajectories by using mutual feedback,

$$\ddot{x}=f(x,\dot{x})+Ku\ H(t-T_0)$$
$$\ddot{u}=f(u,\dot{u})+Kx\ H(t-T_0). \tag{11}$$

We consider again the symmetric coupling with a common driving constant K.

The synchronization domain for this symmetrically driving system has two parts: $0.1999<K<7.8$ and $-0.8255<K<0.0$. Critical slowing down occurs for negative K near $K_c=0.0$. This is shown by curve (*b*) in Fig. 6. The synchronization phenomenon is much more complicated for this mutual feedback system, since both oscillators could be synchronized on various attractors with $x-u=y-v=0$. Within the synchronization domain, both (u,v) and (x,y) approach a new and smaller limit cycle and phase lock there, as shown

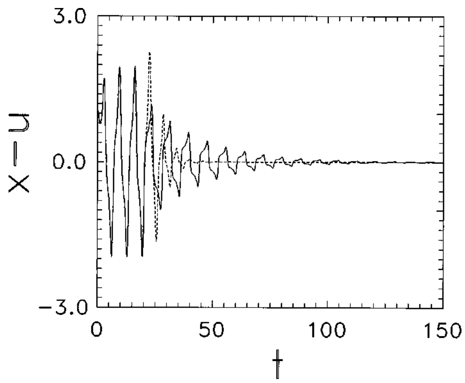

FIG. 8. Two-way periodic feedback defined by Eq. (11) results in a genuine synchronization. Solid and dashed curves stand for $K=0.3$ and $K=-0.3$, respectively.

in Fig. 7(c). For $K=7.8$, both trajectories approach a fixed point at (2.2, 0.0) and become stationary. Another strange synchronization pattern is a nonstop drift, in which both trajectories merge together and drift unbound, similar to that shown in Fig. 4(d).

For those K that lie outside the synchronization domain, trajectories (x,y) and (u,v) are deformed to different limit cycles. Therefore, synchronization between them is impossible. A typical situation is shown in Fig. 7(d).

By comparing these results with those in the previous section, we can see that two-way coupling is more effective in synchronizing two oscillators. It is interesting to find that, unlike the case of one-way coupling, reverse feedback with $K<0$ achieves the synchronization in a shorter time. In Fig. 8, we compare this situation for $K=0.3$ with that for $K=-0.3$.

VI. CONCLUSIONS AND DISCUSSIONS

The van der Pol limit cycle is a simple attractor that is periodic and is easy to keep track of. It is thus a practical model for the investigation of synchronization dynamics. This study examines the transient processes leading to synchronization, and attempts to determine the synchronization domains for various coupled systems. With this simple model, we are able to probe the phase transition properties in nonequilibrium systems.

The critical slowing-down phenomenon observed near the boundaries of synchronization domains reflects the unique transient dynamics when a phase transition point is driven toward its instability. By treating the synchronization and the desynchronization phenomena as two distinct *phases*, a transition between them could be considered as a special class of dynamic phase transition, which is unique only to nonequilibrium systems. For equilibrium systems, phases are usually defined in terms of stationary order parameters with theory of phase transition and critical phenomena well established, while the nonequilibrium phases are defined in terms of dynamic attractors, and theoretical studies of the nonequilibrium transition are much more intricate. Linear stability analysis is a standard technique used to evaluate the critical exponent γ for a transition starting from a fixed point. Determination of the critical exponent γ for a transition starting from nonstationary attractors relies on numerical computations; results are so far meager for nonequilibrium transitions. In fact, it is still an open question concerning the possibility of critical slowing down in noisy systems in which a fixed point attractor could only be defined stochastically [7,17]. More numerical investigations are needed to clarify the nature of nonequilibrium transitions, especially the critical slowing down in dynamic systems. In this report, we provide more numerical evidence and extend the definition of nonequilibrium phases to include the synchronization and nonsynchronization modes.

We find that synchronization can be achieved more effectively with two-way coupling, and by using the driving with differences in positions. On the other hand, the feedback interaction is less effective. In fact, there is no real synchronization for a one-way feedback case, in which a response system is kicked periodically by a nonsinusoidal function representing nonuniform oscillation of the drive system. For the case of two-way feedback, both systems are mutually kicked so that both trajectories deform and it is possible for them to synchronize on a new attractor, which may be topologically different from the original limit cycle. Monitoring the complete transient process is therefore essential for synchronization studies.

The instantaneous synchronization that also exhibits critical slowing down is itself an interesting phenomenon, since this state is not an asymptotic attractor defined in a traditional way. It is only a transient process that occurs momentarily and once only. This phenomenon merits more investigation.

Synchronization is usually detected by monitoring the vanishing of both $x-u$ and $y-v$. The choice of criterion δ should be flexible, and must be considered together with the choice of time step dt in solving the equations numerically. This is especially true if one is to detect the transient synchronization, or to distinguish the transient one from the genuine one. We emphasize again the importance of the transient processes leading to synchronization, since we find that two systems could be synchronized on various states, which might be limit cycle, fixed point, nonstationary drift, or probably chaotic motion.

Synchronization in chaotic systems is much more intricate. Indeed it was found that identifying a chaotic synchronization might depend on the choice of time step [18], and on the computational precision [19,20]. Results in the present simple systems could help us to understand the mechanism and the dynamics of chaotic synchronization. Research along this line is under way.

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