Turbulent transport of a tracer: An electromagnetic formulation

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We present a formulation of the problem of advection and diffusion of a passive tracer by an arbitrary, incompressible velocity field. A Wiener path integral is employed to prove that the problem is *identical* to the diffusive dynamics of a charged particle in electromagnetic fields constructed from the velocity field. The case of zero diffusion has characteristics that coincide with the integral curves of the velocity field. This case is, of course, structurally unstable, and the limit of small diffusion is correctly described by the Wenzel-Kramers-Brillouin limit of our path-integral principle, wherein the tracer dynamics equals the orbits of point charges in electromagnetic fields. To lowest order, diffusive effects are accounted for within a Hamiltonian framework, and the limit of zero diffusion emerges as an unstable submanifold embedded in a six-dimensional phase space. We illustrate these ideas by considering the simple case of tracer advection-diffusion in the flow field of a time-independent, straight vortex line. We also briefly discuss generalization of the path-integral principle for the case where tracer sources and/or sinks are included. When the velocity field obeys the Navier-Stokes equation, the associated electromagnetic fields satisfy the equations of magnetohydrodynamics for a fluid with resistivity that is equal to the viscosity of the (real) fluid. [S1063-651X(98)07507-2]

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The transport and diffusion of a tracer (such as temperature fluctuations or concentration variations of a dye) in a turbulent fluid can often be modeled by assuming that the tracer does not influence the dynamics of the turbulence itself [1]. In the simplest cases, the evolution of the concentration of such a passive tracer, $n(\mathbf{x},t)$, is described by the following advection-diffusion equation (ADE):

$$\frac{\partial n}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla} n=\frac{\kappa}{2}\boldsymbol{\nabla}^2 n,$$

$$\boldsymbol{\nabla}\cdot\mathbf{v}=0\,, \tag{1}$$

where $\mathbf{v}(\mathbf{x},t)$ is the given incompressible velocity field, and $\kappa>0$ is the tracer diffusivity. The ADE is linear in n, and the random character of the solutions arises from either randomness of $\mathbf{v}(\mathbf{x},t)$ or from the initial distribution of n. The multiplicative nature of the randomness, arising from the advective term, leads to intermittency in the statistical distribution functions describing the tracer [2].

We note that the ADE is identical to the Fokker-Planck equation describing the overdamped Brownian motion of a particle subject to an external force $\mathbf{v}(\mathbf{x},t)$ (for unit mobility)—cf. references in [3] for a review of the latter. One difference is that the Fokker-Planck equation is only an approximation in which the lowest two moments are retained, whereas the ADE is an ''exact'' equation in the context of passive tracer advection. A more considerable difference is that we do not have to view $\mathbf{v}(\mathbf{x},t)$ as a given force field. In this paper we demonstrate that it is more fruitful to construct electromagnetic (EM) potentials,

$$\mathbf{A}=-\mathbf{v},\quad \varphi=-\frac{v^2}{2}, \tag{2}$$

from the given $\mathbf{v}(\mathbf{x},t)$, and formulate the ADE as a Wiener path integral (WPI) for the diffusive dynamics of a point charge $(q=m=c=1)$. $\mathbf{A}$ and φ are the vector and scalar potentials for the EM fields, and incompressibility translates to working in the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. The WPI has received considerable attention in the literature on Brownian motion [cf. Refs. [3(b)] and [4]], and we will develop analogies, as well as point out differences as they arise. Below we provide a brief sketch of the derivation of the ADE from the WPI principle before discussing the nature of the EM fields, as well as their action on the tracer. Of particular interest is the case when $\mathbf{v}(\mathbf{x},t)$ obeys the Navier-Stokes equation.

The electric and magnetic fields corresponding to equation (2) are

$$\mathbf{E}=\partial_t\mathbf{v}+\boldsymbol{\nabla}\left(\frac{v^2}{2}\right),$$

$$\mathbf{B}=-\boldsymbol{\nabla}\times\mathbf{v}=-\boldsymbol{\omega}=-\text{vorticity}. \tag{3}$$

The dynamics of the (fictitious) charge in the (fictitious) EM fields is governed by the usual equation of motion,

$$\ddot{\mathbf{x}}=\mathbf{E}+\dot{\mathbf{x}}\times\mathbf{B}, \tag{4}$$

The Galilean invariance of the ADE (1) is reflected in the invariance of Eqs. (3) and (4) with respect to boosts. This property might not be immediately obvious, but can be readily verified by boosting to a (primed) frame that travels with uniform velocity $\mathbf{u}$ with respect to the laboratory frame. The EM fields in the boosted frame are $\mathbf{E}'=[\partial_{t'}\mathbf{v}'+\boldsymbol{\nabla}'(v'^2/2)]$ and $\mathbf{B}'=[-\boldsymbol{\nabla}'\times\mathbf{v}']$, respectively. Applying the transformation laws, $(\mathbf{x}'=\mathbf{x}-\mathbf{u}t, t'=t, \mathbf{v}'=\mathbf{v}-\mathbf{u})$, it can be verified that $\mathbf{E}'=\mathbf{E}+\mathbf{u}\times\mathbf{B}$ and $\mathbf{B}'=\mathbf{B}$, which are the well-known (Galilean) transformation laws for EM fields; it hardly needs to be stressed that the Galilean invariance of Eq. (4) follows automatically. There is also a basic differ-

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ence between potential flows ($\boldsymbol{\omega}=\mathbf{0}$), and vortical flows ($\boldsymbol{\omega}\neq\mathbf{0}$); the former produce no magnetic field, whereas the magnetic fields of the latter have significant effect on particle dynamics.

The above equation of motion may also be derived from an action principle. For any path $\mathbf{x}(t)$ that goes from $(\mathbf{x}_i,t_i)$ to $(\mathbf{x}_f,t_f)$, we define the action functional

$$S[\mathbf{x}(t)]=\int L(\mathbf{x},\dot{\mathbf{x}},t)dt, \tag{5}$$

where the Lagrangian

$$L=\frac{\dot{\mathbf{x}}^2}{2}+\mathbf{A}\cdot\dot{\mathbf{x}}+\frac{A^2}{2}=\frac{1}{2}|\dot{\mathbf{x}}+\mathbf{A}|^2 \tag{6}$$

$$=\frac{\dot{\mathbf{x}}^2}{2}-\mathbf{v}\cdot\dot{\mathbf{x}}+\frac{v^2}{2}=\frac{1}{2}|\dot{\mathbf{x}}-\mathbf{v}|^2. \tag{7}$$

The equations of motion (4) are obtained, as usual, by extremizing the action with respect to variations of the path $\mathbf{x}(t)$, keeping the end points fixed. Both Lagrangian and action are non-negative definite, and their minimum value of zero occurs for the special solution, $\dot{\mathbf{x}}=\mathbf{v}(\mathbf{x},t)$ of the equations (4); borrowing a term from a related work on overdamped Brownian motion, we will refer to this solution as the *optimal* solution.

Now we introduce the WPI principle: the relative probability of going from $(\mathbf{x}_i,t_i)$ to $(\mathbf{x}_f,t_f)$ by path $\mathbf{x}(t)$ is assumed to be given by $\exp[-S/\kappa]$, where S is the action given by Eq. (5). Therefore, the probability of going from $(\mathbf{x}_i,t_i)$ to $(\mathbf{x}_f,t_f)$ by *any* path is given by the Green function

$$G(\mathbf{x}_f,t_f;\mathbf{x}_i,t_i)=\sum_{\text{paths}} e^{-S/\kappa}, \tag{8}$$

where the sum over paths requires employment of the Wiener measure (cf. Ref. [5]) in the space of paths. We may write this explicitly using Feynman's method (c.f. Ref. [6]) of splitting the time interval into a large number $(N+1)$ of thin slices of equal sizes, $\epsilon=(t_f-t_i)/(N+1)$. At each intermediate time step, $t_1,t_2,\ldots,t_N$, we choose spatial coordinates, $\mathbf{x}_1,\ldots,\mathbf{x}_N$, and define the Green function as the limit [7],

$$G(\mathbf{x}_f,t_f;\mathbf{x}_i,t_i)=\lim_{N\to\infty}\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_N\left(\frac{1}{2\pi\kappa\epsilon}\right)^{3(N+1)/2} \times\exp\left[-\frac{\epsilon}{\kappa}\sum_{j=0}^{N}\left\{\frac{1}{2}\left|\frac{\mathbf{x}_{j+1}-\mathbf{x}_j}{2}\right|^2+\frac{v^2(\mathbf{x}_j)}{2}\right\} +\frac{1}{\kappa}\sum_{j=0}^{N}(\mathbf{x}_{j+1}-\mathbf{x}_j)\cdot\mathbf{v}\left(\frac{\mathbf{x}_j}{2}\right)\right]. \tag{9}$$

If we are given $n(\mathbf{x}_i,t_i)$, then the Green function propagates this initial distribution to time t_f,

$$n(\mathbf{x}_f,t_f)=\int G(\mathbf{x}_f,t_f;\mathbf{x}_i,t_i)n(\mathbf{x}_i,t_i). \tag{10}$$

Standard manipulations of Eqs. (9) and (10) (cf. Ref. [5], Chap. 4) can be used to show that $n(\mathbf{x},t)$ satisfies the ADE [Eq. (1)].

Paths for which S is minimum contribute most to G in Eq. (8). Thus the optimal path $\dot{\mathbf{x}}=\mathbf{v}(\mathbf{x},t)$ together with neighboring paths lying inside a tube of width proportional to $\kappa^{1/2}$ make the dominant contributions to G. The other classical solutions of Eq. (4), for which S is an extremum but not a minimum, do not directly contribute to G. However, as experiments with electronic devices show, these extremal paths are not only real, but contribute fundamentally to large fluctuations away from the optimal solution [8]. To explore their significance, let us first cast our Eq. (4) in Hamiltonian form. The Hamiltonian

$$H(\mathbf{x},\mathbf{p},t)=\frac{\mathbf{p}^2}{2}+\mathbf{p}\cdot\mathbf{v}(\mathbf{x},t) \tag{11}$$

gives rise to the equations of motion,

$$\dot{x}_i=p_i+v_i, \quad \dot{p}_i=-p_j\frac{\partial v_j}{\partial x_i}, \tag{12}$$

which are identical to Eqs. (4); the optimal solution corresponds to $\mathbf{p}=\mathbf{0}$.

A blob of the tracer introduced into even a nonturbulent fluid will be teased out into whorls and tendrils. The dynamics of the strong variations of n can be described in the Wenzel-Kramers-Brillouin (WKB) (or ''eikonal'') limit of the ADE, wherein the physical significance of $\mathbf{p}\neq\mathbf{0}$ dynamics is clearly revealed (c.f. Ref. [3(a)] for other applications). Let us write $n=F\exp[-W/\kappa]$, where both W and F have far gentler spatial variations than n itself [9]. Substituting for n in the ADE, terms of order κ^{-1} give us the following equation for W:

$$\frac{\partial W}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}W+\tfrac{1}{2}|\boldsymbol{\nabla}W|^2=0\,. \tag{13}$$

Thus W obeys the same (Hamilton-Jacobi) equation as the action S. However, it is better to view Eq. (13) as the evolution equation of a three-dimensional Lagrangian manifold, defined by $\mathbf{p}=\boldsymbol{\nabla}W$, in a six-dimensional phase space $(\mathbf{x},\mathbf{p})$; the Hamiltonian equations (12) determine the ''rays'' of this system. The meaning of $\mathbf{p}$ is evident from

$$\mathbf{p}=\boldsymbol{\nabla}W=-\kappa\frac{\boldsymbol{\nabla}n}{n}=2\times(\text{diffusion velocity})+O(\kappa). \tag{14}$$

If we are given $n(\mathbf{x},0)$ at some initial time $t=0$, then $\mathbf{p}(\mathbf{x},0)$ is also known, and this Lagrangian manifold can be evolved forward in time using the Hamiltonian equations (12). The case $\kappa=0$ corresponds to $\mathbf{p}=\mathbf{0}$. Rays with these initial conditions always trace the integral curves of the velocity field. But this motion is unstable to small, diffusive fluctuations when κ is small. To lowest order in κ, the characteristics of the ADE are described by the dynamics of a point charge in the EM fields of Eqs. (3); *thus the effects of a small amount* *of diffusion on larger advective motions can be described within a Hamiltonian framework, by constructing EM fields from the velocity field*.

A simple example to work out is the encounter between the tracer and a time-independent, straight vortex tube of circular cross section. The particle dynamics is integrable, and we can readily solve the special case of a line vortex, for which $\mathbf{v}=(\Gamma/2\pi R)\hat{\theta}$ and $\boldsymbol{\omega}=\Gamma\delta(\mathbf{R})\hat{z}$, where (R,θ,z) are cylindrical polar coordinates, Γ is the circulation around the vortex, and $\delta(\mathbf{R})$ is a Dirac δ function in the plane perpendicular to the line. The canonical momenta are $p_R=\dot{R}$, $p_z=\dot{z}$, and $p_\theta=(\ell-\Gamma/2\pi)$—where $\ell=R^2\dot{\theta}$ is the particle angular momentum—and the Hamiltonian is

$$H=\tfrac{1}{2}\left(p_R^2+p_z^2+\frac{p_\theta^2}{R^2}\right)+\frac{\Gamma p_\theta}{2\pi R^2}. \tag{15}$$

Both p_z and p_θ are conserved, so the dynamics reduces to $\ddot{R}=C/R^3$, where

$$C=p_\theta\left(p_\theta+\frac{\Gamma}{\pi}\right) \tag{16}$$

is a constant that determines the sign of the force felt by the particle.

Optimal paths follow $\mathbf{v}$, going around in circles in the plane perpendicular to $\hat{z}$ (these paths have $\mathbf{p}=\mathbf{0}$, from which $\dot{R}=0$, $\dot{z}=0$, and $p_\theta=0$, the latter implying that $C=0$). The smallest of deviations away from $\mathbf{p}=\mathbf{0}$ makes $C\neq 0$, and the force is either attractive or replusive. *Thus the optimal solution* $\dot{\mathbf{x}}=\mathbf{v}$ is unstable to diffusive fluctuations (much like what happens for large fluctuations in overdamped brownian dynamics [8]), illustrating the structural instability of the case of zero diffusion, as was discussed earlier. It is straightforward to verify that $C>0$ when $|\ell|>(\Gamma/2\pi)$, so that the particle spirals outward—the corresponding parts of the tracer are spun out to infinity. For smaller $|\ell|$, $C<0$, the force is attractive, and those parts of the tracer for which $|\dot{R}|<|C|/R$ are eventually captured by the vortex line. Thus the vortex line will shred the tracer, cloaking itself with the low angular momentum parts and casting off the rest to the vorticity-free, straining velocity field outside. The physical meaning of the point particle dynamics is as follows: as tracer particles go around in circles, following the velocity field, they also diffuse. The diffusion in the radial directions brings some tracer particles to the center, whereas others drift outward.

In a turbulent fluid, vorticity appears to be concentrated in filaments [10], so our fictitious charged particle (i.e., a ray) will see magnetic ropes immersed in an electric sea. The particle dynamics of mutually noninteracting charges in such a disordered environment should reveal much about the behavior of solutions to the ADE. An array of straight vortex tubes, all parallel to one another, is a two-dimensional problem. The velocity field $\mathbf{v}=\boldsymbol{\nabla}\times(\psi\hat{z})$, where $\psi(x,y,t)$ is a stream function is not only independent of z, but has no component along $\hat{z}$. In this case,

$$\mathbf{A}=\boldsymbol{\nabla}\times(\psi\hat{z}),\qquad \varphi=-\frac{|\boldsymbol{\nabla}\psi|^2}{2}. \tag{17}$$

Even when ψ is independent of time, charged particle orbits in the above EM potentials can be chaotic. The optimal paths follow the isocontours of ψ; an interesting unsolved problem is to determine the conditions under which the optimal paths are unstable to diffusive fluctuations. A related problem is to make a connection between the regular or chaotic dynamics of charged particles, and macroscopic transport coefficients. The general case of three-dimensional, turbulent flows requires following the orbits of charged particles in disordered EM fields (Ref. [1] also discusses this for *real* EM fields), a task that appears numerically less forbidding than solving the ADE itself. Reference [1] reviews the extensive literature on tracer transport with emphasis on percolation properties of the networks formed by the channels.

When sources and sinks of n are also present, the following source-and-sink ADE (sADE) describes advection and diffusion of the tracer:

$$\frac{\partial n}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}n=\frac{\kappa}{2}\boldsymbol{\nabla}^2 n+\gamma n,\qquad \boldsymbol{\nabla}\cdot\mathbf{v}=0, \tag{18}$$

where $\gamma(\mathbf{x},t)$ is the local rate of generation. If we generalize Eq. (2) to include a source field $\kappa\gamma$,

$$\mathbf{A}=-\mathbf{v},\qquad \varphi=-\frac{v^2}{2}+\kappa\gamma, \tag{19}$$

then the Lagrangian of Eq. (6), with the above expressions for $\mathbf{A}$ and φ, used in the WPI principle—Eqs. (9) and (10)—leads to the sADE. The magnetic field remains unaffected, whereas the electric field picks up an additional term equal to $-\kappa\boldsymbol{\nabla}\gamma$. Charged particle dynamics now has an interesting coupling to this extra electric field that combines both diffusion and source rate. The Peclet number may be defined as $\mathrm{Pe}=V^2T/\kappa$, where V and T are flow velocity and time scales. The limit of large Pe is commonly encountered in turbulent mixing, and when $\mathrm{Pe}\gg\gamma T$, we have $v^2\gg\kappa\gamma$, which implies that the source field in Eq. (19) is a small perturbation on the basic advective-diffusive dynamics.

Let us return to general considerations of the EM fields themselves. From the definitions in Eqs. (3), the EM fields obey the source-free Maxwell equations:

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0,\qquad \boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}. \tag{20}$$

It is interesting to see what constraints are imposed on the EM fields when $\mathbf{v}$ obeys the Navier-Stokes (NS) equations,

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}=-\boldsymbol{\nabla}P+\nu\boldsymbol{\nabla}^2\mathbf{v},\qquad \boldsymbol{\nabla}\cdot\mathbf{v}=0. \tag{21}$$

Here P is the sum of pressure and potential forces (e.g., gravitational potential) per unit mass, and is determined by the incompressibility condition on $\mathbf{v}$: $\nabla^2 P = -\nabla\cdot[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}]$. If we take the curl of the NS equation, we obtain an equation for the vorticity. Recalling that $\boldsymbol{\omega} = -\mathbf{B}$, we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) + \nu\boldsymbol{\nabla}^2\mathbf{B}, \tag{22}$$

which is identical to the induction equation of magnetohydrodynamics for a fluid with resistivity equal to ν. Not all solutions of Eq. (22) are of interest; only the special solution, $\mathbf{B} = -\boldsymbol{\omega}$, noted by Batchelor [11] is needed. Using the NS equations in the expression for $\mathbf{E}$ in Eqs. (3), we obtain

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B} + \nu\boldsymbol{\nabla}\times\mathbf{B} - \boldsymbol{\nabla}P. \tag{23}$$

If we regard $\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B}$ as the electric field in the frame of the fluid element, then Eq. (23) bears resemblance to Ohm's law, where ν equals resistivity, $(4\pi)^{-1}\boldsymbol{\nabla}\times\mathbf{B} = \mathbf{J}$ is current density, and $-\boldsymbol{\nabla}P$ is a ''battery'' term [12].

In conclusion, (fictitious) EM fields associated with the flow of an incompressible fluid—not necessarily obeying the NS equations—couple to (fictitious) charges, whose diffusive dynamics is exactly equivalent to the dynamics of a passive tracer; ''diffusive dynamics'' refers to a Wiener path integral generalization of classical dynamics. The **E** and **B** fields, together with the equations of motion of the charge are Galilean invariant (and this is consistent with working in the Coulomb gauge). An advantage of this new formulation is that, to lowest order, diffusive effects are accounted for by a Hamiltonian formulation of the dynamics of a point charge in EM fields. When the velocity obeys the NS equations, the EM fields are carried around as if the fluid was a conducting medium whose resistivity equals its viscosity.

Note added in proof. It has been brought to the author's attention that EM fields, similar to Eq. (3), were earlier considered by M. Berry [13] in the fascinating context of male moths chasing female moths.

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