Scaling laws for a two-dimensional vibro-fluidized granular material

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A simple model for a two-dimensional vibrofluidized granular material is presented. The choice of a mean free path proportional to $1/n^{1/2}$, where n is the local number density, leads to scaling laws for the granular temperature, E_0, which fit experimental data significantly better than previously published models. In particular, E_0 is predicted to scale as $V^{4/3}$, where V is the velocity amplitude of the vibration, and as $N^{1/3}$, where N is the total number of grains. [S1063-651X(98)16309-2]

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The fluidization of a granular material by vibration has been the subject of many experimental, theoretical and numerical studies in recent years [1–10]. Under conditions of high excitation with reasonably elastic collisions, the probability density function for velocity and number density is found to be closely approximated by the classical Maxwell-Boltzmann distribution [3,9]. The question then arises: how does the parameter analogous to temperature (the so-called granular temperature E_0) in such a distribution relate to the degree of excitation provided by the vibrating base of the cell?

Molecular dynamics simulations presented in [1,2] showed that the increase in the height of the center of mass of the system varied as

$$\Delta h \propto (A_0\omega)^{\alpha}/X^{\beta}, \tag{1}$$

where α and β are constants, A_0 and ω are, respectively, the amplitude and angular frequency of vibration, and X is a dissipation parameter given by

$$X=(N/n_b)(1-\varepsilon). \tag{2}$$

In Eq. (2), N is the total number of grains, n_b is the average number of grains per layer in the condensed phase, and ε is the restitution coefficient for grain-grain collisions. Luding *et al.* found that α and β took the values 2 and 1, respectively, for a one-dimensional system, but that α dropped to 1.5 for a two-dimensional system. Δh is proportional to E_0, provided the granular temperature distribution is uniform [3]. Experimental results suggested α values in the range 1.3–1.4 (depending on the method used to calculate E_0) and β values between 0.3 and 0.6 [3]. By contrast, the simple theoretical analysis described in [3] predicted α and β values of 2 and 1, respectively, even in two dimensions. An α value differing from 2 implies that the mean particle speed does not scale linearly with the mean speed of the vibrating base.

The purpose of this Brief Report is to describe a simple modification to the theory in [3] that results in scaling constants that are much closer to the experimentally-observed ones. It is based on the observation that at high packing densities the mean free path between collisions, λ_h, deviates from the low-density value λ_l assumed in [3]:

$$\lambda_l=\frac{1}{2nd}. \tag{3}$$

Here n is the number density and d is the grain diameter. Equation (3) follows from simple two-dimensional gas kinetic theory: a rectangle of length λ_l and width $2d$ swept out by a moving grain will contain on average one other particle and result in a single collision. This assumes, however, that the grains are randomly distributed. At high densities this is no longer true: each grain is surrounded by on average six nearest neighbors whose relative positions become progressively more correlated as the density increases. In effect the nearest neighbors act as a cage, the size of which defines λ_h:

$$\lambda_h \approx \frac{B}{n^{1/2}}, \tag{4}$$

where B is a numerical constant of order unity [9]. The different dependence on n and d for the two mean free paths results in differing scaling laws for the granular temperature. In particular, as the number density increases, the collision frequency increases more slowly when evaluated by Eq. (4) than by Eq. (3). This suggests that, for a given granular temperature in the dense phase, the energy dissipation rate will be lower than expected from Eq. (3), in qualitative agreement with the simulation results presented by McNamara and Luding [10].

The derivation of the granular temperature appropriate to λ_h will be outlined next. $n(y)$ is given by [3]

$$n(y)=\frac{mgN}{E_0W}\exp\left(\frac{-mgy}{E_0}\right), \tag{5}$$

where m is the mass of each grain, g is the acceleration due to gravity, y is distance in the vertical direction, and W is the width of the cell. The collision rate is

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$$Z=\frac{\overline{c}}{\lambda_h}, \tag{6}$$

where $\overline{c}$ is the mean particle speed. $\overline{c}$ is related to E_0 through the equipartition of energy theorem:

$$\frac{m\overline{c^2}}{2}=E_0. \tag{7}$$

The fractional energy lost per collision is proportional to $(1-\varepsilon^2)$; combining this with Eqs. (4)–(7) allows the total energy dissipation rate due to grain-grain collisions to be calculated as in [3], giving

$$\dot{E}_h^T \propto (1-\varepsilon^2)\overline{c}\left(\frac{mgE_0N^3}{W}\right)^{1/2}. \tag{8}$$

This may be compared with the energy dissipation rate calculated in [3] using λ_l:

$$\dot{E}_l^T=\frac{C(1-\varepsilon^2)\overline{c}dmgN^2}{W}, \tag{9}$$

where C is a numerical constant.

The approximate rate of energy input provided by the vibrating base can be easily calculated for the case of triangular sawtooth excitation (base moving up and down at constant speed V) [3]:

$$\dot{E}^I=\frac{n(0)\overline{c}}{2}WV^2m. \tag{10}$$

Equating Eqs. (8) and (10) results in an expression for the granular temperature

$$E_0=\frac{DmV^{4/3}}{(1-\varepsilon)^{2/3}}\left(\frac{Wg}{N}\right)^{1/3}, \tag{11}$$

where D is another numerical constant. It seems reasonable to expect that the scaling laws for sinusoidal and sawtooth excitation differ only through a constant of proportionality. For sinusoidal motion of the base, $A_0\omega$ is the parameter equivalent to V. The analysis therefore predicts an α value of 4/3, which is very close to the experimental values of 1.3–1.4 [3], though somewhat lower than the value 1.5 obtained from simulations [2]. The $1/N^\beta$ (where $\beta=1/3$) dependence is also closer to the experimental exponent values that were found to lie in the range 0.3–0.6.

In a recent communication [10], McNamara and Luding presented results from an intensive numerical investigation into the validity of Eq. (9). Parameters that were varied included g (by factors of 25 and 1/25) and N/W (by factors of 5 and 1/5). However, one of the interesting observations was that, when C was plotted against Δh, all the results fell onto a single master curve, even in the dense region. The fact that Eqs. (8) and (9) have different scaling laws for the variation of $\dot{E}^T$ with g and N would appear at first sight to bring into question the validity of using Eq. (8) in preference to Eq. (9). However, Eq. (8) may not in fact be inconsistent with McNamara and Luding's results. We consider the scaling with g first. If g is varied then E_0 must also scale in proportion to keep Δh constant, since as shown in [3]

$$\Delta h=\frac{E_0}{mg}. \tag{12}$$

$\overline{c}$ scales as $E_0^{1/2}$ and therefore both $\dot{E}_h^T$ and $\dot{E}_l^T$ scale identically (as $g^{3/2}$) for fixed Δh. The different dependence on N ($N^{3/2}$ versus N^2) cannot be dismissed so easily, however. One possible explanation is that the distribution of E_0 throughout the cell is changing with N: one would expect a steeper temperature gradient for large N to provide sufficient energy flux for the increased dissipation rate within the bulk material [9]. Analysis of a nonuniform granular temperature is beyond the scope of the present Brief Report, but could form the basis for future research.

It is also worth pointing out that because of the exponential tail of the Boltzmann distribution, there will always be some point in the cell above which the mean free path is more appropriately described by Eq. (3) than Eq. (4). However, the region of the cell with the highest packing density will contribute the most to the correctly weighted calculation of energy dissipation rate, and if this is sufficiently dense then Eq. (11) is likely to be more appropriate than the corresponding Eq. (24) from [3]. At sufficiently high excitation levels, even the densest region of the medium will be sufficiently dilute for the low density mean free path λ_l to be used and one would therefore expect a transition to an α value of 2. Such a transition was in fact observed in simulations carried out in containers with smooth elastic sidewalls [11]. Sidewalls of this type exert the least perturbation on the system and are the closest configuration to the one analyzed here. The transition point between the two exponents was investigated in Ref. [10] and was found to correspond to a given value of Δh, whereas intuitively one would expect it to occur at a particular value of maximum density. Some of the assumptions made in the analysis (uniform granular temperature; isotropic velocity distribution; no rotational degrees of freedom; number density given by the Boltzmann distribution) are only approximately satisfied [3,9] and these may account at least in part for the remaining discrepancies between the theory presented here and the experimental and numerical results from the literature. Nevertheless it is hoped that this simple model will provide a useful starting point for improved theories of vibrofluidized granular materials.

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[1] S. Luding, E. Clément, A. Blumen, J. Rajchenbach, and J. Duran, Phys. Rev. E **49**, 1634 (1994).
[2] S. Luding, H. J. Herrmann, and A. Blumen, Phys. Rev. E **50**, 3100 (1994).
[3] S. Warr, J. M. Huntley, and G. T. H. Jacques, Phys. Rev. E **52**, 5583 (1995).
[4] S. Warr and J. M. Huntley, Phys. Rev. E **52**, 5596 (1995).
[5] S. Warr, W. Cooke, R. C. Ball, and J. M. Huntley, Physica A **231**, (4) 551 (1996).
[6] K. M. Aoki and N. Ito, Phys. Rev. E **54**, 1990 (1996).
[7] H. Hayakawa and D. C. Hong, Phys. Rev. Lett. **78**, 2764 (1997).
[8] G. H. Ristow, G. Strassburger, and I. Rehberg, Phys. Rev. Lett. **79**, 833 (1997).
[9] K. Helal, T. Biben, and J.-P. Hansen, Physica A **240**, 361 (1997).
[10] S. McNamara and S. Luding, Phys. Rev. E **58**, 813 (1998).
[11] S. Luding, Phys. Rev. E **52**, 4442 (1995).