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Hydrodynamics for granular flow at low density

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The hydrodynamic equations for a gas of hard spheres with dissipative dynamics are derived from the Boltzmann equation. The heat and momentum fluxes are calculated to Navier-Stokes order and the transport coefficients are determined as explicit functions of the coefficient of restitution. The dispersion relations for the corresponding linearized equations are obtained and the stability of this linear description is discussed. This requires consideration of the linear Burnett contributions to the energy balance equation from the energy sink term. Finally, it is shown how these results can be imbedded in simpler kinetic model equations with the potential for analysis of more complex states.
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I. INTRODUCTION

The rapid flow of granular media is frequently described at the macroscopic level by the equations for fluid dynamics, modified to account for dissipation among the interacting particles [1]. These equations are generally phenomenological with unknown transport coefficients and with unknown limits of validity. The details and basis for such macroscopic balance equations have their origins in the more fundamental microscopic kinetic theory description of granular flow. To isolate the most important distinguishing feature of granular media, dissipative dynamics, we consider here the simplest case of smooth hard spheres at low density. The system considered is well described by the Boltzmann kinetic equation modified to account for inelastic two-particle collisions. Derivations of the hydrodynamic equations based on this and related kinetic equations have been considered for some time [2–4]. However, the complexity of the equation has led to the use of various approximations not required for elastic fluids and a complete identification of the transport coefficients as a function of the dissipation parameter to Navier-Stokes order has not yet been given. However, let us mention that very recently Sela and Goldhirsch [5] have numerically obtained a perturbative solution of the inelastic Boltzmann equation to Burnett order, i.e., one order in the gradients beyond the Navier-Stokes approximation, although restricted to the small inelasticity limit.

The consideration of the dependence of the transport coefficients on the dissipation parameter can be significant since a primary application of hydrodynamics is to discover the conditions of stability for various states. Also, the rheology of granular fluids under large shear is determined from the dependence of the transport coefficients on the dissipation parameter. The objective here is to provide a derivation of the hydrodynamic equations from the Boltzmann equation using an extension of the Chapman-Enskog method to granular media. The transport coefficients in the heat and momentum fluxes at Navier-Stokes order are calculated as functions of the coefficient of restitution using a first Sonine polynomial approximation, as is usual for elastic fluids. Additional transport coefficients associated with contributions from the sink term in the energy equation are discussed and those required for a linear stability analysis at Navier-Stokes order (up through second order in the gradients) are explicitly calculated in the same approximation. They characterize contributions that are proportional to the Laplacian of the temperature or the density.

In the next section the Boltzmann equation for inelastic collisions is given and the exact balance equations for mass, energy, and momentum are obtained from it. The Chapman-Enskog method for obtaining a ‘‘normal’’ solution as an expansion in spatial gradients is described and the results through Navier-Stokes order are given. The analysis is more complete than previous studies [2–4]; it is exact to leading order in the dissipation but not limited to weak dissipation. Some details of the calculations are given in Appendixes A and B. The contributions coming from the energy sink term are also analyzed, and the coefficients associated with linear terms are explicitly evaluated. This requires consideration of the distribution function to second order in the gradients (Burnett order). Appendixes C and D provide a sketch of the calculation of these coefficients. The hydrodynamic equations and their linearization about the homogeneous cooling state are discussed in Sec. III. The conditions for stability are identified as functions of the wave vector and the coefficient of restitution.

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The derivation of hydrodynamics is based on a special ''normal'' solution expanded to low order in the spatial gradients. For more complex states the Boltzmann equation is generally intractable and traditionally (elastic collisions) it has been replaced by model kinetic equations for such applications [6–8]. The spirit of a kinetic model is not to capture all details of the underlying Boltzmann equation, but rather to preserve only the critical features responsible for the properties of interest. For example, a kinetic model should imply the same macroscopic balance equations for mass, energy, and momentum density as derived from the Boltzmann equation. In the present case this should include the energy sink due to inelastic collisions [9,10]. This is discussed in Sec. IV, where it is shown to what extent the transport properties of the Boltzmann equation can be embedded in a kinetic model. Three different versions of kinetic models for granular flow are indicated depending on the degree of compromise between quantitative accuracy and tractability. The final section contains some further comments on hydrodynamics and kinetic modeling of granular flows.

II. BOLTZMANN EQUATION AND CHAPMAN-ENSKOG SOLUTION

The distribution function, $f(\mathbf{r},\mathbf{v};t)$, for the positions and velocities of a low density gas of smooth hard spheres is assumed to be well described by the Boltzmann kinetic equation [4,9],

$$(\partial_t+\mathbf{v}\cdot\nabla)f=J[f,f]. \tag{1}$$

The Boltzmann collision operator $J[f,f]$ describing the scattering of two particles is

$$J[f,f]=\sigma^2\int d\mathbf{v}_1 \int d\hat{\boldsymbol{\sigma}}\Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}}) \times(\alpha^{-2}b^{-1}-1)f(\mathbf{r},\mathbf{v},t)f(\mathbf{r},\mathbf{v}_1,t), \tag{2}$$

where $\mathbf{g}=\mathbf{v}-\mathbf{v}_1$, σ is the diameter of the particles, $\hat{\boldsymbol{\sigma}}$ is a unit vector along their line of centers, and Θ is the Heaviside function. The operator for ''restituting'' collisions, b^{-1}, is defined by

$$b^{-1}h(\mathbf{G},\mathbf{g})=h(\mathbf{G},b^{-1}\mathbf{g}),$$

$$b^{-1}\mathbf{g}=\mathbf{g}-\alpha^{-1}(1+\alpha)(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}. \tag{3}$$

The center of mass velocity $\mathbf{G}=(\mathbf{v}+\mathbf{v}_1)/2$ is invariant under the action of b. It is easily verified that the total energy change in collision is $\Delta E=-(1-\alpha^2)m/4(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})^2$, which identifies α as the coefficient of restitution, in the range $0<\alpha\leqslant 1$. A useful identity, for an arbitrary function h, is given by

$$\int d\mathbf{v}h(\mathbf{v})J[f,f] =\sigma^2\int d\mathbf{v}_1 \int d\mathbf{v}\, f(\mathbf{r},\mathbf{v},t)f(\mathbf{r},\mathbf{v}_1,t)\int d\hat{\boldsymbol{\sigma}}\, \Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}}) \times(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(b-1)h(\mathbf{v}), \tag{4}$$

where the inverse operator b for direct collisions is given by

$$b\mathbf{g}=\mathbf{g}-(1+\alpha)(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}. \tag{5}$$

With this identity the following properties of $J[f,f]$ follow directly,

$$\int d\mathbf{v}\begin{pmatrix}1\\ m\mathbf{v}\\ \frac{1}{2}mv^2\end{pmatrix}J[f,f]=\begin{pmatrix}0\\ \mathbf{0}\\ -(1-\alpha^2)\omega[f,f]\end{pmatrix}. \tag{6}$$

The zeros on the right-hand side represent conservation of mass and momentum for the two-particle collisions. The term $-(1-\alpha^2)\omega[f,f]$ provides the rate of energy loss due to dissipation, with $\omega[f,h]$ given by

$$\omega[f,h]=\frac{m\pi\sigma^2}{16}\int d\mathbf{v}_1 \times\int d\mathbf{v}_2\, |\mathbf{v}_1-\mathbf{v}_2|^3 f(\mathbf{r},\mathbf{v}_1,t)h(\mathbf{r},\mathbf{v}_2,t). \tag{7}$$

The balance equations for mass, momentum, and energy follow by taking moments of the Boltzmann equation with respect to 1, $\mathbf{v}$, and v^2, [2,3]

$$D_t n+n\nabla\cdot\mathbf{u}=0, \tag{8}$$

$$D_t u_i+(mn)^{-1}\nabla_j P_{ij}=0, \tag{9}$$

$$D_t T+\frac{2}{3nk_B}(P_{ij}\nabla_j u_i+\nabla\cdot\mathbf{q})+T\zeta=0, \tag{10}$$

where $D_t=\partial_t+\mathbf{u}\cdot\nabla$ is the material derivative, and we have introduced the cooling rate

$$\zeta[f]=(1-\alpha^2)\frac{2}{3nk_BT}\omega[f,f]. \tag{11}$$

The pressure tensor P_{ij} and heat flux $\mathbf{q}$ are given by

$$P_{ij}[\mathbf{r},t|f]=\frac{n}{\beta}\delta_{ij}+\int d\mathbf{v}D_{ij}(\mathbf{V})f(\mathbf{r},\mathbf{v},t),$$

$$\mathbf{q}[\mathbf{r},t|f]=\int d\mathbf{v}\, \mathbf{S}(\mathbf{V})f(\mathbf{r},\mathbf{v},t), \tag{12}$$

$$D_{ij}(\mathbf{V})\equiv m\left(V_iV_j-\tfrac{1}{3}V^2\delta_{ij}\right),\quad \mathbf{S}(\mathbf{V})\equiv\left(\frac{m}{2}V^2-\frac{5}{2\beta}\right)\mathbf{V}. \tag{13}$$

Here $\mathbf{V}=\mathbf{v}-\mathbf{u}$ is the velocity relative to the local flow, $\beta=(k_BT)^{-1}$, and $n(\mathbf{r},t)$, $\mathbf{u}(\mathbf{r},t)$, and $T(\mathbf{r},t)$ are the local density, flow velocity, and temperature, respectively. The Boltzmann equation admits a basic solution describing the homogeneous cooling state, in which the system is spatially uniform and the temperature decreases monotonically in time.

The balance equations become a closed set of hydrodynamic equations for the fields n, $\mathbf{u}$, and T once $P_{ij}[\mathbf{r},t|f]$, $\mathbf{q}[\mathbf{r},t|f]$, and $\zeta[\mathbf{r},t|f,]$ are expressed as functionals of these fields. The latter are explicit functionals of f so the desired forms are obtained from a solution to the kinetic equation that expresses the space and time dependence of f entirely in terms of the hydrodynamic fields. Such a solution is called a ‘‘normal’’ solution, and a practical means to obtain it for small spatial gradients is given by the Chapman-Enskog method [11]. The hydrodynamic equations depend on gradients of the pressure tensor and heat flux, and on the pressure tensor multiplied by a gradient. Thus a calculation of f to first order in the gradients (Navier-Stokes order) gives contributions to the hydrodynamic equations up through second order in the gradients. However, $\zeta[f]$ appears in the hydrodynamic equations without additional gradients and requires calculation of f to second order in the gradients (Burnett order) for consistency. In this way, the balance equations become approximate closed hydrodynamic equations valid for long wavelength phenomena (length scales large compared to the mean free path).

The solutions for homogeneous states are easily seen from Eqs. (8)–(10) to be four ‘‘modes’’ with infinite relaxation time, corresponding to the conserved density and momentum density, and one with a relaxation time equal to the inverse cooling rate of $T(t)$. For small but finite spatial perturbations, the relaxation times for the first four modes become finite, proportional to the square of the wavelength. The remaining mode has a relaxation time proportional to the inverse cooling rate of the homogeneous state perturbed by terms proportional to the square of the wavelength. The appearance of two time scales within the hydrodynamic description is a new feature of granular media. However, the fact that these time scales may be quite different does not compromise the existence of a hydrodynamic description. This is because the reference state for the Chapman-Enskog expansion incorporates the dynamics of cooling, so the residual corrections are all proportional to the small spatial gradients.

The Chapman-Enskog method assumes a solution to the kinetic equation of the form

$$f(\mathbf{r},\mathbf{v},t)=f[\mathbf{v}|n,\mathbf{u},T]. \tag{14}$$

This means that the space and time dependence all occurs through a functional dependence on the hydrodynamic fields. This functional dependence can be made local in space and time through an expansion in the gradients of these fields, for states with small spatial variations. In this case, the distribution function is represented as a series in a formal ‘‘uniformity’’ parameter ϵ (set equal to unity at the end),

$$f=f^{(0)}+\epsilon f^{(1)}+\epsilon^2 f^{(2)}+\cdots. \tag{15}$$

The series is generated by assigning a factor of ϵ to every gradient operator. Use of this expansion in the definitions for the fluxes and the dissipation function $\omega[f,f]$ gives a corresponding expansion for these quantities. Finally, use of these in the hydrodynamic equations leads to an identification of the time derivatives of the fields as an expansion in the gradients, $\partial_t=\partial_t^{(0)}+\epsilon\partial_t^{(1)}+\cdots$. This is the usual Chapman-Enskog method for solving kinetic equations [11,12]. The problem is more complex here than for the case $\alpha=1$ since the reference state about which gradients are considered is not stationary, and the terms from $\partial_t^{(0)}$ are not zero.

To zeroth order in the gradients the kinetic equation (1) becomes

$$\partial_t^{(0)}f^{(0)}=J[f^{(0)},f^{(0)}]. \tag{16}$$

The macroscopic balance equations to this order read

$$\partial_t^{(0)}n=0,\quad \partial_t^{(0)}\mathbf{u}=0,\quad T^{-1}\partial_t^{(0)}T=-\zeta^{(0)}, \tag{17}$$

where the cooling rate $\zeta^{(0)}$ is given by

$$\zeta^{(0)}=(1-\alpha^2)\frac{2\omega[f^{(0)},f^{(0)}]}{3nk_BT}. \tag{18}$$

The left side of Eq. (16) can be evaluated more explicitly to give

$$-\zeta^{(0)}T\partial_T f^{(0)}=J[f^{(0)},f^{(0)}], \tag{19a}$$

or

$$\frac{1}{2}\zeta^{(0)}\frac{\partial}{\partial\mathbf{V}}\cdot(\mathbf{V}f^{(0)})=J[f^{(0)},f^{(0)}]. \tag{19b}$$

The second form follows from the fact that $f^{(0)}$ is a function of the velocity only through V/v_0, where $v_0=(2/\beta m)^{1/2}$ is the thermal velocity. Thus its dependence on the temperature is of the form $T^{-3/2}\overline{f}^{(0)}(v/T^{1/2})$, and consequently

$$-T\partial_T f^{(0)}=\frac{1}{2}\frac{\partial}{\partial\mathbf{V}}\cdot(\mathbf{V}f^{(0)}). \tag{20}$$

In the following both forms, Eq. (19a) and Eq. (19b), will be used. Dimensional analysis requires that $\zeta^{(0)}$ is proportional to $nT^{1/2}$. The solution $f^{(0)}=f^{(0)}(V)$ is isotropic so that the zeroth order pressure tensor and heat flux are found to be

$$P_{ij}^{(0)}=p\delta_{ij},\quad \mathbf{q}^{(0)}=0, \tag{21}$$

where $p=nk_BT$ is the hydrostatic pressure.

To first order in the gradients Eq. (1) leads to an equation for $f^{(1)}$,

$$(\partial_t^{(0)}+L)f^{(1)}=-(\partial_t^{(1)}+\mathbf{v}\cdot\nabla)f^{(0)}=-(D_t^{(1)}+\mathbf{V}\cdot\nabla)f^{(0)}, \tag{22}$$

with $D_t^{(1)}=\partial_t^{(1)}+\mathbf{u}\cdot\nabla$ and

$$Lf^{(1)}=-J[f^{(0)},f^{(1)}]-J[f^{(1)},f^{(0)}]. \tag{23}$$

The macroscopic balance equations give

$$D_t^{(1)}n=-n\nabla\cdot\mathbf{u},\quad D_t^{(1)}u_i=-(mn)^{-1}\nabla_i p,$$

$$D_t^{(1)}T=-\frac{2T}{3}\nabla\cdot\mathbf{u}-\zeta^{(1)}T, \tag{24}$$

with

$$\zeta^{(1)}=(1-\alpha^2)\frac{4\omega[f^{(0)},f^{(1)}]}{3nk_BT}. \tag{25}$$

Therefore, Eq. (22) becomes

$$\begin{aligned}(\partial_t^{(0)}+L)f^{(1)}-\zeta^{(1)}T\partial_T f^{(0)} &= f^{(0)}(\nabla\cdot\mathbf{u}-\mathbf{V}\cdot\nabla\ln n)+(\partial_T f^{(0)})\left(\frac{2T}{3}\nabla\cdot\mathbf{u}-\mathbf{V}\cdot\nabla T\right)\\ &\quad+\left(\frac{\partial}{\partial V_i}f^{(0)}\right)[-(mn)^{-1}\nabla_i p+\mathbf{V}\cdot\nabla u_i].\end{aligned} \tag{26}$$

The solution to this equation has the form

$$f^{(1)}=\boldsymbol{\mathcal{A}}(\mathbf{V})\cdot\nabla\ln T+\boldsymbol{\mathcal{B}}(\mathbf{V})\cdot\nabla\ln n+\mathcal{C}_{ij}(\mathbf{V})\nabla_i u_j. \tag{27}$$

Substitution of Eq. (27) into Eq. (26) and identifying coefficients of independent gradients gives the set of equations that determine the functions $\boldsymbol{\mathcal{A}}(\mathbf{V})$, $\boldsymbol{\mathcal{B}}(\mathbf{V})$, and $\mathcal{C}_{ij}(\mathbf{V})$. These equations and the details of their analysis are given in Appendix A. The contributions to the pressure tensor, heat flux, and cooling rate of first order in the gradients can be calculated from Eqs. (12) and (27) with the results

$$P_{ij}^{(1)}=-\eta(\nabla_i u_j+\nabla_j u_i-\tfrac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}), \tag{28}$$

$$\mathbf{q}^{(1)}=-\kappa\boldsymbol{\nabla}T-\mu\boldsymbol{\nabla}n, \tag{29}$$

$$\zeta^{(1)}=0, \tag{30}$$

where η is the shear viscosity and κ is the thermal conductivity. The other transport coefficient μ has no analogue for the elastic scattering case and is nonvanishing only for $\alpha<1$. The term $\zeta^{(1)}$ vanishes from symmetry since it is a scalar that does not couple linearly to the vector functions $\mathcal{A}_i(T,\mathbf{V})$ and $\mathcal{B}_i(T,\mathbf{V})$, nor to the traceless tensor $\mathcal{C}_{ij}(T,\mathbf{V})$. The transport coefficients are given by

$$\eta^*(\alpha)\equiv\frac{\eta(\alpha)}{\eta_0}=\left[\nu_\eta^*(\alpha)-\frac{1}{2}\zeta^*(\alpha)\right]^{-1}, \tag{31}$$

$$\kappa^*(\alpha)\equiv\frac{\kappa(\alpha)}{\kappa_0}=\tfrac{2}{3}[\nu_\kappa^*(\alpha)-2\zeta^*(\alpha)]^{-1}[1+c^*(\alpha)], \tag{32}$$

$$\begin{aligned}\mu^*(\alpha)&\equiv\frac{n}{T\kappa_0}\mu(\alpha)\\ &=2\zeta^*(\alpha)\left[\kappa^*(\alpha)+\frac{1}{3}\frac{c^*(\alpha)}{\zeta^*(\alpha)}\right]\\ &\quad\times[2\nu_\mu^*(\alpha)-3\zeta^*(\alpha)]^{-1}.\end{aligned} \tag{33}$$

Here $\eta_0=5m^{1/2}/16\sigma^2(\pi\beta)^{1/2}$ and $\kappa_0=15k_B\eta_0/4m$ are the elastic limit values of the shear viscosity and thermal conductivity, respectively. The dimensionless functions of α in these expressions are

$$\zeta^*(\alpha)\equiv\frac{\zeta^{(0)}}{\nu_0}=(1-\alpha^2)\frac{2\beta\omega[f^{(0)},f^{(0)}]}{3n\nu_0}, \tag{34}$$

$$c^*(\alpha)=\frac{8}{15}\left[\left(\frac{\beta m}{2}\right)^2\frac{1}{n}\int d\mathbf{V}\,V^4f^{(0)}-\frac{15}{4}\right], \tag{35}$$

$$\nu_\eta^*(\alpha)=\frac{\int d\mathbf{V}D_{ij}(\mathbf{V})L\mathcal{C}_{ij}(\mathbf{V})}{\nu_0\int d\mathbf{V}\,D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V})},$$

$$\nu_\kappa^*(\alpha)=\frac{\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot L\boldsymbol{\mathcal{A}}(\mathbf{V})}{\nu_0\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{A}}(\mathbf{V})}, \tag{36}$$

$$\nu_\mu^*(\alpha)=\frac{\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot L\boldsymbol{\mathcal{B}}(\mathbf{V})}{\nu_0\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{B}}(\mathbf{V})}, \tag{37}$$

where $D_{ij}(\mathbf{V})$ and $\mathbf{S}(\mathbf{V})$ are defined in (13) and $\nu_0=(16/5)n\sigma^2(\pi/m\beta)^{1/2}$ is a characteristic collision frequency introduced for normalization. The function $c^*(\alpha)$ is related to the deviation of the fourth moment $f^{(0)}$ from the corresponding value for a Maxwellian. These results are still exact. To obtain more explicit expressions for the dependence of the transport coefficients on α we use a first Sonine polynomial expansion for $\boldsymbol{\mathcal{A}}(\mathbf{V})$, $\boldsymbol{\mathcal{B}}(\mathbf{V})$, $\mathcal{C}_{ij}(\mathbf{V})$, and $f^{(0)}$. This is the usual approximation for a gas with elastic collisions and there is no *a priori* reason to question its accuracy here as well. It follows from symmetry that the leading terms in this expansion are $\boldsymbol{\mathcal{A}}(\mathbf{V})\propto\boldsymbol{\mathcal{B}}(\mathbf{V})\propto f_M(V)\mathbf{S}(\mathbf{V})$, $\mathcal{C}_{ij}(\mathbf{V})\propto f_M(V)D_{ij}(\mathbf{V})$, and $f^{(0)}(V)-f_M(V)\propto f_M(V)[(\beta mV^2/2)^2-5\beta mV^2/2+15/4]$. Here $f_M(V)$ is the Maxwellian weight factor defining the scalar product with respect to which the orthogonal polynomials are defined. With these expressions the following results hold (see Appendix B)

$$\zeta^*(\alpha)=\frac{5}{12}(1-\alpha^2)\left[1+\frac{3}{32}c^*(\alpha)\right], \tag{38}$$

$$c^*(\alpha)=\frac{32(1-\alpha)(1-2\alpha^2)}{81-17\alpha+30\alpha^2(1-\alpha)}, \tag{39}$$

$$\nu_\eta^*(\alpha)=\left[1-\frac{1}{4}(1-\alpha)^2\right]\left[1-\frac{1}{64}c^*(\alpha)\right], \tag{40}$$

$$\begin{aligned}\nu_\kappa^*(\alpha)=\nu_\mu^*(\alpha)&=\tfrac{1}{3}(1+\alpha)\\ &\quad\times\left[1+\frac{33}{16}(1-\alpha)+\frac{19-3\alpha}{1024}c^*(\alpha)\right].\end{aligned} \tag{41}$$

Equations (38) and (39) have been shown to be in very good agreement with the numerical results obtained by a direct Monte Carlo simulation of the Boltzmann equation [13]. This completes the identification of all terms in the expressions (31)–(33) for the transport coefficients. The corresponding contribution to the distribution function $f^{(1)}$ is calculated in Appendix C with the result

$$\begin{aligned}f^{(1)}(\mathbf{V})=-\beta^3n^{-1}\Big\{&\frac{2m}{5}\mathbf{S}(\mathbf{V})\cdot[\kappa(\alpha)\boldsymbol{\nabla}T+\mu(\alpha)\boldsymbol{\nabla}n]\\ &+\beta^{-1}\eta(\alpha)D_{ij}(\mathbf{V})\nabla_i u_j\Big\}f_M(V).\end{aligned} \tag{42}$$

The most frequently used expressions for transport coefficients to date are those obtained by Lun *et al.* [3] using an approximate moment method based on the Boltzmann equation (low density limit of their results) for small α. In the current terminology this method calculates the contributions

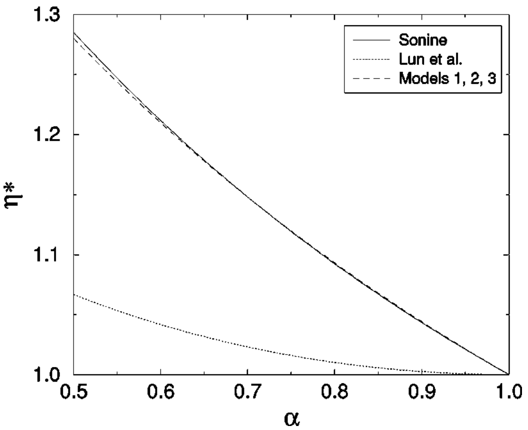

FIG. 1. Shear viscosity as a function of the coefficient of restitution α. The solid line is the result obtained here by using the Sonine expansion and the dotted line is the one reported in Ref. [3]. The dashed lines correspond to the model equations discussed in Sec. IV. The shear viscosity is reduced in each case by its value in the elastic limit $\alpha=1$.

from $\nu^*_{\eta,\kappa,\mu}$ only to leading order in $(1-\alpha)$ and neglects completely the contributions from $\zeta^*(\alpha)$ and $c^*(\alpha)$. Also, this method predicts $\mu(\alpha)=0$ in contrast to the finite result obtained here. Goldshtein and Shapiro [4] use the correct Chapman-Enskog method but report no results for any of the three transport coefficients at Navier-Stokes order. Figures 1–3 show a comparison of our results with those of [3] for the coefficients η, κ, and μ, respectively, as functions of α. There are both qualitative and quantitative differences. Preliminary Monte Carlo simulations of the shear viscosity confirm the accuracy of our results. Further details will be provided elsewhere. Also shown in these figures are results from kinetic models discussed in Sec. IV.

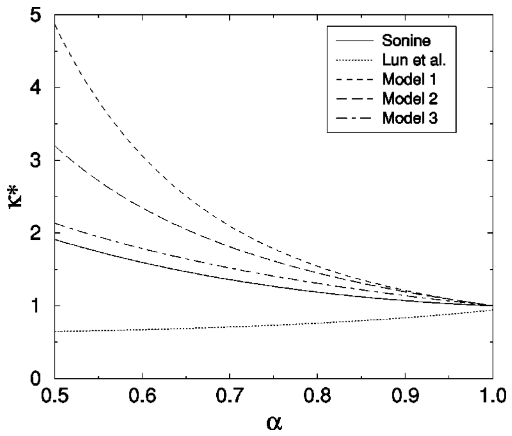

FIG. 2. Thermal conductivity as a function of the coefficient of restitution α. Symbols are the same as in Fig. 1. The heat conductivity is reduced in each case by its value in the elastic limit $\alpha=1$.

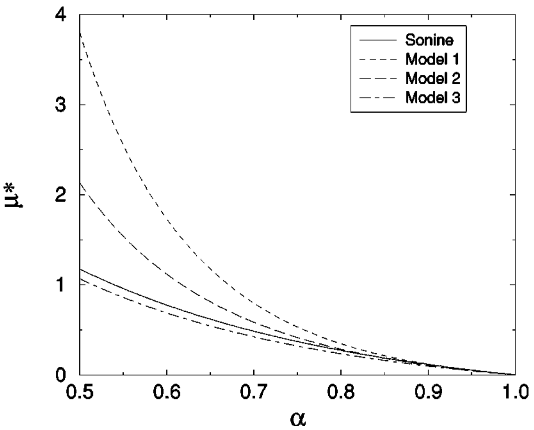

FIG. 3. The transport coefficient μ as a function of the coefficient of restitution α. Symbols are the same as in Fig. 1. The theory in Ref. [3] predicts $\mu=0$. The transport coefficient is reduced by $\kappa_0 T/n$, where κ_0 is the elastic thermal conductivity.

We have already seen that there is no contribution to the heating rate $\zeta[\mathbf{r},t|f]$ of first order in the gradients. However, for consistency it is necessary to include the contributions from second-order gradients in the hydrodynamic equations since the latter depend on the gradients of $P_{ij}^{(1)}$ and $\mathbf{q}^{(1)}$, which also are of second order. Symmetry dictates the form of $\zeta^{(2)}$,

$$\zeta^{(2)}=\zeta_1\nabla^2 T+\zeta_2\nabla^2 n+\zeta_3(\nabla T)^2+\zeta_4(\nabla n)^2+\zeta_5(\boldsymbol{\nabla} T)\cdot(\boldsymbol{\nabla} n) + \zeta_6(\nabla_i u_j)(\nabla_i u_j)+\zeta_7(\nabla_i u_j)(\nabla_j u_i). \tag{43}$$

Calculation of these coefficients requires knowledge of $f^{(2)}$. The contributions to $f^{(2)}$ that determine the coefficients of the linear terms in Eq. (43) are discussed in Appendix D. In the same Sonine polynomial approximation as discussed above it is found that ζ_1 and ζ_2 are given by Eqs. (D19)–(D21). In order to provide an idea of the relevance of these transport coefficients, κ^*/ζ_1^* and μ^*/ζ_2^* are plotted as functions of α in Figs. 4 and 5, respectively. The reduced coefficients ζ_1^* and ζ_2^* are defined in Eq. (D18). For not very inelastic particles, ζ_1^* (ζ_2^*) is much smaller than κ^* (ν^*). That means that the terms $\zeta_1\nabla^2 T$ and $\zeta_2\nabla^2 n$ can be accurately neglected in the transport equations [see Eqs. (58) and (65) below].

As mentioned in the Introduction, Sela and Goldhirsch [5] have obtained expressions for the Boltzmann transport coefficients up to Burnett order in the gradients and first order in

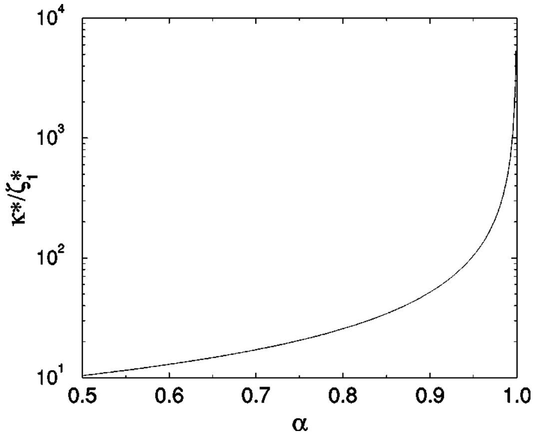

FIG. 4. Ratio of the thermal conductivity κ to the coefficient ζ_1 associated with the contribution from the energy sink term proportional to the Laplacian of the temperature, as a function of the coefficient of restitution α. Both quantities have been reduced to dimensionless form. The precise definitions are given in the main text.

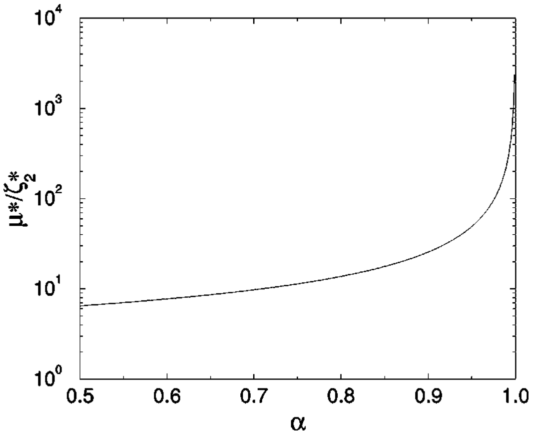

FIG. 5. Ratio of the transport coefficient μ to the coefficient ζ_2 associated with the contribution from the energy sink term proportional to the Laplacian of the density, as a function of the coefficient of restitution α. Both quantities have been reduced to dimensionless form. The precise definitions are given in the main text.

the inelasticity parameter $\epsilon = 1-\alpha^2$. In order to compare with the results presented here, we have carried out an expansion of our results in powers of ϵ,

$$\eta^* = 1+\tfrac{157}{768}\,\epsilon + O(\epsilon^2), \tag{44}$$

$$\kappa^* = 1+\tfrac{57}{256}\,\epsilon + O(\epsilon^2), \tag{45}$$

$$\mu^* = \frac{\epsilon}{2} + O(\epsilon^2), \tag{46}$$

$$\zeta^* = \tfrac{5}{12}\,\epsilon - \tfrac{5}{512}\,\epsilon^2 + O(\epsilon^3), \tag{47}$$

$$\zeta_1^* = \tfrac{3}{32}\,\epsilon + O(\epsilon^2), \tag{48}$$

$$\zeta_2^* = \tfrac{27}{256}\,\epsilon^2 + O(\epsilon^3). \tag{49}$$

The corresponding results in Ref. [5] written in the units we are using read

$$\eta_{\mathrm{SG}}^* \approx 1.0160 + 0.1801\epsilon + O(\epsilon^2), \tag{50}$$

$$\kappa_{\mathrm{SG}}^* \approx 1.0259 + 0.2682\epsilon + O(\epsilon^2), \tag{51}$$

$$\mu_{\mathrm{SG}}^* \approx 0.5278\epsilon + O(\epsilon^2), \tag{52}$$

$$\zeta_{\mathrm{SG}}^* \approx \tfrac{5}{12}\,\epsilon - 0.0107\epsilon^2 + O(\epsilon^3), \tag{53}$$

$$\zeta_{1,\mathrm{SG}}^* \approx 0.0866\epsilon + O(\epsilon^2), \tag{54}$$

$$\zeta_{2,\mathrm{SG}}^* = O(\epsilon^2). \tag{55}$$

We see that there is a quite good quantitative agreement with no qualitative discrepancies. In particular, the order in ϵ and the sign of the first corrections to the leading terms are the same in both theories. The quantitative differences are probably due to the use of different trial functions and also to the numerical evaluation of the integrals in Ref. [5].

III. HYDRODYNAMIC EQUATIONS AND STABILITY

The results obtained in the previous section for the pressure tensor, heat flux, and energy sink provide the necessary constitutive equations to convert the balance equations (8)–(10) into a closed set of hydrodynamic equations for n, T, and $\mathbf{u}$,

$$D_t n + n\nabla\cdot\mathbf{u} = 0, \tag{56}$$

$$D_t u_i + (nm)^{-1}\nabla_i p - (nm)^{-1}\nabla_j[\eta(\nabla_i u_j + \nabla_j u_i - \tfrac{2}{3}\,\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u})] = 0, \tag{57}$$

$$\begin{aligned} D_t T &+ 2(3nk_B)^{-1} p\boldsymbol{\nabla}\cdot\mathbf{u} - 2(3nk_B)^{-1}(\nabla_i u_j)\\ &\times[\eta(\nabla_i u_j + \nabla_j u_i - \tfrac{2}{3}\,\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u})]\\ &- 2(3nk_B)^{-1}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T) - 2(3nk_B)^{-1}\boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}n)\\ &= -T\zeta^{(0)} - T\zeta^{(2)}. \end{aligned} \tag{58}$$

In general, the energy sink $\zeta^{(2)}$ has many contributions, as indicated in Eq. (43). Our analysis of the hydrodynamic equations will be limited to their linearization about the homogeneous cooling state and consequently only the first two terms of Eq. (43) are needed.

The linearization about the homogeneous state leads to partial differential equations with coefficients that are independent of the space variable but dependent on the time, since the reference state is cooling. This time dependence can be eliminated through a change in the time and space variables, and a scaling of the hydrodynamic fields. Let $\delta y_\alpha(\mathbf{r},t) \equiv y_\alpha(\mathbf{r},t) - y_{H\alpha}(t)$ denote the deviation of the hydrodynamic fields from their value in the homogeneous state. A set of Fourier transformed dimensionless variables are then defined by

$$\delta y_{\mathbf{k}\alpha}(\tau) = \int d\mathbf{l}\; e^{-i\mathbf{k}\cdot\mathbf{l}}\delta y_\alpha(\mathbf{l},\tau), \tag{59}$$

$$\theta_{\mathbf{k}}(\tau) = \frac{\delta T_{\mathbf{k}}(\tau)}{T_H(\tau)}, \quad \mathbf{w}_{\mathbf{k}}(\tau) = \left[\frac{m}{k_B T_H(\tau)}\right]^{1/2}\delta\mathbf{u}_{\mathbf{k}}(\tau),$$

$$\rho_{\mathbf{k}}(\tau) = \frac{\delta n_{\mathbf{k}}(\tau)}{n_H}. \tag{60}$$

The subscript H indicates that the quantity is computed in the homogeneous cooling state, and $\mathbf{l}$ and τ are the dimensionless space and time variables,

$$\tau = \frac{1}{2}\int_0^t dt'\,\nu_H(t'), \quad \mathbf{l} = \frac{\nu_H(t)}{2}\left(\frac{m}{k_B T_H(t)}\right)^{1/2}\mathbf{r}, \tag{61}$$

and $\nu_H(t) \equiv (16/5) n_H \sigma^2[\pi k_B T_H(t)/m]^{1/2}$. In terms of these variables the linearized hydrodynamic equations are

$$\frac{\partial}{\partial\tau}\rho_{\mathbf{k}} + ikw_{\mathbf{k}\parallel} = 0, \tag{62}$$

$$\left(\frac{\partial}{\partial\tau} - \zeta^* + \tfrac{2}{3}\,\eta^* k^2\right) w_{\mathbf{k}\parallel} + ik\theta_{\mathbf{k}} + ik\rho_{\mathbf{k}} = 0, \tag{63}$$

$$\left(\frac{\partial}{\partial\tau}-\zeta^*+\tfrac{1}{2}\,\eta^*k^2\right)\mathbf{w}_{\mathbf{k}\perp}=0, \tag{64}$$

$$\left[\frac{\partial}{\partial\tau}+\zeta^*+\tfrac{5}{4}\,(\kappa^*-\zeta_1^*)k^2\right]\theta_{\mathbf{k}}+[2\zeta^*+\tfrac{5}{4}\,(\mu^*-\zeta_2^*)k^2]\rho_{\mathbf{k}} + \tfrac{2}{3}\,ikw_{\mathbf{k}\parallel}=0. \tag{65}$$

The symbols $w_{\mathbf{k}\parallel}$ and $\mathbf{w}_{\mathbf{k}\perp}$ denote the longitudinal and transversal components of the velocity field relative to the wave vector $\mathbf{k}$, respectively. Moreover, we have introduced the reduced Burnett transport coefficients

$$\zeta_1^*=\frac{3p}{2\kappa_0}\zeta_1,\quad \zeta_2^*=\frac{3n^2k_B}{2\kappa_0}\zeta_2. \tag{66}$$

Equation (64) is decoupled from the rest and can be integrated directly yielding

$$\mathbf{w}_{\mathbf{k}\perp}(\tau)=\mathbf{w}_{\mathbf{k}\perp}(0)\exp(s_\perp\tau), \tag{67}$$

where

$$s_\perp=\zeta^*-\tfrac{1}{2}\,\eta^*k^2. \tag{68}$$

This identifies two degenerate shear modes. The remaining three eigenmodes have the form $\varphi_n(\mathbf{k})\exp[s_n(\alpha,k)\tau]$ for $n=1,2,3$, where $s_n(\alpha,k)$ are the solutions of the equation

$$s^3+\tfrac{5}{4}(\kappa^*+\tfrac{8}{15}\,\eta^*-\zeta_1^*)k^2s^2+\left\{\frac{5}{6}k^4\eta(\kappa^*-\zeta_1^*)+\tfrac{1}{3}\,k^2[5+2\zeta^*\eta^*-\tfrac{15}{4}\,\zeta^*(\kappa^*-\zeta_1^*)]-\zeta^{*2}\right\}s+[\tfrac{5}{4}\,(\kappa^*-\zeta_1^*-\mu^*+\zeta_2^*)k^2-\zeta^*]k^2=0, \tag{69}$$

which follows from Eqs. (62), (63), and (65). Since the latter are valid only for $k\ll 1$ (i.e., for wavelengths long compared to the mean free path) and for $0<\alpha\leqslant 1$, the solutions to Eq. (69) are meaningful only in this range. For the special case of elastic collisions, $\alpha=1$, the solutions are polynomials in k of degree two corresponding to two sound modes and a heat diffusion mode. However, the eigenvalues and eigenvectors are not analytic about the point $\alpha=1,k=0$ so that even in the range close to the elastic limit the hydrodynamic modes must be interpreted with some care. In this range $\zeta^*\propto(1-\alpha^2)$ and k both are small parameters in (69), and the type of modes obtained in the approach to the elastic limit depends on how these parameters approach zero. Analysis of Eq. (69) shows that if $k\to 0$ as ζ^{*y} with $y\leqslant 1/2$ then the solutions have the interpretation of modified sound modes and a heat mode, similar to those in the elastic limit. Otherwise the relationship to the elastic case is not simple and the modes couple to different hydrodynamic variables. More generally, away from the elastic limit the eigenvalues have a more complicated dependence on k and ζ^*. The case $k\propto\zeta^{*y}$ for small ζ^* and $y\leqslant 1/2$ sometimes is referred to as the hydrodynamic limit, which is a misnomer since solutions to Eq. (69) for any values of $k\ll 1$ and $0<\alpha\leqslant 1$ characterize the hydrodynamic modes.

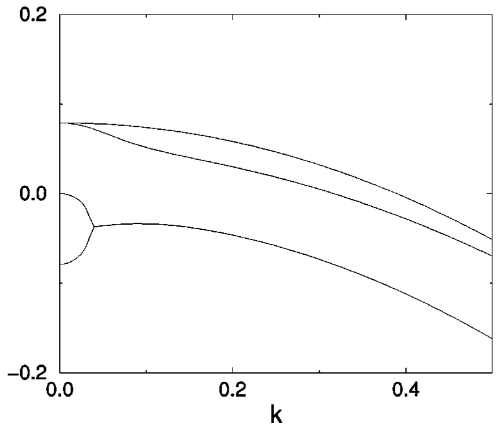

FIG. 6. Dispersion relations for the hydrodynamic modes vs the reduced wavenumber k. From top to bottom the curves correspond to the shear modes ($\perp$), the heat mode (h), and the sound modes ($\pm$). The coefficient of restitution is $\alpha=0.9$. Both s and k are measured in the reduced dimensionless units defined in the main text.

It is straightforward to solve Eq. (69) in this entire parameter space. As an illustration, the dispersion relations for a three-dimensional system with $\alpha=0.9$ are shown in Fig. 6. The modes have been labeled as shear modes ($\perp$), heat mode (h), and sound modes ($\pm$) with the above understanding that, except for the shear modes, the physical meaning of the modes is different from that of elastic fluids. At very small k all modes are real, while at larger k two modes become a complex conjugate pair of propagating modes. Only the common real part of the propagating modes has been plotted. The results are similar at other values of α.

The linear hydrodynamic equations characterize the stability of the homogeneous cooling state. It is seen that the shear and heat modes are positive for $k<k_\perp^c$ and $k<k_h^c$, respectively, where

$$k_\perp^c=\left(\frac{2\zeta^*}{\eta^*}\right)^{1/2},\quad k_h^c=\left[\frac{4\zeta^*}{5(\kappa^*-\zeta_1^*-\mu^*+\zeta_2^*)}\right]^{1/2}. \tag{70}$$

Therefore, initial long wavelength perturbations of the homogeneous cooling state that excite these modes will grow exponentially, representing an instability of the reference state. As the perturbations become larger, the full nonlinear hydrodynamic equations are required to understand the subsequent evolution and possible stabilization. Molecular dynamics simulations show the formation of spatial structures in the fluid at this later stage (e.g., velocity vortices and high density clusters) [14–16]. The analysis here based on kinetic theory provides an alternative and potentially more instructive method to study these effects, using direct simulation Monte Carlo methods to solve the kinetic equation [17]. The advantages of this approach to exploring unstable dynamics has been illustrated recently for elastic fluids [18].

The reference state for the hydrodynamic equations is spatially homogeneous but not stationary, due to the continuous collisional loss of energy. The distribution function for the homogeneous cooling state is a local equilibrium distribution with a time dependent temperature given by $T(\tau)$

$=T(0)e^{-2\zeta^* \tau}$. To analyze spatial perturbations of this homogeneous state, dimensionless hydrodynamic fields have been introduced in this section that include some of this time dependence. For example, the reduced velocity field $\mathbf{w}_{\mathbf{k}}(\tau)$ is obtained by scaling relative to $\sqrt{T(\tau)}$ and this is responsible for the positive values of the modes in Fig. 6 at $k=0$. Positivity of the modes in the linear equations therefore represents instability of the reduced variables only. These are the relevant variables for the time dependent reference state and their instability signals an onset of the nonlinear couplings of these variables that are responsible for the subsequent formation of spatial structures. Another important point to be noted is that the exponential behavior for the modes in the reduced variable τ translates into an algebraic decay in time t,

$$e^{\tau s_n} = \left(1 + \frac{t}{t_0} \right)^{s_n/\zeta^*}, \tag{71}$$

where $t_0^{-1} = \zeta^* \nu_0(0)/2$. A more complete discussion of the stability of the homogeneous state to spatial perturbations will be given elsewhere.

IV. KINETIC MODELS

The Chapman-Enskog method described in Sec. II generates a special solution to the Boltzmann equation suitable for states whose space and time dependence can be described through the hydrodynamic fields. Within this context, the method generates that solution as a perturbation expansion in small spatial gradients relative to the homogeneous cooling state. While the calculations are tractable to the order considered here they become prohibitively complex for states with larger spatial gradients, such as granular flow under shear or driven by other external forces and boundary conditions. For such cases and for more general states outside the hydrodynamic description it is useful to consider kinetic model equations in place of the Boltzmann equation. Such equations are obtained by replacing the complex Boltzmann collision operator with a simpler form, restricted to retain certain fundamental properties such as those leading to the balance equations for mass, energy, and momentum. Kinetic models have been used extensively for gases with elastic collisions, providing quantitative descriptions of states far from equilibrium that have not been possible using the Boltzmann equation [7]. In this section, a brief description of the generalization to inelastic collisions is given. As a first test of kinetic models for this case, it is shown that much of the hydrodynamics obtained in the previous sections from the Boltzmann equation can be imbedded in the models with a suitable choice of parameters.

The class of kinetic models considered are one relaxation time models of the form

$$(\partial_t + \mathbf{v}\cdot\nabla) f = -\nu(f - f_0). \tag{72}$$

The parameter ν is a scalar functional of f only through its dependence on low degree moments of f and possibly the dissipation parameter ζ which characterizes the cooling rate in granular media. The function $f_0 = f_0[f]$ is a functional of f to be specified. To constrain the form of this functional we first require the exact balance equations through [see Eq. (6)]

$$\int d\mathbf{v} \begin{pmatrix} 1 \\ m\mathbf{v} \\ \frac{1}{2}mv^2 \end{pmatrix} (f - f_0) = \begin{pmatrix} 0 \\ \mathbf{0} \\ \frac{3p}{2\nu}\zeta \end{pmatrix}, \tag{73}$$

where ζ is the functional of f defined by Eq. (11). Next we require the existence of a given homogeneous cooling solution f_H. From the discussion leading to Eq. (20) this implies that

$$f_0[f_H] = f_H + \frac{\zeta_H}{2\nu_H} \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v} f_H). \tag{74}$$

A sufficient condition for both Eqs. (73) and (74) is to take

$$f_0[f] = f_{\ell H} + \frac{\zeta}{2\nu} \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f_{\ell H}), \tag{75}$$

where $f_{\ell H}$ is the local version of f_H, with the temperature and density replaced by their true nonequilibrium values associated with f, and with **v** replaced by **V**. The kinetic equation obtained using Eq. (75) reads

$$(\partial_t + \mathbf{v}\cdot\nabla) f = -\nu(f - f_{\ell H}) + \tfrac{1}{2}\zeta \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f_{\ell H}). \tag{76}$$

This form encompasses the kinetic model of Refs. [19] and [10], which differ in the choice of f_H. These will be referred to as models 1 and 2, respectively, and are reviewed briefly in Appendix E.

An important consequence of the choice (75) is that velocity moments of sufficiently high degree diverge. This is most easily seen for the simplest case of homogeneous states using the change of variables [v_0 is defined below Eq. (19b)]

$$d\tau = \frac{\nu_H}{2} dt, \quad \mathbf{c} = \mathbf{v}/v_0(t), \quad \overline{f} = v_0^3(t) f, \tag{77}$$

to get

$$(\partial_\tau + 2\overline{\nu})(\overline{f} - \overline{f}_H) = -\overline{\zeta} \frac{\partial}{\partial \mathbf{c}} \cdot [\mathbf{c}(\overline{f} - \overline{f}_H)], \tag{78}$$

where $\overline{\zeta} = \zeta/\nu_H$, $\overline{\nu} = \nu/\nu_H$, and $\overline{f}_H = v_0^3(t) f_H$. Consider the scalar moments

$$M^{(l)}(\tau) = \int d\mathbf{c}\, c^l \overline{f}(c,\tau). \tag{79}$$

Using Eq. (78) an equation for their time evolution is obtained,

$$\frac{\partial}{\partial \tau} M^{(l)}(\tau) = (-2\overline{\nu} + l\overline{\zeta})[M^{(l)}(\tau) - M_H^{(l)}]. \tag{80}$$

It follows that moments of degree $l > 2\overline{\nu}/\overline{\zeta}$ grow without bound as a function of time, due to the term on the right side of Eq. (75). For model 1 this behavior is expected since the corresponding moments of f_H do not exist; for model 2 it represents an inconsistency between the choice (75) and the additional requirement in this model that $f_H \rightarrow f_M$. Nevertheless, for relevant values of α the values of l for which these problems arise are large and of little physical interest. For instance, for $\alpha = 0.9$ the smallest moment to diverge is for $l = 26$. Even if very large dissipation, $\alpha = 0.6$, is considered divergence only shows up for $l \geqslant 8$. The description of the low order moments relevant for hydrodynamics is quite good for both models, as illustrated below for the transport coefficients.

A better choice than Eq. (75), still consistent with the requirements (73) and (74), is given by

$$f_0[f] = f_{\ell H} + \frac{\zeta}{2\nu} \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f). \tag{81}$$

The model kinetic equation obtained using Eq. (81) in Eq. (72) is

$$(\partial_t + \mathbf{v} \cdot \nabla) f = -\nu (f - f_{\ell H}) + \tfrac{1}{2} \zeta \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f). \tag{82}$$

This will be referred to as model 3. All three models are quite similar although model 3 predicts finite moments of any degree (it leads to an evolution equation of form $[\partial_\tau + 2\overline{\nu})(\overline{f} - \overline{f}_H) = 0]$ and therefore is expected to be more accurate for applications outside the hydrodynamic limit as well. More detailed discussion of the mathematical basis for this choice is given elsewhere.

The Chapman-Enskog analysis of Appendixes A and B applies for these kinetic models as well, and the expressions for the transport coefficients given by Eqs. (31)–(33) are still valid. Only the values of $c^*(\alpha)$ and the linear operator L in the expressions for ν_η^*, ν_κ^*, and ν_μ^* are changed. For the above three models these are found to be

$$c_1^*(\alpha) = \frac{25(1-\alpha^2)^2}{12(1+5\alpha^2)}, \quad c_2^* = c_3^* = 0, \tag{83}$$

$$L_1 = \nu_1, \quad L_2 = \nu_2, \quad L_3 = \nu_3 - \tfrac{1}{2} \zeta_3^{(0)} \left(3 + \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{V}} \right), \tag{84}$$

$$\zeta_i^{(0)} = \zeta[f_M] = \tfrac{5}{12}(1-\alpha^2)\nu_0, \quad i = 1,2,3. \tag{85}$$

The values of $c_i^*(\alpha)$ follow from the calculation of $f_{\ell H}$ in model 1 using a Gaussian form for $f_0[f]$, while in models 2 and 3 the choice $f_{\ell H} = f_M$ is made. These choices are dictated by simplicity but other choices are possible [e.g., the approximation (B1) from the Boltzmann equation]. With these forms for L, Eqs. (36) and (37) are easily evaluated, with the result

$$\nu_{\eta 1}^*(\alpha) = \nu_{\kappa 1}^*(\alpha) = \nu_{\mu 1}^*(\alpha) = \nu_1^*, \tag{86}$$

$$\nu_{\eta 2}^*(\alpha) = \nu_{\kappa 2}^*(\alpha) = \nu_{\mu 2}^*(\alpha) = \nu_2^* \tag{87}$$

$$\nu_{\eta 3}^*(\alpha) = \nu_3^* + \zeta_M^*, \quad \nu_{\kappa 3}^*(\alpha) = \nu_{\mu 3}^*(\alpha) = \nu_3^* + \tfrac{3}{2} \zeta_M^* \tag{88}$$

where $\nu_i^* = \nu_i / \nu_0$, $\zeta_M^* = \zeta[f_M]/\nu_0$. These are still free parameters of the models that can be chosen to optimize the models for any particular application. For example, it is possible to choose them such that the viscosity agrees with that from the Boltzmann equation, aside from the value of c^* given by Eq. (83), which must be kept by consistency, i.e., require that

$$\nu_1^* = \nu_2^* = 1 - \tfrac{1}{4}(1-\alpha)^2, \quad \nu_3^* = 1 - \tfrac{1}{4}(1-\alpha)^2 - \zeta_M^*. \tag{89}$$

The explicit expressions for the transport coefficients for each of the models follow by substituting the above equations into Eqs. (31)–(33). It must be noticed that there the expression for the heat conductivity, $\kappa(\alpha)$ is reduced by its Boltzmann value for the elastic limit. This latter does not agree with the limiting value derived from the model equation, which is $\kappa_{0,i} = 5nk_B^2T/2m\nu_0$. Therefore, if the transport coefficients are reduced by their elastic values as derived from the models the factor of 2/3 disappears in Eq. (32). For the same reason, a factor of 2/3 appears in front of c^* on the right hand side of Eq. (33).

The transport coefficients obtained from the Chapman-Enskog expansion of the different models are shown in Figs. (1)–(3). Model 3 provides the most accurate results as compared with the expressions obtained from the Boltzmann equation in the Sonine approximation. Nevertheless, also models 1 and 2 give a quite fairly qualitative description of the α dependence of the transport coefficients. In practical applications dealing with specific physical situations, the model to be used will be dictated by a compromise between accuracy and tractability.

V. DISCUSSION

The primary objective of this work has been to derive a fluid dynamics description for inelastic hard spheres from an underlying kinetic theory in a systematic analysis. In addition to giving a firm foundation for the hydrodynamic equations, the kinetic equation admits an efficient complementary way to determine the hydrodynamic fields directly through Monte Carlo simulation [17]. The derivation of the hydrodynamic equations consists of two steps. First the macroscopic balance equations for mass, energy, and momentum are obtained from the kinetic equation without approximation. Next, the fluxes and energy sink in these equations are determined from a solution to the kinetic equation expressed in terms of the hydrodynamic variables and their spatial gradients. The Chapman-Enskog method described in the Appendixes yields this solution as an expansion in these gradients. Truncation of this series at zeroth (first) order yields Euler (Navier-Stokes) level fluxes, and associated contributions to the hydrodynamic equation of first (second) order in the gradients. In the case of inelastic collisions there is a sink in the energy equation that requires calculation of the solution to the kinetic equation to one order higher than that used for the fluxes. The small parameter in this expansion is the ratio of the mean free path relative to the wavelength of the hydrodynamic variation. The mean free path is independent of the time and therefore constant even for the homogeneous cooling state so the conditions for Navier-Stokes order hydrodynamics are essentially the same as for elastic collisions.

The Chapman-Enskog solution is not the most general solution to the kinetic equation and consequently the resulting hydrodynamic equations do not give a complete description of the response of these variables to an initial perturbation. To clarify the context in which the hydrodynamic equations apply it is useful to recall the case of small initial perturbations for a gas with elastic collisions. The linearized kinetic equation has a fivefold degenerate spectrum at the origin for asymptotically long wavelengths, corresponding to the five conserved variables (the hydrodynamic modes). In addition there are points, branch cuts, etc. in the spectrum bounded away from the origin by the collision frequency ν_H corresponding to excitations of all other degrees of freedom (the kinetic modes). At finite wavelengths this separation of points in the spectrum persists, except for wavelengths on the order of the mean free path. Consequently, the kinetic modes decay on a time scale of the inverse collision frequency leaving only the hydrodynamic modes at longer times. This explains why a reduced description of the dynamics, given by the hydrodynamic equations, provides a complete description on sufficiently long time scales. Now consider the case of inelastic collisions. Since energy is not conserved the fivefold degenerate set of points in the spectrum become four points at the origin and one at the finite inverse cooling rate, $\zeta^{(0)}$, for long wavelengths. The kinetic modes are again bounded away from the origin by ν_H. Note that here we are referring to the actual time variable t and not to τ defined in Eq. (61). Using the parameters of the previous section it is found that $0 \leqslant \zeta^{(0)}/\nu_H \leqslant 5/12$, with the lower and upper limits corresponding to $\alpha=1$ and $\alpha=0$, respectively. Thus the hydrodynamic modes are again closer to the origin than the kinetic modes and dominate for long times. A more complete description of the relationship of the hydrodynamic and kinetic modes will be given elsewhere. The conclusion is that the hydrodynamic equations apply for sufficiently large space and time scales, just as for the case of elastic collisions.

The hydrodynamic equations as derived by means of the Chapman-Enskog method are restricted to near equilibrium situations. Even in the elastic case there is very little known about solutions of the Boltzmann equation far from equilibrium. It is then useful to consider model kinetic equations that retain the relevant properties of the Boltzmann equation but are simpler than it. In this paper, we have discussed three models with different degree of accuracy (and complexity). All of them have been shown to lead to hydrodynamic equations similar to the Boltzmann equation. Also, the model referred to as model 2 is known to provide a semiquantitative description of the (inelastic) Boltzmann equation for some far from equilibrium states [20,21]. Other applications are being studied now.

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APPENDIX A: CHAPMAN-ENSKOG SOLUTION

The Chapman-Enskog solution to the Boltzmann equation is based on the uniformity parameter expansion of Eq. (15) together with the corresponding expansion for the time derivatives of the hydrodynamic fields. To zeroth order the solution $f^{(0)}$ is determined from Eq. (16). For fluids with elastic collisions the result is a local Maxwellian f_M but the exact solution is not known for $\alpha \neq 1$. To next order, $f^{(1)}$ is the solution to Eq. (26), which can be rewritten using Eq. (20) as

$$\begin{aligned}(\partial_t^{(0)}+L)f^{(1)}-\zeta^{(1)}T\partial_T f^{(0)}\\ =\mathbf{A}\cdot\boldsymbol{\nabla}\ \ln T+\mathbf{B}\cdot\boldsymbol{\nabla}\ \ln n+C_{ij}\nabla_j u_i,\end{aligned} \tag{A1}$$

where L is the linear operator given by Eq. (23) and $\zeta^{(1)}$ is the linear functional of $f^{(1)}$ given by Eq. (25). The coefficients of the field gradients on the right side are functions of $\mathbf{V}$ and the hydrodynamic fields

$$\mathbf{A}(\mathbf{V}|n,T)=\frac{\mathbf{V}}{2}\frac{\partial}{\partial\mathbf{V}}\cdot(\mathbf{V}f^{(0)})-(\beta m)^{-1}\frac{\partial}{\partial\mathbf{V}}f^{(0)}, \tag{A2}$$

$$\mathbf{B}(\mathbf{V}|n,T)=-\mathbf{V}f^{(0)}-(\beta m)^{-1}\frac{\partial}{\partial\mathbf{V}}f^{(0)}, \tag{A3}$$

$$C_{ij}(\mathbf{V}|n,T)=\frac{\partial}{\partial V_i}(V_j f^{(0)})-\frac{1}{3}\delta_{ij}\frac{\partial}{\partial\mathbf{V}}\cdot(\mathbf{V}f^{(0)}). \tag{A4}$$

The solution to Eq. (A1) therefore has the form

$$f^{(1)}=\boldsymbol{\mathcal{A}}\cdot\nabla\ \ln T+\boldsymbol{\mathcal{B}}\cdot\nabla\ \ln n+\mathcal{C}_{ij}\nabla_i u_j. \tag{A5}$$

The left side of Eq. (A1) can be simplified by noting that C_{ij} is traceless and therefore $\mathcal{C}_{ij}$ also is traceless. Since $\zeta^{(1)}$ is a scalar it follows directly from symmetry that $\zeta^{(1)}=0$. Also the time derivative can be expressed in terms of the gradients using

$$\partial_t^{(0)}\boldsymbol{\nabla}\ \ln T=\boldsymbol{\nabla}(T^{-1}\partial_t^{(0)}T)=-\zeta^{(0)}(\boldsymbol{\nabla}\ \ln n+\tfrac{1}{2}\boldsymbol{\nabla}\ \ln T). \tag{A6}$$

The functions $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, and $\mathcal{C}_{ij}$ in the solution (A5) are found by equating coefficients of the various gradients

$$\left(-\zeta^{(0)}T\partial_T+L-\frac{\zeta^{(0)}}{2}\right)\boldsymbol{\mathcal{A}}=\mathbf{A}, \tag{A7}$$

$$(-\zeta^{(0)}T\partial_T+L)\boldsymbol{\mathcal{B}}=\mathbf{B}+\zeta^{(0)}\boldsymbol{\mathcal{A}}, \tag{A8}$$

$$(-\zeta^{(0)}T\partial_T+L)\mathcal{C}_{ij}=C_{ij}. \tag{A9}$$

Consider first the viscosity. The contribution to the pressure tensor (12) from first order in the gradients is

$$1P_{ij}^{(1)}=\int d\mathbf{V}\, D_{ij}(\mathbf{V})\mathcal{C}_{k\ell}(\mathbf{V})\nabla_k u_\ell$$

$$=-\eta(\nabla_j u_i+\nabla_i u_j-\tfrac{2}{3}\,\delta_{ij}\nabla\cdot\mathbf{u}), \tag{A10}$$

$$\eta=-\tfrac{1}{10}\int d\mathbf{V}\, D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V}). \tag{A11}$$

Use of Eq. (A9) gives an equation for η,

$$-\zeta^{(0)}T\partial_T\eta-\tfrac{1}{10}\int d\mathbf{V}\, D_{ij}(\mathbf{V})L\mathcal{C}_{ij}(\mathbf{V})$$

$$=-\tfrac{1}{10}\int d\mathbf{V}\, D_{ij}(\mathbf{V})C_{ij}(\mathbf{V}). \tag{A12}$$

Define

$$\nu_\eta=\frac{\int d\mathbf{V}\, D_{ij}(\mathbf{V})L\mathcal{C}_{ij}(\mathbf{V})}{\int d\mathbf{V}\, D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V})}. \tag{A13}$$

Then Eq. (A12) becomes

$$(-\zeta^{(0)}T\partial_T+\nu_\eta)\,\eta=-\tfrac{1}{10}\int d\mathbf{V}\, D_{ij}(\mathbf{V})C_{ij}(\mathbf{V}). \tag{A14}$$

Dimensional analysis requires that $\zeta^{(0)}$ and ν_η are proportional to $T^{1/2}$ while the integral on the right side is proportional to T . Therefore the viscosity is proportional to $T^{1/2}$ and Eq. (A14) gives immediately

$$\eta=-\frac{1}{10(\nu_\eta-(1/2)\,\zeta^{(0)})}\int d\mathbf{V}\, D_{ij}(\mathbf{V})\frac{\partial}{\partial V_i}(V_j f^{(0)})$$

$$=\frac{1}{3(\nu_\eta-(1/2)\,\zeta^{(0)})}\int d\mathbf{V}\quad mV^2 f^{(0)}=p(\nu_\eta-\tfrac{1}{2}\,\zeta^{(0)})^{-1}. \tag{A15}$$

It is convenient to express this result in dimensionless form by introducing a characteristic average collision frequency ν_0 defined in terms of the Boltzmann viscosity for $\alpha=1$, η_0,

$$\nu_0=p/\eta_0=\frac{16}{5}n\sigma^2\sqrt{\frac{\pi k_B T}{m}}. \tag{A16}$$

The result (A15) then takes the dimensionless form

$$\frac{\eta}{\eta_0}=(\nu_\eta^*-\tfrac{1}{2}\,\zeta^*)^{-1}, \tag{A17}$$

where $\zeta^*\equiv\zeta^{(0)}/\nu_0$ and $\nu_\eta^*\equiv\nu_\eta/\nu_0$.

The heat flux to this order is

$$\mathbf{q}^{(1)}=-\kappa\boldsymbol{\nabla}T-\mu\boldsymbol{\nabla}n, \tag{A18}$$

with transport coefficients given by

$$\kappa=-\frac{1}{3T}\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{A}}(\mathbf{V}),$$

$$\mu=-\frac{1}{3n}\int d\mathbf{V}\ \mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{B}}(\mathbf{V}). \tag{A19}$$

The analysis is similar to that above for the viscosity, with the results

$$\kappa=-\frac{1}{3T(\nu_\kappa-2\zeta^{(0)})}\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\mathbf{A}(\mathbf{V}), \tag{A20}$$

$$\mu=\frac{2}{2\nu_\mu-3\zeta^{(0)}}\left[\zeta^{(0)}\frac{T}{n}\kappa-\frac{1}{3n}\int d\mathbf{V}\ \mathbf{S}(\mathbf{V})\cdot\mathbf{B}(\mathbf{V})\right], \tag{A21}$$

with

$$\nu_\kappa=\frac{\int d\mathbf{V}\ \mathbf{S}(\mathbf{V})\cdot L\boldsymbol{\mathcal{A}}(\mathbf{V})}{\int d\mathbf{V}\ \mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{A}}(\mathbf{V})},\qquad \nu_\mu=\frac{\int d\mathbf{V}\ \mathbf{S}(\mathbf{V})\cdot L\boldsymbol{\mathcal{B}}(\mathbf{V})}{\int d\mathbf{V}\ \mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{B}}(\mathbf{V})}. \tag{A22}$$

Use of the above forms for $\mathbf{A(V)}$ and $\mathbf{B(V)}$ allows further simplification,

$$\frac{1}{3T}\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot\mathbf{A}(\mathbf{V})$$

$$=-\frac{m}{6T}\int d\mathbf{V}V^2\mathbf{V}\cdot\left[\mathbf{V}(T\partial_T f^{(0)})+(\beta m)^{-1}\frac{\partial}{\partial\mathbf{V}}f^{(0)}\right]$$

$$=-\frac{5nk_B}{2m\beta}[1+c^*(\alpha)], \tag{A23}$$

$$\frac{1}{3n}\int d\mathbf{V}\ \mathbf{S}(\mathbf{V})\cdot\mathbf{B}(\mathbf{V})$$

$$=-\frac{m}{6n}\int d\mathbf{V}\ V^2\mathbf{V}\cdot\left[\mathbf{V}f^{(0)}+(\beta m)^{-1}\frac{\partial}{\partial\mathbf{V}}f^{(0)}\right]$$

$$=-\frac{5c^*(\alpha)}{4m\beta^2}\ . \tag{A24}$$

With these results κ and μ become

$$\frac{\kappa}{\kappa_0}=\frac{2}{3}(\nu_\kappa^*-2\zeta^*)^{-1}(1+c^*), \tag{A25}$$

$$\frac{n\mu}{T\kappa_0}=2(2\nu_\mu^*-3\zeta^*)^{-1}\left(\zeta^*\frac{\kappa}{\kappa_0}+\tfrac{1}{3}\,c^*\right), \tag{A26}$$

where $\nu_\kappa^*=\nu_\kappa/\nu_0$, $\nu_\mu^*=\nu_\mu/\nu_0$, and $\kappa_0=75k_B/64\sigma^2(\pi\beta m)^{1/2}$ is the Boltzmann thermal conductivity for $\alpha=1$.

APPENDIX B: EVALUATION OF ζ^*, ν_η^*, ν_κ^*, AND ν_μ^*

The results of Appendix A for the transport coefficients are still formal since they depend on ζ^*, ν_η^*, ν_κ^*, and ν_μ^*. Here approximate expressions for these quantities will be obtained by using an expansion in Sonine polynomials. This is the same approximation as is used in the case of fluids with elastic collisions and it is known in that case to be very accurate. In the above Sonine approximation Eq. (16) can be solved with the result [4,22]:

$$f^{(0)}(V)=f_M(V)\{1+\tfrac{1}{4}c^*(\alpha)[(\tfrac{1}{2}\beta mV^2)^2-\tfrac{5}{2}\beta mV^2+\tfrac{15}{4}]\}. \tag{B1}$$

The function $c^*(\alpha)$ is given to linear order by Eq. (39). Consider now $\zeta^*\equiv\zeta^{(0)}/\nu_0$, with $\zeta^{(0)}$ defined by Eq. (18). Using the approximation (B1) and neglecting again terms nonlinear in c^*, it is obtained:

$$\zeta^*=(1-\alpha^2)\frac{2\beta\omega[f^{(0)},f^{(0)}]}{3n\nu_0}=(1-\alpha^2)\tfrac{5}{192}\sqrt{2\pi}I(\alpha), \tag{B2}$$

where I is the dimensionless integral

$$I=\pi^{-3}\int d\mathbf{v}_1\int d\mathbf{v}_2\, e^{-(v_1^2+v_2^2)}|v_1-v_2|^3[1+\tfrac{1}{2}c^*(\alpha)\times(v_1^4-5v_1^2+\tfrac{15}{4})]. \tag{B3}$$

The integral can be performed with the final result

$$\zeta^*=(1-\alpha^2)\tfrac{5}{12}[1+\tfrac{3}{32}c^*(\alpha)]. \tag{B4}$$

The functions ν_η^*, ν_κ^*, and ν_μ^* are evaluated approximately by expanding $\boldsymbol{\mathcal{A}}(\mathbf{V})$, $\boldsymbol{\mathcal{B}}(\mathbf{V})$, and $\mathcal{C}_{ij}(\mathbf{V})$ as a series in Sonine polynomials and using the lowest order truncation. To lowest-order the velocity dependence is

$$\boldsymbol{\mathcal{A}}(\mathbf{V})\propto f_M\mathbf{S}(\mathbf{V}),\quad \boldsymbol{\mathcal{B}}(\mathbf{V})\propto f_M\mathbf{S}(\mathbf{V}),\quad \mathcal{C}_{ij}(\mathbf{V})\propto f_M D_{ij}(\mathbf{V}), \tag{B5}$$

where $f_M(V)$ is the Maxwellian. Then the expressions for ν_η^*, ν_κ^*, and ν_μ^* become

$$\nu_\eta^*=\frac{\int d\mathbf{V}\,D_{ij}(\mathbf{V})L[f_M(\mathbf{V})D_{ij}(\mathbf{V})]}{\nu_0\int d\mathbf{V}\,f_M(\mathbf{V})D_{ij}(\mathbf{V})D_{ij}(\mathbf{V})}=\frac{\beta^2}{10n\nu_0}\int d\mathbf{V}D_{ij}(\mathbf{V})L[f_M(\mathbf{V})D_{ij}(\mathbf{V})], \tag{B6}$$

$$\nu_\kappa^*=\nu_\mu^*=\frac{\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot L[f_M(\mathbf{V})\mathbf{S}(\mathbf{V})]}{\nu_0\int d\mathbf{V}\,f_M(\mathbf{V})\mathbf{S}(\mathbf{V})\cdot\mathbf{S}(\mathbf{V})}=\frac{2m\beta^3}{15n\nu_0}\int d\mathbf{V}\mathbf{S}(\mathbf{V})\cdot L[f_M(\mathbf{V})\mathbf{S}(\mathbf{V})]. \tag{B7}$$

The linear collision operator L, defined by Eq. (23), is

$$L[X(\mathbf{V})f_M(\mathbf{V})]=-\sigma^2\int d\mathbf{v}_1\int d\hat{\boldsymbol{\sigma}}\,\Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\alpha^{-2}b^{-1}-1)\times[f^{(0)}(\mathbf{V}_1)f_M(\mathbf{V})X(\mathbf{V})+f^{(0)}(\mathbf{V})f_M(\mathbf{V}_1)X(\mathbf{V}_1)], \tag{B8}$$

with the operator b^{-1} given by Eq. (3). Integrals of the form $\int d\mathbf{v}YL[f_MX]$ can be transformed as follows by a change of variables,

$$\begin{aligned}\int d\mathbf{v}Y(\mathbf{v})L[X(\mathbf{V})f_M(\mathbf{V})]&=-\sigma^2\int d\mathbf{v}\int d\mathbf{v}_1\int d\hat{\boldsymbol{\sigma}}\,\Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})Y(\mathbf{v})\\&\quad\times(\alpha^{-2}b^{-1}-1)[f^{(0)}(\mathbf{V}_1)f_M(\mathbf{V})X(\mathbf{V})\\&\quad+f^{(0)}(\mathbf{V})f_M(\mathbf{V}_1)X(\mathbf{V}_1)]\\&=\sigma^2\int d\mathbf{v}\int d\mathbf{v}_1\;f^{(0)}(\mathbf{V})f_M(\mathbf{V}_1)X(\mathbf{V}_1)\\&\quad\times\int d\hat{\boldsymbol{\sigma}}\,\Theta(-\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(b-1)[Y(\mathbf{v})+Y(\mathbf{v}_1)].\end{aligned} \tag{B9}$$

The evaluation of Eqs. (B6) and (B7) is now a well-defined problem. The integrations are straightforward but lengthy, and the exact evaluation is facilitated considerably by the use of symbolic computer programs. For the reader interested in confirming our results some of the intermediate results are given. Consider first ν_η^*. Use of Eq. (B9) in Eq. (B6) leads to

$$\nu_\eta^*=\frac{m\beta^2}{10n\nu_0}\sigma^2\int d\mathbf{V}\int d\mathbf{V}_1\;f^{(0)}(\mathbf{V})f_M(\mathbf{V}_1)D_{ij}(\mathbf{V}_1)\times\int d\hat{\boldsymbol{\sigma}}\Theta(-\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(b-1)[V_iV_j+V_{1i}V_{1j}]. \tag{B10}$$

The solid angle integral over $\hat{\boldsymbol{\sigma}}$ can be performed with the result

$$\nu_\eta^*=\frac{\pi m\beta^2}{80n\nu_0}(3-\alpha)(1+\alpha)\sigma^2\int d\mathbf{V}\int d\mathbf{V}_1\;f^{(0)}(\mathbf{V})\times f_M(\mathbf{V}_1)D_{ij}(\mathbf{V}_1)gg_ig_j\,. \tag{B11}$$

Next, change variables to $\mathbf{V}$ and $\mathbf{g}$, and perform the integration over $\mathbf{g}$ to get

$$\nu_\eta^*=\frac{4\pi}{5\nu_0}\left(\frac{2\pi}{\beta m}\right)^{1/2}[4-(1-\alpha)^2]n\sigma^2\times\int_0^\infty dX\;f^{(0)*}(X)X[(\tfrac{1}{2}+X^2)\sqrt{\pi}\mathrm{erf}(X)+Xe^{-X^2}], \tag{B12}$$

where $f^{(0)*}(X)=n^{-1}(2/\beta m)^{3/2}f^{(0)}[(2/\beta m)^{1/2}X]$ and $\mathrm{erf}(X)$ is the error function of X. The integral over X is performed using the approximation (B1) for the final result

$$\nu_\eta^*=[1-\tfrac{1}{4}(1-\alpha)^2][1-\tfrac{1}{64}c^*(\alpha)]. \tag{B13}$$

In a similar way $\nu_\kappa^*=\nu_\mu^*$ is evaluated as

$$\begin{aligned}\nu_\kappa^* = \nu_\mu^* = {} & \frac{m^2\beta^3}{15n\nu_0}(1+\alpha)\sigma^2 \int d\mathbf{V} \int d\mathbf{V}_1\; f^{(0)}(\mathbf{V}) f_M(\mathbf{V}_1) S_i(\mathbf{V}_1) \int d\hat{\boldsymbol{\sigma}}\; \Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\{\tfrac{1}{2}\overline{G_i}(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})^2(1-\alpha) \\ & +[(g_i\overline{\mathbf{G}}\cdot\hat{\boldsymbol{\sigma}}+\overline{\mathbf{G}}\cdot\mathbf{g}\hat{\sigma}_i)(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})-\hat{\sigma}_i\overline{\mathbf{G}}\cdot\hat{\boldsymbol{\sigma}}(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})^2(1+\alpha)]\} \\ = {} & \frac{m^2\beta^3}{60n\nu_0}(1+\alpha)\sigma^2\pi \int d\mathbf{V} \int d\mathbf{V}_1\; f^{(0)}(\mathbf{V}) f_M(\mathbf{V}_1) S_i(\mathbf{V}_1)[\tfrac{2}{3}g^3\overline{G_i}(1-2\alpha)+(3-\alpha)gg_i\overline{\mathbf{G}}\cdot\mathbf{g}] \\ = {} & \frac{6\pi}{45\nu_0}n\sigma^2\left(\frac{2\pi}{\beta m}\right)^{1/2}(1+\alpha)\int_0^\infty dX\; f^{(0)*}(X)\{4(4-3\alpha)X^2e^{-X^2}+[5(5-3\alpha)X+4(4-3\alpha)X^3)]\sqrt{\pi}\operatorname{erf}(X)\}.\end{aligned} \tag{B14}$$

In the above expressions it is $\overline{\mathbf{G}}=(\mathbf{V}+\mathbf{V}_1)/2$. Finally, integration over X using once again approximation (B1) yields

$$\nu_\kappa^* = \nu_\mu^* = \tfrac{1}{3}(1+\alpha)\left[1+\tfrac{33}{16}(1-\alpha)+\frac{19-3\alpha}{1024}c^*(\alpha)\right]. \tag{B15}$$

It is easily checked that the results given by Eqs. (B13) and (B15) reduce to the correct elastic values for $\alpha=1$.

APPENDIX C: THE DISTRIBUTION FUNCTION $f^{(1)}$

In the first Sonine polynomial approximation described in Appendix B the distribution function to Navier-Stokes order has the form

$$\begin{aligned} f^{(1)}(\mathbf{V}) = {} & [c_T\mathbf{S}(\mathbf{V})\cdot\nabla\ln T + c_n\mathbf{S}(\mathbf{V})\cdot\nabla\ln n \\ & + c_u D_{ij}(\mathbf{V})\nabla_i u_j] f_M(V). \end{aligned} \tag{C1}$$

The coefficients c_T, c_n, and c_u are determined in terms of the transport coefficients directly by use of this form in their definitions (A11) and (A19),

$$\begin{aligned}\eta &= -\frac{1}{10}\int d\mathbf{V}\; D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V}) \\ &= -c_u\tfrac{1}{10}\int d\mathbf{V}\; f_M(V) D_{ij}(\mathbf{V}) D_{ij}(\mathbf{V}) \\ &= -c_u n\beta^{-2},\end{aligned} \tag{C2}$$

$$\begin{aligned}\kappa &= -\frac{1}{3T}\int d\mathbf{V}\; \mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{A}}(\mathbf{V}) \\ &= -c_T\frac{1}{3T}\int d\mathbf{V} f_M(V)\mathbf{S}(\mathbf{V})\cdot\mathbf{S}(\mathbf{V}) \\ &= -c_T\frac{5nk_B}{2m}\beta^{-2},\end{aligned} \tag{C3}$$

$$\begin{aligned}\mu &= -\frac{1}{3n}\int d\mathbf{V}\; \mathbf{S}(\mathbf{V})\cdot\boldsymbol{\mathcal{B}}(\mathbf{V}) \\ &= -c_n\frac{1}{3n}\int d\mathbf{V} f_M(V)\mathbf{S}(\mathbf{V})\cdot\mathbf{S}(\mathbf{V}) \\ &= -c_n\frac{5}{2m}\beta^{-3}.\end{aligned} \tag{C4}$$

The distribution function $f^{(1)}$ is therefore

$$\begin{aligned} f^{(1)}(\mathbf{V}) = -\beta^3 n^{-1}\bigg[& \frac{2m}{5}\mathbf{S}(\mathbf{V})\cdot(\kappa\boldsymbol{\nabla}T+\mu\boldsymbol{\nabla}n) \\ & +\eta\beta^{-1}D_{ij}(\mathbf{V})\nabla_i u_j\bigg] f_M(V). \end{aligned} \tag{C5}$$

APPENDIX D: DETERMINATION OF $\zeta^{(2)}$

The macroscopic balance equations depend on the gradients of the pressure tensor and heat flux. Thus determining the latter to first order (Navier-Stokes) in the gradients leads to contributions of second order. For consistency the source term in the temperature equation must be calculated to second order. The contribution $\zeta^{(2)}$ is given by

$$\zeta^{(2)} = (1-\alpha^2)\frac{2\beta}{3n}\{\omega[f^{(1)},f^{(1)}]+2\omega[f^{(0)},f^{(2)}]\}, \tag{D1}$$

where $\omega[f,h]$ is defined by Eq. (7). The general form of $\zeta^{(2)}$ is given in Eq. (43). In this appendix the contributions to $\zeta^{(2)}$ that are linear in the gradients are considered, since only these contribute to the linear stability analysis. Therefore, the quantity we are interested in is

$$\zeta_L^{(2)} \equiv (1-\alpha^2)\frac{4\beta}{3n}\omega[f^{(0)},f_L^{(2)}]. \tag{D2}$$

Here $f_L^{(2)}$ denotes the part of $f^{(2)}$ that is linear in the gradients. The Chapman-Enskog equation for $f^{(2)}$ is

$$(\partial_t^{(0)}+L)f^{(2)} = -\partial_t^{(2)}f^{(0)}-(D_t^{(1)}+\mathbf{V}\cdot\nabla)f^{(1)}+J[f^{(1)},f^{(1)}], \tag{D3}$$

which simplifies for $f_L^{(2)}$ to

$$
\begin{aligned}
(\partial_t^{(0)}+L)f_L^{(2)}-\zeta_L^{(2)}T\partial_T f^{(0)}
&=-2(3nk_B)^{-1}(\kappa\nabla^2T+\mu\nabla^2 n)\partial_T f^{(0)}\\
&\quad+\frac{2m}{5}\beta^3 n^{-1}\mathbf{S}(\mathbf{V})\mathbf{V}:(\kappa\nabla\nabla T+\mu\nabla\nabla n)f_M\\
&\quad+(nm)^{-1}\eta[\tfrac{1}{3}\nabla(\nabla\cdot\mathbf{u})+\nabla^2\mathbf{u}]\cdot\frac{\partial}{\partial\mathbf{V}}f^{(0)}\\
&\quad-\frac{2m}{5}\beta^3 n^{-1}\left(\frac{2T}{3}\kappa+n\mu\right)\mathbf{S}(\mathbf{V})\cdot(\nabla\nabla\cdot\mathbf{u})f_M\\
&\quad-\beta^2 n^{-1}\eta D_{ij}(\mathbf{V})[(nm)^{-1}\nabla_i\nabla_j p-\mathbf{V}\cdot\nabla(\nabla_i u_j)]f_M .
\end{aligned}
\tag{D4}
$$

Further simplifications occur by noting that $\zeta_L^{(2)}$ is a scalar so that any contributions to $f_L^{(2)}$ that are vector or traceless functions of the velocity will not contribute. Let $\delta f_L^{(2)}$ denote the residual part of $f_L^{(2)}$ that gives contributions to $\zeta_L^{(2)}$. The vector and traceless parts on the right side of Eq. (D4) can be neglected for the purposes of calculating $\delta f_L^{(2)}$ and we get

$$
\begin{aligned}
(\partial_t^{(0)}+L)\delta f_L^{(2)}-\zeta_L^{(2)}T\partial_T f^{(0)}
&=\left[\frac{2m}{15}\beta^3 n^{-1}f_M(V)\mathbf{S}(\mathbf{V})\cdot\mathbf{V}-2\beta(3n)^{-1}(T\partial_T f^{(0)})\right]\\
&\quad\times(\kappa\nabla^2T+\mu\nabla^2 n).
\end{aligned}
\tag{D5}
$$

The solution to this equation has the form

$$
\delta f_L^{(2)}=M(T,\mathbf{V})\nabla^2T+N(T,\mathbf{V})\nabla^2 n,
\tag{D6}
$$

leading after substitution into Eq. (D2) to

$$
\zeta_L^{(2)}=\zeta_1\nabla^2T+\zeta_2\nabla^2 n.
\tag{D7}
$$

Then using the fact that

$$
\partial_t^{(0)}\nabla^2T=-\frac{3\zeta^{(0)}}{2}\nabla^2T-\frac{T\zeta^{(0)}}{n}\nabla^2 n+\text{nonlinear terms},
\tag{D8}
$$

the equations for M and N are found to be

$$
\begin{aligned}
\left(-\zeta^{(0)}T\partial_T-\frac{3\zeta^{(0)}}{2}+L\right)M-\zeta_1T\partial_T f^{(0)}
&=\kappa\left[\frac{2m}{15}\beta^3 n^{-1}f_M(V)\mathbf{S}(\mathbf{V})\cdot\mathbf{V}-2\beta(3n)^{-1}(T\partial_T f^{(0)})\right],
\end{aligned}
\tag{D9}
$$

$$
\begin{aligned}
(-\zeta^{(0)}T\partial_T+L)N-\zeta_2T\partial_T f^{(0)}
&=\frac{T\zeta^{(0)}}{n}M+\mu\left[\frac{2m}{15}\beta^3 n^{-1}f_M(V)\mathbf{S}(\mathbf{V})\cdot\mathbf{V}\right.\\
&\quad\left.-2\beta(3n)^{-1}(T\partial_T f^{(0)})\right].
\end{aligned}
\tag{D10}
$$

The functions $M(\mathbf{V})$ and $N(\mathbf{V})$ are scalars and are orthogonal to 1, $\mathbf{V}$, and V^2. Thus their lowest order Sonine expansion should be

$$
M(\mathbf{V})\to c_T^{(2)}P(V)f_M(V),\quad N(\mathbf{V})\to c_n^{(2)}P(V)f_M(V),
\tag{D11}
$$

where

$$
P(V)=(\tfrac{1}{2}\beta mV^2)^2-\tfrac{5}{2}\beta mV^2+\tfrac{15}{4}
\tag{D12}
$$

is the third Sonine polynomial except for a normalization factor, which is not relevant for our purposes. Use of Eqs. (D11) into Eq. (D2) yields

$$
\zeta_1=\frac{5\nu_0}{32}(1-\alpha^2)c_T^{(2)}\left(1+\frac{3c^*}{64}\right),
\tag{D13}
$$

$$
\zeta_2=\frac{5\nu_0}{32}(1-\alpha^2)c_n^{(2)}\left(1+\frac{3c^*}{64}\right).
\tag{D14}
$$

Now we substitute Eqs. (D11) into Eqs. (D9) and (D10), multiply by V^4, and integrate over $\mathbf{V}$. In this way it is obtained that

$$
(-\tfrac{5}{2}\zeta^*+\nu_\zeta^*)c_T^{(2)}=\frac{\zeta_1}{\nu_0}\left(1+\frac{c^*}{2}\right)+\frac{4\kappa}{15p\nu_0}\left(1-\frac{5c^*}{4}\right),
\tag{D15}
$$

$$
\begin{aligned}
(-2\zeta^*+\nu_\zeta^*)c_n^{(2)}
&=\frac{\zeta_2}{\nu_0}\left(1+\frac{c^*}{2}\right)+c_T^{(2)}\zeta^*\frac{T}{n}+\frac{4\mu}{15p\nu_0}\left(1-\frac{5c^*}{4}\right),
\end{aligned}
\tag{D16}
$$

where

$$
\begin{aligned}
\nu_\zeta^*&=\frac{\int d\mathbf{V}V^4L[f_M(V)P(V)]}{\nu_0\int d\mathbf{V}V^4f_M(V)P(V)}\\
&=\left(\frac{30n\nu_0k_B^2T^2}{m^2}\right)^{-1}\int d\mathbf{V}\,V^4L[f_M(V)P(V)].
\end{aligned}
\tag{D17}
$$

Upon deriving the above equations use has been made of the fact that $c_T^{(2)}\propto T^{-1}$ and $c_n^{(2)}\propto T^0$. Taking into account Eqs. (D13) and (D14), the above equations can be converted into two closed equations for ζ_1 and ζ_2. Introducing the dimensionless transport coefficients

$$
\zeta_1^*=\frac{3p}{2\kappa_0}\zeta_1,\quad \zeta_2^*=\frac{3n^2k_B}{2\kappa_0}\zeta_2,
\tag{D18}
$$

one gets

$$\zeta_1^* = \tfrac{1}{16}(1-\alpha^2)\kappa^*\left(1+\frac{3c^*}{64}\right)\left(1-\frac{5c^*}{4}\right) \times\left[\nu_\zeta^* - \tfrac{5}{2}\zeta^* - \tfrac{5}{32}(1-\alpha^2)\left(1+\frac{3c^*}{64}\right)\left(1+\frac{c^*}{2}\right)\right]^{-1}$$

$$\simeq \tfrac{1}{16}(1-\alpha^2)\kappa^*\left(1-\frac{77c^*}{64}\right)\left[\nu_\zeta^* - \tfrac{5}{2}\zeta^* - \tfrac{5}{32}(1-\alpha^2) \times\left(1+\frac{35c^*}{64}\right)\right]^{-1}, \tag{D19}$$

$$\zeta_2^* = \left[\zeta_1^*\zeta^* + \tfrac{1}{16}(1-\alpha^2)\mu^*\left(1+\frac{3c^*}{64}\right)\left(1-\frac{5c^*}{4}\right)\right] \times\left[\nu_\zeta^* - 2\zeta^* - \tfrac{5}{32}(1-\alpha^2)\left(1+\frac{3c^*}{64}\right)\left(1+\frac{c^*}{2}\right)\right]^{-1}$$

$$\simeq\left[\zeta_1^*\zeta^* + \tfrac{1}{16}(1-\alpha^2)\mu^*\left(1-\frac{77c^*}{64}\right)\right] \times\left[\nu_\zeta^* - 2\zeta^* - \tfrac{5}{32}(1-\alpha^2)\left(1+\frac{35c^*}{64}\right)\right]^{-1}. \tag{D20}$$

In the last transformations we have consistently neglected those contributions that are nonlinear in c^*. The only remaining task is to compute the characteristic frequency ν_ζ^* defined by Eq. (D17). This can be done by using the same procedure as discussed in Appendix B. We merely quote the result,

$$\nu_\zeta^* = \frac{1+\alpha}{192}\left[271 - 207\alpha + 30\alpha^2 - 30\alpha^3 + \frac{c^*}{64}(137 - 9\alpha - 30\alpha^2 + 30\alpha^3)\right]. \tag{D21}$$

APPENDIX E: TWO PREVIOUS KINETIC MODELS

Model 1 was introduced in Ref. [19] (see also the discussion in Ref. [23]). It was formulated by means of a kinetic equation of the same form as Eq. (72) with

$$f_0 = n\left(\frac{m}{2\pi k_B T\Delta}\right)^{3/2}\exp\left(-\frac{mV^2}{2k_B T\Delta}\right), \tag{E1}$$

where

$$\Delta = 1 - a(1-\alpha^2), \tag{E2}$$

a being a constant. This choice for f_0 leads to Eq. (73) with ζ approximated by

$$\zeta_1 = \nu_1 a(1-\alpha^2). \tag{E3}$$

The value of the constant a can be fixed by requiring the above expression to agree with the one obtained from the Boltzmann equation, Eq. (11), in the local equilibrium approximation, i.e. with f replaced by f_M. This gives

$$a = \frac{5\nu_0}{12\nu_1}. \tag{E4}$$

The solution of the model for the homogeneous cooling state reads

$$f_H(\mathbf{v},t) = \int_1^\infty dx\; P(x) n_H\left(\frac{m}{2\pi k_B T_H(t) x\Delta}\right)^{3/2} \times\exp\left(-\frac{mv^2}{2k_B T_H(t) x\Delta}\right), \tag{E5}$$

$$P(x) = p x^{-(1+p)}, \quad p = \frac{2}{1-\Delta}. \tag{E6}$$

It is now trivially seen that Eq. (E1) is equivalent to Eq. (75) with appropriate choices for $f_{\not H}$ and ζ.

Model 2 was formulated at the level of the Enskog equation in Ref. [10]. In the low density limit, it reduces directly to Eq. (75) with $f_{\not H} = f_M$, the Maxwellian local equilibrium distribution, while ζ is given by the exact functional of f of Eq. (11). Then it follows that f_H is also a Maxwellian for this model.

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