Dynamic magnetization-reversal transition in the Ising model

Arkajyoti Misra and Bikas K. Chakrabarti
Saha Institute of Nuclear Physics, 1/AF Bidhannangar, Calcutta 700 064, India

(Received 22 May 1998)

We report the results of mean field and the Monte Carlo study of the dynamic magnetization-reversal transition in the Ising model, brought about by the application of an external field pulse applied in opposition to the existing order before the application of the pulse. The parameters controlling the transition are the strength h_p and the duration Δt of the pulse. In the mean field case, an approximate analytical expression is obtained for the phase boundary which agrees well with that obtained numerically in the small Δt and large T limit. The order parameter of the transition has been identified, and is observed to vary continuously near the transition. The order parameter exponent β was estimated both for the mean field ($\beta=1$) and the Monte Carlo ($\beta=0.90\pm0.02$ in two dimensions) cases. The transition shows a ''critical slowing down'' type behavior near the phase boundary with diverging relaxation time. The divergence was found to be logarithmic in the mean field case and exponential in the Monte Carlo case. The finite size scaling technique was employed to estimate the correlation length exponent ν ($=1.5\pm0.3$ in two dimensions) in the Monte Carlo case.
[S1063-651X(98)06010-3]

PACS number(s): 05.50.+q

I. INTRODUCTION

The dynamic response of pure Ising systems to time dependent magnetic fields is being studied intensively these days (see, e.g., Refs. [1,2] and references therein). In particular, the response of Ising systems to pulsed fields has recently been investigated [3,4]. The pulse can be either ''positive'' or ''negative.'' At temperatures T below the critical temperature T_c of the corresponding static case (without any external field), the majority of the spins orient themselves to a particular direction giving rise to the prevalent order. If the external field pulse is applied along the direction of the existing order, it is called a positive pulse, and if the pulse is applied opposite to the existing order, it is called a negative pulse. The effect of positive pulse was studied in Ref. [3], whereas the occurrence of a magnetization-reversal transition as a result of the application of the negative pulse was reported in an earlier work [4]. Here we report the results of a detailed investigation of this dynamic magnetization-reversal transition for pure Ising models under a pulsed field.

In the absence of any symmetry breaking field, for temperatures below the critical temperature of the corresponding static case ($T<T_c$), there are two equivalent free energy minima with average magnetizations $+m_0$ and $-m_0$. If in the ordered state the equilibrium magnetization is $+m_0$ (say), and the pulse is applied in the direction opposite to the existing order, then temporarily during the pulse period the free energy minimum with magnetization $-m_0$ will be brought down compared to that with $+m_0$. If this asymmetry is made permanent, then any nonzero field (strength), which is responsible for the asymmetry, would eventually induce a transition from $+m_0$ to $-m_0$. Instead, if the field is applied in the form of a pulse, the asymmetry in the free energy wells is removed after a finite period of time. In that case, the point of interest lies in the combination of the pulse height or strength (h_p) and its width or duration (Δt) that can give rise to the transition from $+m_0$ to $-m_0$. We call this a magnetization-reversal transition. A crucial point about the transition is that it is not necessary that the system attains its final equilibrium magnetization $-m_0$ during the presence of the pulse; the combination of h_p and Δt should be such that the final equilibrium state is attained at any subsequent time, even long time after the pulse is withdrawn. The phase boundary, giving the minimal combination of h_p and Δt necessary for the transition, depends on the temperature. As $T \to T_c$, the magnetization-reversal transition occurs at lower values of h_p and/or Δt and the transition disappears at $T \geqslant T_c$.

In this paper we have given both the mean field (MF) and the Monte Carlo (MC) results for the transition. The MC studies have been carried out for a two-dimensional ($d=2$) lattice of Ising spins. The phase boundaries in the $h_p-\Delta t$ plane (each for a fixed T) are obtained for both the MF and MC cases. Approximate analytical expressions are also obtained and compared to these phase boundaries. The order parameter (O) for the dynamic transition has been identified, and at a fixed T its variation with the driving parameters h_p and Δt has been studied. The observed continuous variation of O indicates the nature of the transition to be comparable to the second order type static transitions. The critical exponents for the order parameter variations near the phase boundary has been obtained for both the MF and MC cases. We also employed the finite size scaling method (see, e.g., Ref. [5]) to estimate the correlation length exponent ν for the transition in the MC studies. We observe a significant ''critical slowing down'' near the phase boundary, and the behavior of the relaxation time τ has been studied in both the cases. In the MF case, we have obtained an approximate analytical expression for τ, indicating clearly a different kind of divergence of τ as compared to that in the static case.

II. MODEL

We have taken the Ising model for both the numerical simulation and the mean field study. The Hamiltonian of nearest neighbor Ising system without any disorder is

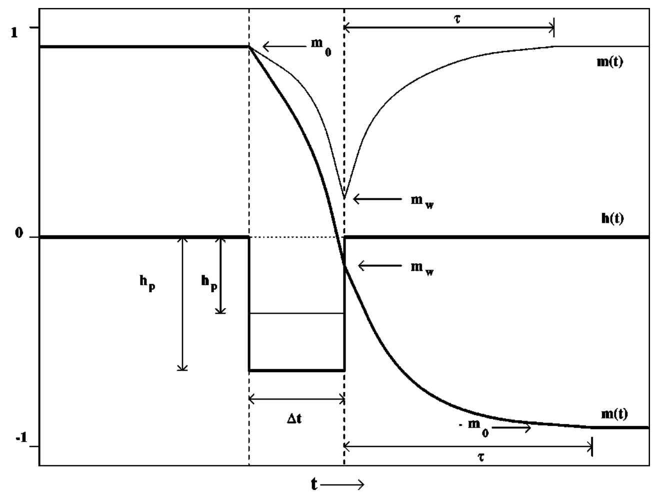

FIG. 1. Schematic time variation of the external field $h(t)$ and of the corresponding response magnetization $m(t)$. The thin line (smaller h_p) shows the case of no transition, whereas the thicker line (larger h_p) shows the occurrence of transition. The various quantities studied in this paper like the relaxation time τ, magnetization at the time of withdrawal of the field m_w, etc., are also indicated.

$$H=-J\sum_{\langle ij\rangle} S_iS_j-h(t)\sum_i S_i, \tag{1}$$

where $S_i=\pm 1$ represents the Ising spins at lattice site i, and $J(>0)$ denotes the nearest neighbor ferromagnetic interaction strength. The time dependent external magnetic field $h(t)$ is applied in the form of a pulse of duration Δt,

$$\begin{aligned} h(t)&=-h_p \quad \text{for } t_0<t<t_0+\Delta t \\ &=0 \qquad \text{otherwise.} \end{aligned} \tag{2}$$

t_0 is taken to be much larger than the relaxation time of the unperturbed system, so that the system is guaranteed to reach a state of equilibrium before the pulse is applied. The average magnetization m is given by $\langle S_i\rangle$, where the angular brackets represent a thermal average. By the time $t=t_0$, the system has reached its equilibrium state with the magnetization $m(t)=\pm m_0$. This is the state before the application of the pulse, which is applied at time $t=t_0$ to compete with the initial magnetization. We then want to look at the dynamics of the system under a pulsed field starting with the initial condition $m(t_0)=+m_0$ (say). Depending on the strength (h_p) or duration (Δt) of the pulse, the system has two choices after the pulse is withdrawn: it can either go back to the original ordered state $[m(t=\infty)=+m_0]$, or it can switch to the other equivalent equilibrium ordered state $[m(t=\infty)=-m_0]$. Figure 1 shows these behaviors schematically. The result, at any finite temperature below T_c, naturally depends on the strength and the duration of the pulse. Specifically, we observe that the result depends on $m_w\equiv m(t_0+\Delta t)$, the average magnetization at the time of withdrawal of the field. The sign of m_w, which in turn depends on the combination of h_p and Δt, governs the transition. $|m_w|$ was found out to be the appropriate candidate for the order parameter: if m_w becomes negative, on an average one observes a magnetization-reversal transition; however if m_w is positive,

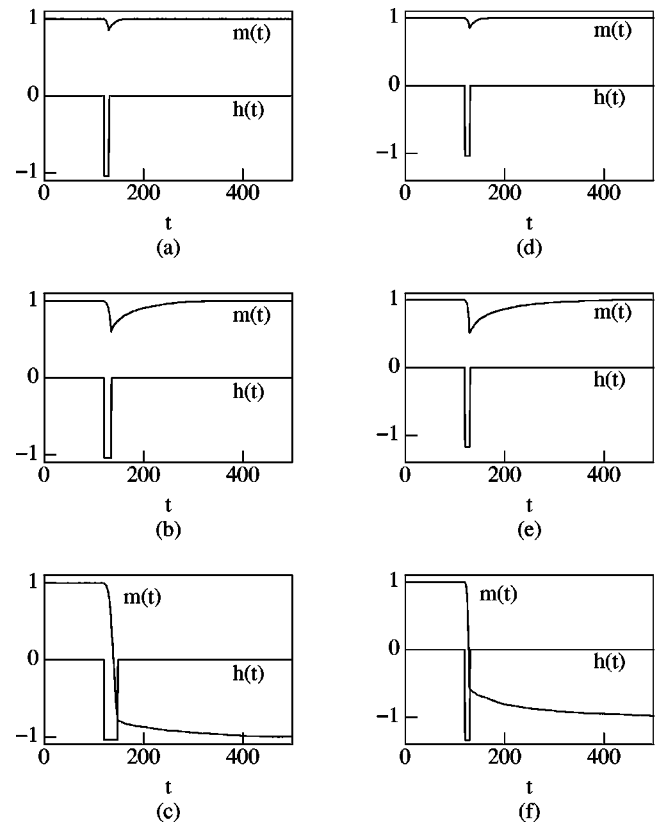

FIG. 2. Monte Carlo results for the time variation of the magnetization $m(t)$ against the time variation of the field $h(t)$ at $T=1.0$ for (a) $h_p=1.04$, $\Delta t=10$; (b) $h_p=1.04$, $\Delta t=15$; (c) $h_p=1.04$, $\Delta t=28$; (d) $h_p=1.04$, $\Delta t=10$; (e) $h_p=1.17$, $\Delta t=10$; and (f) $h_p=1.34$, $\Delta t=10$. Time is measured in units of Monte Carlo steps.

the system is most likely to return back to its original equilibrium state. In fact, it is guaranteed to be so in the mean field case. However in MC simulations, there exists occasional fluctuations due to which the system may finally arrive at the $+m_0$ state starting with negative m_w, and vice versa. We have identified the $h_p-\Delta t$ phase boundary at a particular temperature which gives the optimum combination of the driving parameters (h_p and Δt) that can force the system to a final state with magnetization $-m_0$, starting from an initial state with magnetization $+m_0$, or vice versa. We define the relaxation time τ as the time taken by the system to reach its final equilibrium state from the time of withdrawal of the field (at $t=t_0+\Delta t$). The relaxation time increases as the value of $|m_w|$ approaches zero, or equivalently as one approaches the phase boundary from either side. In Fig. 2 the typical MC results (on a square lattice) show how the transition can be brought about by either increasing Δt [see Figs. 2(a)–2(c)] or h_p [Figs. 2(d)–2(f)]. One can also note from these figures how the relaxation time τ increases as one approaches the phase boundary.

III. MEAN FIELD STUDY

The mean field equation of motion for the average magnetization $m(t)$ of the system is

$$\frac{dm}{dt}=-m+\tanh\left(\frac{m+h(t)}{T}\right), \tag{3}$$

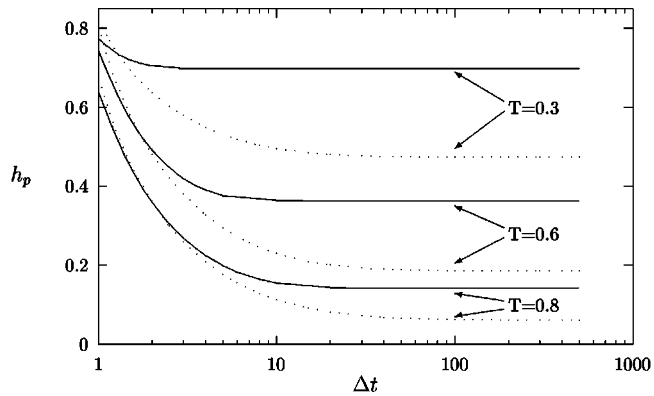

FIG. 3. Mean field phase boundary in the $h_p-\Delta t$ plane for three different values of T. The dotted lines correspond to the numerical solution of Eq. (3), while the solid lines give the corresponding theoretical prediction [from Eq. (6)].

where $h(t)$ is given by Eq. (2). Here we have assumed $Jn=1$, where n is the lattice coordination number. With this choice, the critical temperature in the static limit (T_c) becomes unity. All the mean field calculations are performed, therefore, at a temperature $T<1$. The equation was solved numerically to obtain the phase boundaries, shown in Fig. 3. A point on a particular phase boundary gives the optimal combination of h_p and Δt that can induce the transition from a state with magnetization $+m_0$ to $-m_0$, where $m_0=\tanh(m_0/T)$. The axes side of the boundaries correspond to the return to original equilibrium state, whereas one obtains a magnetization-reversal transition for combinations of h_p and Δt beyond the phase boundary. Because of the absence of any fluctuations in the mean field equation (3), there exists a finite coercive field for $T<T_c$ and therefore one cannot bring about the transition just by increasing Δt if h_p does not exceed the coercive field value. Hence the phase boundary becomes parallel to Δt axis for large values of Δt.

For large T, the mean field phase boundary can be estimated approximately by solving the linearized mean field equation

$$\frac{dm}{dt}=-\epsilon m+\frac{h(t)}{T}, \quad \epsilon=\frac{1-T}{T}, \tag{4}$$

where $h(t)$ is given by Eq. (2). The solution of Eq. (4) for $t>t_0$ can be written as

$$m(t)=\left(m_0-\frac{h_p}{\epsilon T}\right)\exp[\epsilon(t-t_0)]+\frac{h_p}{\epsilon T}. \tag{5}$$

It may be noted from solution (5) that t has to be close to t_0 in order to keep the value of $m(t)$ small, so that the linearization in Eq. (4) is valid. The magnetization-reversal transition occurs if $m(t_0+\Delta t)\leqslant 0$. The phase boundary can therefore be obtained from Eq. (5) by putting $m_w=m(t_0+\Delta t)=0$. The resulting equation for the phase boundary is then

$$h_p^c=\frac{m_0\epsilon T}{1-\exp(-\epsilon\Delta t)}. \tag{6}$$

In Fig. 3, this analytic result for the phase boundary is compared with those obtained by numerically solving Eq. (3). As one can clearly see from the figure, the agreement is good in the small Δt region of the phase boundaries for large values of T.

As mentioned above, the magnetization m_w at the time of withdrawal of the pulse, seems to be the crucial quantity governing the transition. The sign of m_w solely decides the final equilibrium state of the system out of the two equilibrium choices. Therefore we define the mean field order parameter (O) as following

$$O=m_w\theta(m_w), \tag{7}$$

where the step function θ is defined as

$$\theta(x)=1, \quad \text{for } x>0$$
$$=0, \quad \text{otherwise.}$$

The nature of variation of O with $|h_p-h_p^c|$ at different values of Δt and T is shown in Fig. 4. Here $h_p^c=h_p^c(\Delta t,T)$ is ob-

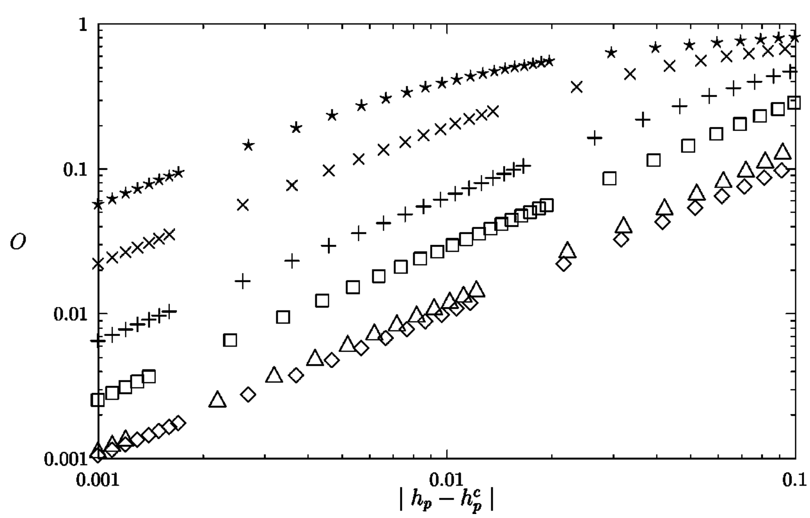

FIG. 4. Log-log plot of the variation of order parameter O against $|h_p-h_p^c|$ for $\Delta t=1$, $T=0.9$ ($\Diamond$); $\Delta t=1$, $T=0.6$ ($\triangle$); $\Delta t=2$, $T=0.8$ ($\square$); $\Delta t=5$, $T=0.9$ ($+$); $\Delta t=10$, $T=0.8$ ($\times$); and $\Delta t=10$, $T=0.6$ ($\star$) in the mean field case. The h_p^c values are obtained from the corresponding phase boundaries (Fig. 3).

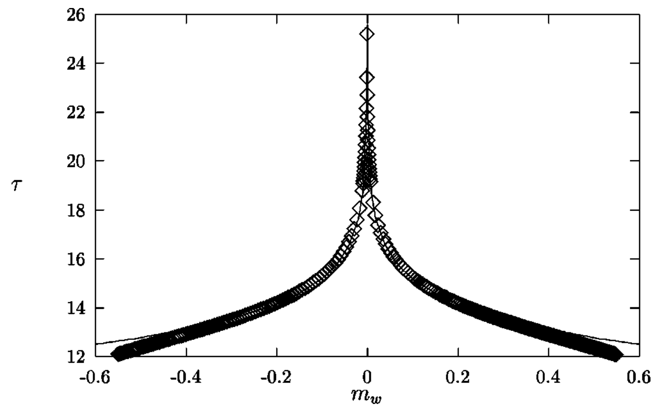

FIG. 5. Divergence of the relaxation time τ in the MF case. The solid line shows the logarithmic fit to the numerical solution of Eq. (3).

tained from the phase boundary. We fitted the order parameter variations to the power law form

$$O \sim |h_p - h_p^c|^\beta, \tag{8}$$

and found the value for the exponent $\beta \simeq 1$ by fitting the numerical results for different values of Δt and T.

The relaxation time (τ) grows as the phase boundary is approached from either side, and shows a divergence on the boundary. We measure the relaxation time by measuring the time required by $m(t)$ to reach the final equilibrium value $\pm m_0$, with an accuracy of $O(10^{-4})$, from the time of withdrawal of the pulse. According to Eqs. (7) and (8), for a fixed Δt, $m_w \sim |h_p - h_p^c|$. Therefore, τ is also expected to diverge as m_w vanishes. Figure 5 shows that this is indeed the case, and the growth of the relaxation time was found to be logarithmic in nature. That the relaxation time will diverge logarithmically at the phase boundary at any temperature below the static critical temperature ($T<1$) can be shown analytically. Let us follow the mean field dynamics of the system after the withdrawal of the field. The system starts with a magnetization m_w, and evolves according to Eq. (3) with $h(t)=0$ to reach the final state of equilibrium characterized by magnetization $\pm m_0$. Keeping the value of m/T small, we expand tanh of Eq. (3) up to the cubic term

$$\frac{dm}{dt} = \epsilon m + \alpha m^3, \tag{9}$$

where $\alpha = -1/(3T^3)$. Thus we can write

$$\int_0^\tau dt = \int_{m_w}^{m_0} \frac{dm}{\epsilon m + \alpha m^3}$$

or

$$\tau = -\frac{\ln m_w}{\epsilon} + \frac{\ln(\epsilon + \alpha m_w^2)}{2\epsilon} + C(T), \tag{10}$$

where $C(T)$ is a constant depending on temperature only. Now, if the combination of h_p and Δt is such that one starts with very small value of m_w, then one obtains

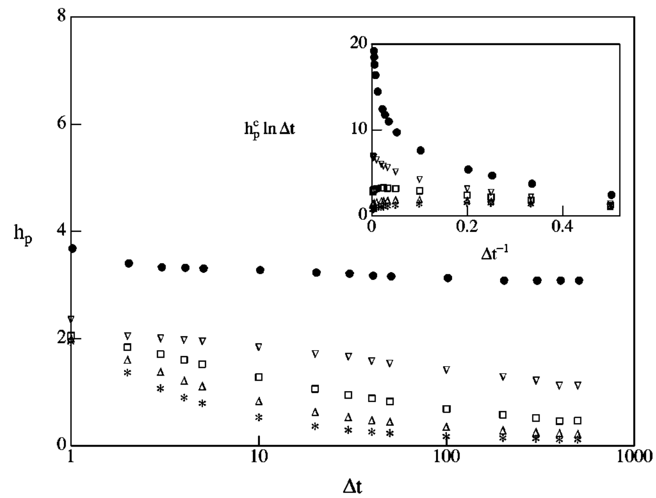

FIG. 6. Monte Carlo phase diagram in the $h_p - \Delta t$ plane for temperatures $T=0.1$ (●), $T=0.5$ (∇), $T=1.0$ (□), $T=1.5$ (△), and $T=2.0$ (⋆). The inset shows the variation of the quantity $h_p^c \ln \Delta t$ with Δt^{-1} for the same values of T.

$$\tau \sim -\frac{1}{\epsilon} \ln|m_w|. \tag{11}$$

It can be easily verified that even if the $O(m^5)$ term in the expansion of Eq. (9) is kept, the final result (11) is not modified in the $m_w \to 0$ limit. The above form for the variation of τ fits accurately the numerical results: at $T=0.6$, $1/\epsilon = 1.5$, and this matches exactly with the numerical result as shown in Fig. 5. The above analysis also makes it very clear that the logarithmic divergence of m_w implies a similar divergence with $|h_p - h_p^c|$ for a fixed Δt. It may be mentioned here that for the dynamic magnetization-reversal transition, the above divergence in τ occurs at any $T<T_c$. The same equation (11) also gives the well known $|T-T_c|^{-1}$ divergence of τ for the static transition at $T=T_c=1$. It may also be noted from Eq. (11) that the logarithmic dependence of τ on $|h_p - h_p^c|$, through its dependence on m_w, for this dynamic transition at $h_p^c(\Delta t, T<T_c)$, is qualitatively different from the divergence of τ ($\sim |T-T_c|^{-1}$) for the static transition in the same model [Eq. (3)] at $T=T_c$.

IV. MONTE CARLO STUDY

For the MC simulation we have taken Ising spins on a square lattice of size $L \times L$ with periodic boundary condition. We have taken $L=200$ for typical studies. Each spin of the lattice was updated sequentially using the Glauber single spin flip dynamics [6]. One complete sweep through the entire lattice is taken as one Monte Carlo Step (MCS). Phase boundaries at different temperatures below the static critical temperature ($T_c \simeq 2.27$) shows (Fig. 6) a qualitatively similar nature to that in the mean field case. However, unlike the mean field phase diagrams, the presence of fluctuations causes the MC phase diagrams to touch the abscissa asymptotically.

When the contributions of fluctuations become important at higher values of T and small values of h_p, the above MF theory fails. If $h_p \to 0$ (as in the large Δt region of the phase boundary), one can use the picture of nucleation of a single

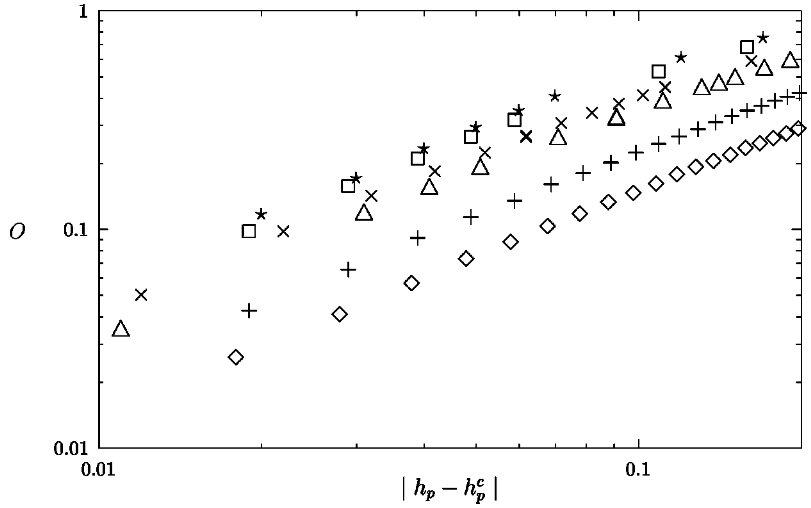

FIG. 7. Log-log plot of the variation of order parameter O against $|h_p - h_p^c|$ for $\Delta t = 2$, $T = 2.0$ (◇); $\Delta t = 1$, $T = 1.5$ (+); $\Delta t = 5$, $T = 1.5$ (△); $\Delta t = 10$, $T = 2.0$ (×); $\Delta t = 5$, $T = 1.0$ (□); and $\Delta t = 1$, $T = 1.0$ (⋆) in the MC case. The h_p^c values are obtained from the corresponding phase boundaries in the MC case.

domain. The classical nucleation theory of Becker and Döring (see, e.g., Ref. [7]) suggests that the nucleation rate $I \sim \exp[-F(l_c)/T]$ is given by the optimality condition $l_c = [\sigma(d-1)/2d]/h_p$ of free energy $F(l) = 2h_p l^d + \sigma l^{d-1}$ for the formation of a droplet or domain of linear size l under field h_p. Here σ is proportional to the surface tension for the formation of a droplet. Equating the growth rate given by the Becker-Döring nucleation rate I with the inverse pulse width, one obtains $\Delta t \simeq \exp(1/h_p^{d-1})$, suggesting

$$h_p^c \ln \Delta t = C, \tag{12a}$$

along the phase boundary in two dimensions, where C is a constant. It agrees qualitatively with the MC estimated phase diagram in the low h_p (i.e., large Δt) limit [4,8]. For large values of h_p, and hence for small values of Δt, the critical droplet size $l_c \sim 1/h_p$ being much smaller than the system size L, many droplets grow simultaneously, and the transition occurs due to coalescence of the droplets and not by the above-mentioned process of growth of a single droplet. In such coalescence regime, one expects that the nucleation time will be given by I^{-x}, where $x = 1/(d+1)$ (instead of unity as in the single droplet regime), resulting in

$$h_p^c \ln \Delta t = C/3 \tag{12b}$$

for the phase boundary in $d = 2$. In order to check results (12a) and (12b), we estimated the product $h_p^c \ln \Delta t$ along the MC phase boundaries at different values of T, as shown in the inset of Fig. 6. It is seen that the product does not approach a constant value [as suggested by Eq. (12a)] for large Δt (single domain region) at low values of T. For higher values of T, however, the product remains more or less a constant for large values of Δt, but the detailed nature of the variation of the product with Δt indicates a nonmonotonic variation. The ratio of the limiting values of this product (for large and small Δt) is observed to vary in the range 1.5–6.5, compared to the value 3 predicted by Eqs. (12a) and (12b).

Similar to the MF case [Eq. (7)], we define the order parameter in the same way: $O = m_w \theta(m_w)$, where θ is the step function and m_w is the average magnetization at the time of withdrawal of the pulse. Figure 7 shows the variation of this order parameter with $|h_p - h_p^c|$ for different values of Δt and T. Typical number of initial seed values taken to generate the configurations over which each data point is averaged is 500. The variation of O was again fitted to the power law form (8) with the order parameter exponent β, and we find $\beta \sim 1$ by fitting the MC results for four different values of Δt and four different values of T.

Here also we estimate the relaxation time τ by measuring the time (MC steps) required for $m(t)$ to reach the final equilibrium value $\pm m_0$ from the time of withdrawal of the pulse with a predefined accuracy $O(10^{-2})$. Again τ was found to diverge as the phase boundary (at any fixed T) is approached from either side. Figure 8 shows that the divergence of τ occurs at the point where m_w vanishes at the phase boundary. It is not possible to average the relaxation time for different realizations of the dynamics for a particular m_w, as different realizations produce different values of m_w. Therefore a small range of m_w was taken, instead of a particular value, to take an average of τ. A typical value of the range of m_w over which averaging has been done is 0.01. It was observed that τ depends on the driving parameters h_p, Δt, and temperature T only implicitly through the quantity m_w. The divergence of τ was found to be very sharp, and it fitted with an exponential function ($\tau \sim \exp[-c|m_w|]$, where c is a constant depending on temperature) unlike the mean field case where τ was seen to behave logarithmically with $|m_w|$.

We fitted above the MC results for $L = 200$, and extracted the value of the exponent β, assuming the finite size effects to be negligible. As $h_p \to h_p^c$, for finite L, one should consider the effective h_p^c to be a function of L. Therefore, one should consider an additional scaling function of L/ξ in Eq. (7), where $\xi \sim |h_p - h_p^c|^{-\nu}$ is the correlation length with exponent ν. Assuming that $\xi \sim L$ for such MC cases, we can write the appropriately modified form of (7) as

$$O \sim L^{-\beta/\nu} f[(h_p - h_p^c) L^{1/\nu}]. \tag{13}$$

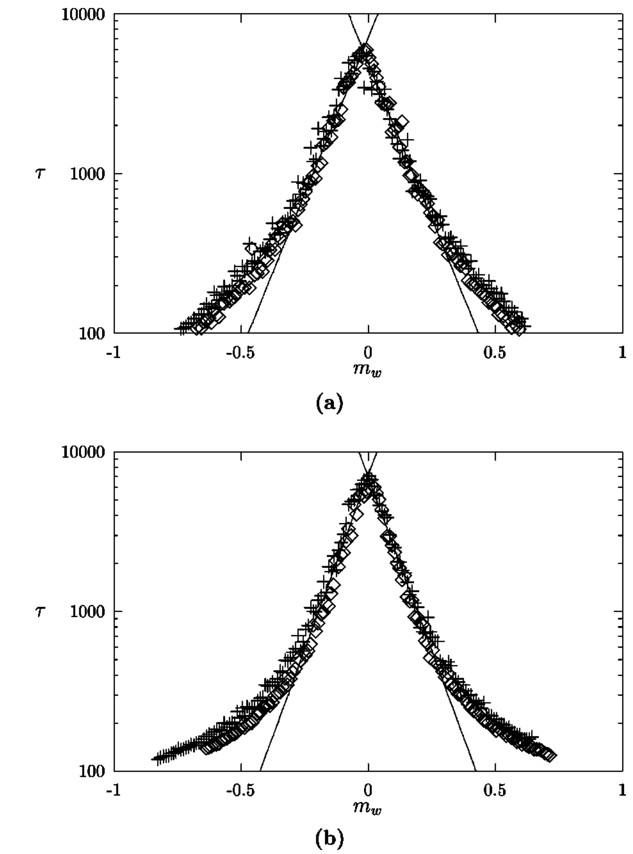

FIG. 8. Divergence of the relaxation time τ for (a) $T=1.5$; $\Delta t=3$ (◇), and $\Delta t=5$ (+), and (b) $T=2.0$; $\Delta t=1$ (◇), and $\Delta t=3$ (+) in the MC case. The solid lines indicates the exponential fit to the MC data. τ is measured in units of MCS.

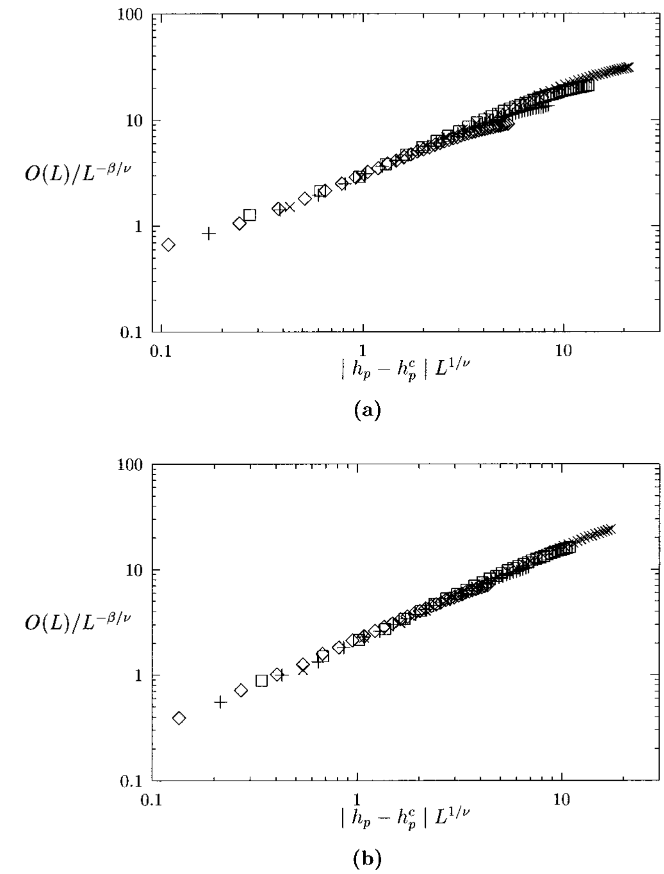

FIG. 9. Finite size scaling analysis of the MC data: $O(L)/L^{\beta/\nu}$ plotted against $|h_p-h_p^c|L^{1/\nu}$ for a fixed Δt. (a) $T=1.5$, $L=100$ (◇), $L=200$ (+) and $L=400$ (□); (b) $T=2.0$, $L=50$ (◇), $L=100$ (+), $L=200$ (□), and $L=400$ (×). The best collapsed data are shown with fitting values $\beta=0.90$, $\nu=1.5$, and $h_p^c=1.088$ in (a), and $h_p^c=0.720$ in (b).

We have fitted the data for different L to the scaling form (13). The data for $L=50$, 100, 200, and 400 for a particular value of Δt $(=5)$ and two different values of T were scaled to obtain the fitting values of the exponents as shown in Fig. 9. A typical number of averages over different initial seeds taken for the MC data was 2500 for $L=50$, and 50 for $L=400$. With such a scaling fit procedure, h_p^c could be obtained with an accuracy of 0.001. This gives $\beta=0.90\pm0.02$ and $\nu=1.5\pm0.3$. It was observed that the quality of fitting does not change appreciably for a variation of the fitting parameters within the error limits specified above.

V. DISCUSSION

The recently reported [4] dynamic magnetization-reversal transition in a pure Ising model under a pulsed (negative) magnetic field has been studied extensively in this paper employing the mean field approximation as well as the Monte Carlo technique in two dimensions. Accurate estimates have been made for the phase boundary $h_p^c(\Delta t,T)$, both in MF and MC cases. Approximate analytic estimates of the phase boundary have been made in both cases, and compared with the numerical results obtained. The order parameter for this dynamic transition has been identified, and is given by the average magnetization at the time of withdrawal of the pulse. In the mean field approximation the order parameter was found to vary linearly with $|h_p-h_p^c|$, indicating the order parameter exponent $\beta=1$; while in the MC case β was found to be around 0.9 for $d=2$. The application of the finite size scaling method gives the value of the critical exponent ν for the correlation length to be nearly 1.5 for $d=2$. Besides the length scale, a time scale was also found to diverge in both cases of mean field and Monte Carlo studies. In the mean field case, the relaxation time τ was found to diverge logarithmically with m_w, or equivalently with $|h_p-h_p^c|$, as the phase boundary is approached. This has also been demonstrated to be true analytically. However, the divergence of τ was found to be stronger in the Monte Carlo case, where τ diverges exponentially with $|m_w|$ or with $|h_p-h_p^c|^\beta$ (with $\beta\simeq0.90$ in $d=2$) as the phase boundary is approached. The finite size scaling results suggested that the correlation length diverges as $|h_p-h_p^c|^{-\nu}$ at the phase boundary with $\nu\simeq1.5$ in $d=2$. The existence of divergences of both the length and time scales indicates the thermodynamic nature of this intriguing dynamic magnetization-reversal transition in the Ising model.

ACKNOWLEDGMENTS

We would like to thank M. Acharyya, D. Chowdhury, C. Dasgupta, D. Dhar, A. K. Sen, and D. Stauffer for some useful comments and suggestions.

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