Moment ratios for absorbing-state phase transitions

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(Received 20 May 1998)

We determine the first through fourth moments of the order parameter, and various ratios, for several one- and two-dimensional models with absorbing-state phase transitions. We perform a detailed analysis of the system-size dependence of these ratios and confirm that they are indeed universal for three models, the contact process, the *A* model, and the pair contact process, belonging to the directed percolation universality class. Our studies also yield a refined estimate for the critical point of the pair contact process. [S1063-651X(98)02410-6]

PACS number(s): 05.50.+q, 02.50.−r, 05.70.Ln

I. INTRODUCTION

Testing the universality hypothesis and establishing the universality class of the critical points exhibited by various spin models and field theories have been a major preoccupation of statistical physics for some time [1]. In this ongoing project, the utility of studying a variety of universal quantities beyond the critical exponents, such as amplitude ratios and scaling functions, is well established. In particular, obtaining values for cumulant ratios of the order parameter has proved an efficient method for identifying the universality class [2]. In many cases, Binder's "reduced fourth cumulant" $q_4 \equiv 1-\langle\rho^4\rangle/3\langle\rho^2\rangle$ (here ρ represents the order parameter and the angular brackets denote a stationary average) is remarkably insensitive to finite-size effects, permitting one to determine the universality class with a modest computational effort [3].

While critical phenomena in nonequilibrium systems have been under intensive study for a good 20 years and controversies regarding the universality classes have frequently arisen, moment ratios have not, to our knowledge, been applied in this study. Our aim in this paper is to extend the method to nonequilibrium models. We focus on one- and two-dimensional models exhibiting an *absorbing-state phase transition* (i.e., between an active phase and one admitting no escape or evolution) [4]; all belong to the universality class of directed percolation [5,6]. In the following section we define the models. Sections III and IV present our results for one- and two-dimensional models, respectively. Section V contains a brief summary.

TABLE I. Density moments for the critical CP. Numbers in parentheses denote statistical uncertainties in the last figure(s).

L	m_1	m_2	m_3	m_4
20	0.47427(7)	0.26000(6)	0.15654(6)	0.10075(5)
40	0.39754(5)	0.18399(4)	0.09392(3)	0.05144(3)
80	0.33367(8)	0.13011(5)	0.05608(3)	0.02599(2)
160	0.28001 (10)	0.09181 (5)	0.03333 (3)	0.01302 (1)
320	0.23505(9)	0.06477(4)	0.01977(2)	0.006503(5)

II. MODELS

The models considered here are all examples of *interacting particle systems*: Markov processes whose state space is a set of particle configurations on a lattice, with transitions involving local processes of particle creation or annihilation [7–9]. In the *contact process* (CP) [10], each site of the hypercubic lattice $\mathcal{Z}^d$ is either vacant or occupied by a particle. Particles are created at vacant sites at a rate $\lambda n/2d$, where n is the number of occupied nearest neighbors, and are annihilated at a unit rate, independent of the surrounding configuration. The order parameter is the particle density ρ; it vanishes in the vacuum state, which is absorbing. As λ increases beyond λ_c, there is a continuous phase transition from the vacuum to an active steady state; for $\lambda>\lambda_c$, $\rho \sim(\lambda-\lambda_c)^\beta$. The *A* model was devised as a simplified description of surface catalysis [11]; in the present notation we may define it as a generalized CP in which the creation rate (at a vacant site) is λ for $n>0$, i.e., as long as the site has at

TABLE II. Cumulants and ratios for the critical CP.

L	K_2	K_2/m_1^2	K_4/K_2^2	m_4/m_2^2	m_3/m_1^3	m_3/m_1m_2
20	0.03507(12)	0.1559(12)	−0.641(4)	1.490(1)	1.467(1)	1.270(1)
40	0.02596(1)	0.1643(1)	−0.577(2)	1.520(2)	1.495(1)	1.284(1)
80	0.01877(1)	0.1686(2)	−0.544(3)	1.535(2)	1.510(2)	1.292(2)
160	0.01341(1)	0.1710(2)	−0.525(3)	1.545(3)	1.518(3)	1.297(2)
320	0.009517(3)	0.1723(3)	−0.515(4)	1.550(2)	1.522(3)	1.299(3)
∞		0.1736(2)	−0.505(3)	1.554(2)	1.526(3)	1.301(3)

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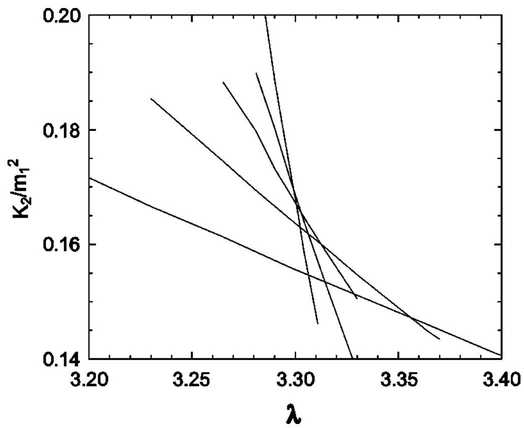

FIG. 1. Ratio K_2/m_1^2 versus creation rate λ in the one-dimensional CP. System sizes $L=20,40,\ldots,320$ are in order of increasing steepness.

least one occupied neighbor. Since creation occurs more readily in the A model than in the CP, the critical creation rate λ_c is smaller (1.7417 for the A model versus 3.2978 for the CP, in one dimension), but the two models (and indeed the whole family of generalized contact processes) share the same critical behavior [12], namely, that of directed percolation (DP).

The *pair contact process* (PCP) has a somewhat more complicated dynamics [13,14]; transitions occur only in the presence of a nearest-neighbor pair of particles. Specifically, in one dimension, if sites i and $i+1$ are both occupied, then the particles at these sites mutually annihilate at a rate $p \leq 1$, while at a rate $1-p$ they create a new particle at either $i-1$ or $i+2$ (chosen at random, each with probability 1/2), provided the chosen site is vacant. The PCP exhibits an active phase for $p<p_c$; above this value the system falls into one of an infinite number of absorbing configurations. (Any arrangement of particles devoid of nearest-neighbor pairs is absorbing.) While the presence of infinitely many absorbing configurations is associated with nonuniversal dynamics, the static critical behavior, which concerns us here, again falls in the DP class [14,17].

In equilibrium spin systems the magnetization (or its projection along a chosen direction) takes positive and negative values with equal likelihood, but in the present case the order parameter is non-negative. This means that we can study odd as well as even moments of the order parameter. In the following sections we report the first through fourth moments, $m_1,\ldots,m_4$, where $m_n \equiv \langle \rho^n \rangle$, and the ratios m_4/m_2, m_3/m_1^3, m_3/m_1m_2, K_2/m_1^2, and K_4/K_2^2, where K_n is the nth cumulant of the order parameter; in particular,

$$K_2 = m_2 - m_1^2 \tag{1}$$

and

$$K_4 = m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4. \tag{2}$$

The argument for universality among moment or cumulant ratios follows the same lines as in equilibrium [3]. We first note that at the critical point $m_1 \simeq AL^{-\beta/\nu_\perp}$, where β is the order parameter critical exponent, $\nu_\perp$ is the critical exponent governing the correlation length, L is the linear extent of the system, and A is a nonuniversal constant. From finite-size scaling, we have that at the critical point, the probability density for ρ satisfies $P(\rho,L;\lambda_c)=P(\rho/m_1) \simeq (L^{\beta/\nu_\perp}/A)\mathcal{P}(\rho L^{\beta/\nu_\perp}/A)$, where $\mathcal{P}$ is a universal scaling function. It follows that $m_n \simeq A^n L^{-n\beta/\nu_\perp} I_n$, where

$$I_n = \int_0^\infty u^n \mathcal{P}(u)\,du \tag{3}$$

is model and size independent. Thus ratios of the form $m_n/m_r^i m_s^j$ are universal for $ir+js=n$, as are cumulant ratios such as K_4/K_2^2.

TABLE III. Density moments for the critical A model.

L	m_1	m_2	m_3	m_4
20	0.43792(10)	0.22144(8)	0.12305(5)	0.07316(3)
40	0.36672(3)	0.15648(2)	0.07366(2)	0.03721(1)
80	0.30770(3)	0.11059(2)	0.04395(2)	0.01878(2)
160	0.25812(5)	0.07800(3)	0.02609(1)	0.009398(5)
320	0.21691(4)	0.05513(3)	0.01552(1)	0.004707(4)

III. RESULTS FOR ONE-DIMENSIONAL MODELS

A. Contact process

We studied the one-dimensional CP on periodic lattices of $L=20, 40,\ldots, 320$ sites, at the critical point $\lambda=\lambda_c = 3.297\,848$ [15]. The number of trials in our sample ranges

TABLE IV. Cumulants and ratios for the critical A model.

L	K_2	K_2/m_1^2	K_4/K_2^2	m_4/m_2^2	m_3/m_1^3	m_3/m_1m_2
20	0.02972(6)	0.1550(4)	−0.608(4)	1.492(2)	1.465(2)	1.269(1)
40	0.021995(4)	0.1636(1)	−0.566(1)	1.520(1)	1.494(1)	1.284(1)
80	0.015914(4)	0.1681(1)	−0.538(1)	1.536(1)	1.509(1)	1.292(1)
160	0.011375(5)	0.1707(1)	−0.522(2)	1.545(2)	1.517(2)	1.296(1)
320	0.008082(5)	0.1718(2)	−0.512(8)	1.549(3)	1.521(2)	1.298(2)
∞		0.1732(2)	−0.503(3)	1.554(2)	1.525(2)	1.300(2)

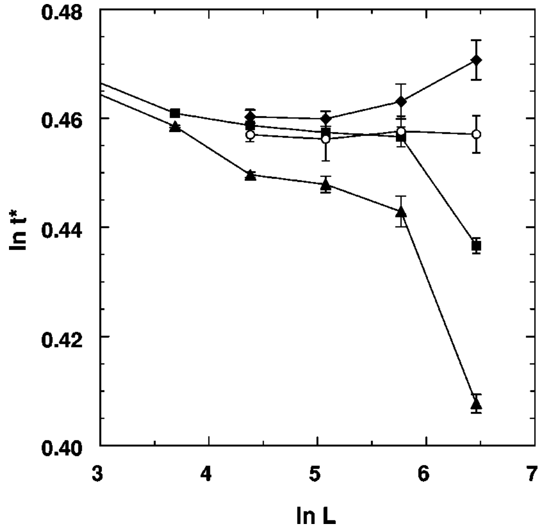

FIG. 2. Scaled lifetime $t^* \equiv \tau/L^{\nu_{\|}/\nu_{\perp}}$ versus system size L in the PCP. Diamonds, $p=0.07708$; circles, 0.07709; squares, 0.07710; triangles, 0.07712.

from 2×10^6 for $L=20$ to 2×10^5 for $L=320$; the maximum duration of a trial extends from 400 ($L=20$) to 16 000 ($L=320$). Each trial starts from a fully occupied lattice; we extract moments from the results for the surviving sample when it has relaxed to the quasistationary state.

The CP density moments are listed in Table I. Ratios are listed in Table II; the entries for $L=\infty$ represent extrapolations from least-squares quadratic fits to the data (for $L \geqslant 40$) for the various ratios as functions of L^{-1}. From Table II we see that K_2/m_1^2 converges more rapidly than K_4/K_2^2: between $L=160$ and 320 the value of the former changes by about 0.8% and the latter by about 2%. The ratios m_4/m_2^2, m_3/m_1^3, and m_3/m_1m_2 are remarkably stable, changing by only about 0.3% between the two largest L values. Similar trends are seen for the A model and the PCP (see below).

We also determined moment ratios for off-critical values of λ; in Fig. 1 we plot K_2/m_1^2 versus λ. As L increases, the point at which the curves intersect rapidly approaches λ_c. [We find that the crossing points $\lambda_{cr}(L,2L)$ for each pair of successive L values follows $\lambda_{cr}(L,2L) \simeq \lambda_{cr}(\infty) + \text{const}/L^2$, with $\lambda_{cr}(\infty)=3.29785(8)$, consistently with the best available estimate for λ_c.]

TABLE V. Density moments for the critical PCP.

L	m_1	m_2	m_3	m_4
20	0.51654(8)	0.31178(4)	0.20815(4)	0.14927(4)
40	0.43101(3)	0.21800(2)	0.12228(2)	0.07390(1)
80	0.36123(3)	0.15327(2)	0.07215(1)	0.03663(1)
160	0.30308(4)	0.10789(1)	0.04262(1)	0.01815(1)
320	0.25448(4)	0.07604(1)	0.02521(1)	0.009010(3)
640	0.21373(10)	0.05363(3)	0.01493(2)	0.004480(5)

B. *A* model

We studied the one-dimensional A model on periodic lattices of $L=20, 40, \ldots, 320$ sites, at the critical point $\lambda=\lambda_c=1.741\,73$ [15]. Sample sizes range from 2×10^6 for $L=20$ to 2×10^5 for $L=320$; the maximum duration of each sample realization extends from 200 ($L=20$) to 2×10^4 ($L=320$). The density moments are listed in Table III; cumulant ratios are listed in Table IV. The latter take values very similar to those found for the CP.

C. Pair contact process

In this model, the order parameter ρ is the density of nearest-neighbor particle *pairs*. We studied the one-dimensional PCP on periodic lattices of $L=20, 40, \ldots, 640$ sites, for $p=0.077\,08$, 0.077 09, 0.077 10, and 0.077 12. The sample sizes range from 10^7 for $L=20$ to 2×10^5 for $L=640$; the maximum duration of each realization extends from 200 to 10^4. Previous work yielded $p_c=0.0771(1)$ [13,14], but we decided to try to sharpen this estimate. To this end we analyzed the scaling of the lifetime $\tau(p,L)$, defined as follows. Starting with all sites occupied, the probability that a trial survives (remains active) until time t decays in proportion to $\exp[-t/\tau(p,L)]$. At the critical point, the lifetime has a power-law dependence on the system size $\tau(p_c,L)\sim L^{\nu_{\|}/\nu_{\perp}}$, while for $p\neq p_c$ deviations from a power law are seen. In Fig. 2 we plot $\ln t^* \equiv \ln \tau(p_c,L) - (\nu_{\|}/\nu_{\perp})\ln L$ versus $\ln L$, using $\nu_{\|}/\nu_{\perp}=1.5822$, the value for the DP class in 1+1 dimensions [15]. It is evident that the data for $p=0.077\,09$ are consistent with a power law, while those for the other p values are not, allowing us to conclude that $p_c=0.077\,090(5)$. The order parameter moments, for $p=p_c=0.077\,09$, are listed in Table V; cumulant ratios are

TABLE VI. Cumulants and ratios for the critical PCP.

L	K_2	K_2/m_1^2	K_4/K_2^2	m_4/m_2^2	m_3/m_1^3	m_3/m_1m_2
20	0.044921(3)	0.16836(6)	−0.66102(3)	1.5356(8)	1.510(1)	1.2925(6)
40	0.032230(4)	0.17349(4)	−0.55536(3)	1.5550(5)	1.527(1)	1.3014(4)
80	0.022779(5)	0.17457(7)	−0.5152(5)	1.5593(8)	1.5307(6)	1.3032(5)
160	0.016038(6)	0.1746(1)	−0.5014(4)	1.559(1)	1.531(1)	1.3034(6)
320	0.011279(5)	0.1742(1)	−0.496(2)	1.558(1)	1.530(1)	1.3028(9)
640	0.007949(5)	0.1740(3)	−0.497(3)	1.558(4)	1.529(4)	1.303(3)
∞		0.1738(2)	−0.493(3)	1.558(2)	1.529(3)	1.303(3)

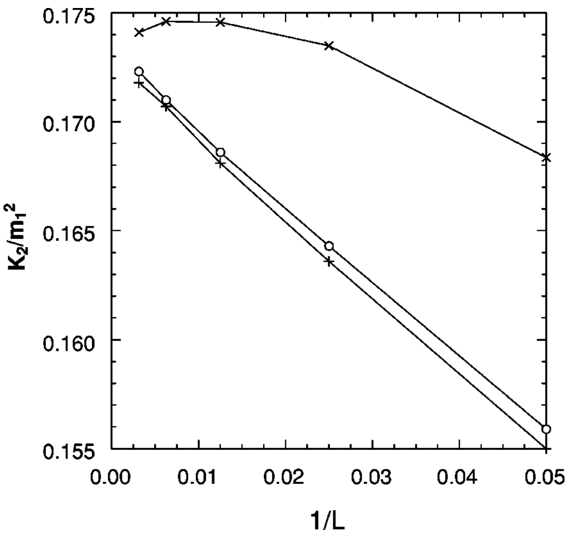

FIG. 3. K_2/m_1^2 versus $1/L$ for one-dimensional models at their critical points. ○, CP; +, A model; ×, PCP.

given in Table VI.

In Fig. 3 we plot the ratios K_2/m_1^2 versus L^{-1} for each of the three one-dimensional models. For the CP and the A model we observe a very similar, nearly linear approach to a limit. The PCP ratio initially approaches its limit more rapidly than the CP and A model do, overshoots it, and for large L, approaches the limit from above. Thus it would appear that the dominant correction to scaling for the size dependence of $P(\rho,L)$ is different in the PCP from that in the CP and the A model. Based on the extrapolations listed in Tables II, IV, and VI, we conclude that for the DP class in 1+1 dimensions, $K_2/m_1^2=0.1735(5)$, with the uncertainty figure a subjective assessment based on the degree of regularity of the data.

The ratios m_3/m_1^3 and m_4/m_2^2 in the three one-dimensional models are plotted versus L^{-1} in Fig. 4. As before, the CP and A model exhibit very similar trends and the PCP ratios have a nonmonotonic approach to their apparent limits. (Though not shown in Fig. 4, the same in fact applies to m_3/m_1m_2.) Based on our extrapolations, we estimate $m_4/m_2^2=1.556(3)$, $m_3/m_1^3=1.527(3)$, and $m_3/m_1m_2=1.301(2)$, for the DP class in 1+1 dimensions. Our results for the cumulant ratio K_4/K_2^2 are more scattered; we estimate the value of this ratio as $-0.50(1)$ for the DP class in 1+1 dimensions.

TABLE VII. Density moments for the critical CP in two dimensions.

L	m_1	m_2	m_3	m_4
10	0.15706(6)	0.03228(2)	0.00781(1)	0.002120(3)
20	0.09027(6)	0.01078(1)	0.001522(2)	0.0002416(4)
40	0.05200(3)	0.003587(3)	0.0002929(3)	0.00002693(4)
80	0.03003(2)	0.001197(2)	0.0000564(1)	0.00000300(1)
160	0.01736(1)	0.0003998(5)	0.00001090(2)	$0.335(1)\times10^{-6}$

IV. CONTACT PROCESS IN TWO DIMENSIONS

In this section we report results for the two-dimensional CP at the critical point $\lambda=\lambda_c=1.6488$ [16]. We studied systems of $L\times L$ sites with L ranging from 10 to 160 and sample sizes ranging from 3×10^6 for $L=10$ to 10^6 for $L=160$. The maximum duration of a trial runs from 1000 for $L=10$ to 10^5 for $L=160$. We calculate moments from a sample of 10^4 maximum-duration trials for $L=10$ to $L=80$ and of 5000 such trials for the largest L. Density moments and ratios are given in Tables VII and VIII, respectively. As in one dimension, the moment ratios m_3/m_1^3, m_3/m_1m_2, and m_4/m_2^2 are quite stable. Estimated limiting values, obtained from quadratic least-squares fits to the data for $L\geqslant20$, are given in Table VIII. Figure 5 shows that m_3/m_1m_2 and m_4/m_2^2 have a similar, nonmonotonic L dependence, resembling that of the one-dimensional PCP.

V. SUMMARY

We have determined order parameter moments and cumulant ratios for one- and two-dimensional models in the directed percolation universality class. In one dimension, we

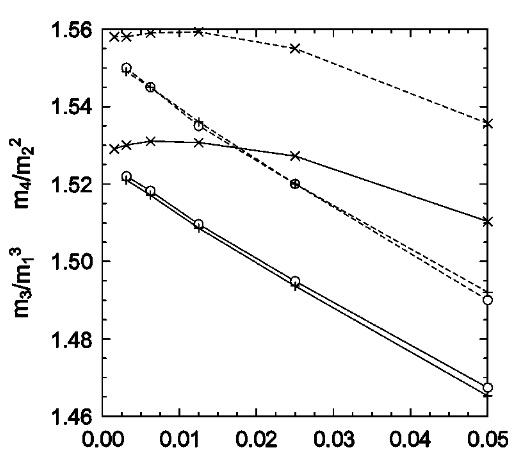

FIG. 4. Same as Fig. 3, but for m_3/m_1^3 (points joined by solid lines) and m_4/m_2^2 (dashed lines).

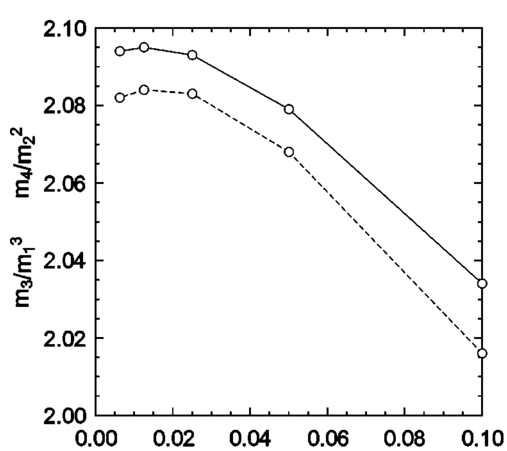

FIG. 5. Moment ratios m_3/m_1^3 (points joined by solid lines) and m_4/m_2^2 (dashed lines), for the CP in two dimensions.

TABLE VIII. Cumulants and ratios for the critical CP in two dimensions.

L	K_2	K_2/m_1^2	K_4/K_2^2	m_4/m_2^2	m_3/m_1^3	$m_3/(m_1m_2)$
10	0.00761(1)	0.3085(3)	−0.169(3)	2.034(6)	2.016(1)	1.541(4)
20	0.002631(2)	0.3229(4)	−0.115(3)	2.079(8)	2.068(1)	1.564(5)
40	0.0008832(7)	0.3267(4)	−0.090(4)	2.093(7)	2.083(2)	1.570(4)
80	0.0002946(3)	0.3269(4)	−0.089(5)	2.095(8)	2.084(2)	1.571(6)
160	0.0000983(1)	0.3264(5)	−0.088(4)	2.094(8)	2.082(2)	1.570(5)
∞		0.3257(5)	−0.088(4)	2.093(8)	2.080(1)	1.569(1)

studied three different models, the contact process, the closely related *A* model, and the pair contact process, and confirmed the universality of their cumulant and odd- and even-moment ratios. We also derived an improved estimate $p_c = 0.077\,090(5)$ for the critical point of the one-dimensional pair contact process. In two dimensions, we determined cumulant and moment ratios for the basic contact process.

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