Anomalous diffusion in disordered media and random quantum spin chains

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Using exact expressions for the persistence probability and for the leading eigenvalue of the Fokker-Planck operator of a random walk in a random environment, we establish a fundamental relation between the statistical properties of anomalous diffusion and the critical and off-critical behavior of random quantum spin chains. Many exact results are obtained from this correspondence, including the space and time correlations of surviving random walks and the distribution of the gaps of the corresponding Fokker-Planck operator. In turn we derive analytically the dynamical exponent of the random transverse-field Ising spin chain in the Griffiths-McCoy region. [S1063-651X(98)02010-8]

Ultraslow dynamics is a common feature of low-dimensional systems with quenched disorder in particular in the vicinity of a critical point. One of the well known examples in this respect is the one-dimensional diffusion process in a random media, when, in the absence of an average drift δ_{RW}, the mean-square displacement grows very slowly like [1]

$$[\langle X^2(t) \rangle]_{av} \sim \ln^4 t, \qquad (1)$$

in contrast to the normal diffusive behavior $\langle X^2(t) \rangle \sim t$ in the homogeneous case. The diffusion process remains anomalous for sufficiently small average drifts $0 < \delta_{RW} < \delta_{RW}^+$, when the average displacement has an algebraic time dependence

$$[\langle X(t) \rangle]_{av} \sim t^\mu, \qquad (2)$$

where the exponent $0 < \mu = \mu(\delta_{RW}) \leq 1$ is a continuous function of the drift [2].

Another class of systems with ultraslow dynamical properties is represented by random quantum spin chains at very low temperatures. For example, the asymptotic decay of the zero-temperature (imaginary time) autocorrelation function $G(t) = [\langle \sigma_i^x(t) \sigma_i^x \rangle]_{av}$ at the quantum critical point ($\delta = 0$) is given by [3]

$$G(t, \delta = 0) \sim [\ln t]^{-2x_m}, \qquad (3)$$

where x_m is the anomalous dimension of the average magnetization. Away from the critical point, in the Griffiths-McCoy region [4,5] with $0 < \delta \leq \delta_G$ the decay of the autocorrelations is of a power-law form

$$G(t, \delta) \sim t^{-1/z}, \qquad (4)$$

where the dynamical exponent $z(\delta) \geq 1$ is a continuous function of the quantum control parameter δ.

Comparing the basic dynamical properties of random walks in disordered environments and of random quantum spin chains one can easily notice close similarities, which hold in both the critical and off-critical situations. A connection between the critical properties of directed walks and quantum spin chains has been known for quite some time [6]. It was also demonstrated recently that many surprising properties of the one-dimensional random transverse-field Ising model (RTIM), a prototype of random quantum spin chains, can be obtained very simply through random walk arguments [7].

In this paper we go further and show that behind the similarities observed before there is a deep connection between the statistical properties of anomalous diffusion and the critical and off-critical behavior of the RTIM. We demonstrate this relation by comparing *exact* expressions for the random walk (RW) and that of the RTIM. In particular we show that the persistence probability of the RW and the surface magnetization of the RTIM have analogous forms and that the expressions for the leading eigenvalue of the Fokker-Planck (FP) operator of the RW and the gap of the Hamiltonian of the RTIM are closely related to each other. We use then this correspondence to obtain exact relations for the two systems; among others we present analytical results about the dynamical exponent $z(\delta)$ in Eq. (4).

We start by considering the one-dimensional random walk with nearest neighbor hopping, which is characterized by the transition probabilities $w_{i,i\pm 1} = w(i \rightarrow i \pm 1)$ for a random walker to jump from site i to site $i \pm 1$. Here we are particularly interested in the general case, in which the transition probabilities are not necessarily symmetric [8], i.e., $w_{i,i+1} \neq w_{i+1,i}$. Moreover, the random walker is confined to a finite number of sites $i = 1, \ldots, L$. At the two ends of this interval, i.e., at $i = 0$ and $i = L+1$, we put *adsorbing walls*, which is simply modeled by setting $w_{0,1} = w_{L+1,L} = 0$ (i.e., the walker cannot jump back into the system once it has landed on 0 or $L+1$). The time evolution of the probability distribution of the walk $P_{i,j}(t)$, which is the probability for the walker to be at time t on site j once started at time 0 on site i, is fully determined by the master equation

$$\frac{d}{dt}\underline{P}_i(t)=\underline{\underline{M}}\cdot\underline{P}_i(t). \qquad (5)$$

Here $\underline{P}_i(t)=(P_{i,0}(t),P_{i,1}(t),\ldots,P_{i,L}(t),P_{i,L+1}(t))^T$ and the transition matrix or the Focker-Planck operator is $(\underline{\underline{M}})_{i,j}=w_{i,j}$ for $i\neq j$ and $(\underline{\underline{M}})_{i,i}=-\Sigma_j w_{i,j}$, while the initial condition is $P_{i,j}(0)=\delta_{i,j}$. The eigenvalue problem of the FP operator in Eq. (5) is defined by

$$\underline{\underline{M}}\,\underline{v}_k=\lambda_k\underline{v}_k, \quad \underline{u}_k^T\underline{\underline{M}}=\underline{u}_k^T\lambda_k \qquad (6)$$

and all physical properties of the model can be expressed in terms of the left and right eigenvectors $\underline{u}_k$ and $\underline{v}_k$, respectively, and the eigenvalues λ_k. For instance, the probabilities $P_{i,j}(t)$ are given by

$$P_{i,j}(t)=\sum_k u_k(i)v_k(j)\exp(\lambda_k t). \qquad (7)$$

With adsorbing boundaries the two leading eigenvalues are zero and the corresponding eigenvectors are $v_1(i)=\delta_{L+1,i}$,

$$u_1(0)=0, \qquad (8)$$

$$u_1(i)=u_1(1)\left[1+\sum_{j=1}^{i-1}\prod_{l=1}^j\frac{w_{l,l-1}}{w_{l,l+1}}\right], \quad i=2,3,\ldots,L+1,$$

while for the other zero mode $v_2(i)=v_1(L+1-i)$ and similarly $u_2(i)=u_1(L+1-i)$. The value of $u_1(1)$ in Eq. (8) is fixed by the normalization condition $u_1(L+1)=1$.

We consider first a quantity that has gained considerable interest recently in related models for anomalous diffusion [9,10]: the *persistence probability* $P_{pr}(L,t)$, which is the probability that a walker starting at site $i=1$ does not cross its starting point until time t. Working with adsorbing sites at $i=0$ and $i=L+1$, we have $P_{pr}(L,t)=P_{1,L+1}(t)$ and its long time limit $p_{pr}(L)=\lim_{t\to\infty}P_{pr}(L,t)$ is simply given, via Eq. (7), by $p_{pr}(L)=u_1(1)v_1(L+1)$, where we used the fact that there is no contribution from the second zero mode since $v_2(L+1)=0$. Now with Eq. (8) we have the simple *exact* relation

$$p_{pr}(L)=\left(1+\sum_{i=1}^L\prod_{j=1}^i\frac{w_{j,j-1}}{w_{j,j+1}}\right)^{-1}. \qquad (9)$$

Note that $p_{pr}(L)$ is the total fraction of walkers adsorbed by the right wall ($i=L+1$) without ever having crossed the starting point.

In a *homogeneous* medium with $w_{i,i+1}=w_{i+1,i}=$ const we have $p_{pr}^{hom}=(L+1)^{-1}$, whereas in an *inhomogeneous* environment with *symmetric* transition probabilities $w_{i,i+1}=w_{i+1,i}$,

$$p_{pr}^{sym}(L)=\left[1+\sum_{i=1}^L\frac{w_{1,0}}{w_{i,i+1}}\right]^{-1}\propto\frac{D}{L}, \qquad (10)$$

where $D=[L\Sigma_{i=1}^L w_{i,i+1}^{-1}]^{-1}$ is the diffusion constant [11]. Thus the average persistence probability in the random sym-

metric case scales as $[p_{pr}^{sym}]_{av}\propto[D]_{av}/L$, similarly to the homogeneous case. From here on we use $[\ ]_{av}$ to denote average over quenched disorder.

In the general case, with *nonsymmetric* transition probabilities, we define the control parameter

$$\delta_{RW}=\frac{[\ln w_\leftarrow]_{av}-[\ln w_\rightarrow]_{av}}{\text{var}[\ln w_\leftarrow]+\text{var}[\ln w_\rightarrow]}, \qquad (11)$$

where $w_\rightarrow$ ($w_\leftarrow$) stands for transition probabilities to the right (left), i.e., $w_{i,i+1}$ ($w_{i,i-1}$). For $\delta_{RW}<0$ there is an average drift of the walk towards the adsorbing site at $i=L+1$; therefore, the persistence will have a finite value in the large system limit $\lim_{L\to\infty}[p_{pr}(L,\delta_{RW})]_{av}>0$, whereas it goes to zero for $\delta_{RW}\geq0$.

Before we proceed with the analysis of the persistence probability (9) we derive a similar formula for the largest nonzero eigenvalue of the FP operator, the *absolute* value of which we denote by λ_{min}. According to the relation in Eq. (7) the time scale t_r of the diffusion process is set by $t_r\sim\lambda_{min}^{-1}$. It is technically easier to estimate λ_{min} using mixed boundary conditions, which will not, however, change the scaling behavior of λ_{min}: At $i=0$ we assume an adsorbing wall as before, whereas at $i=L$ we impose a reflecting boundary by setting formally $w_{L,L+1}=0$. Now, due to different symmetry of the problem there is only one zero mode of the FP operator and the second smallest eigenvalue in modulus will be λ_{min}. To determine λ_{min} we use a perturbational method. First, we express the eigenvalue problem in Eq. (6) as

$$-\tilde{u}(i)w_{i,i-1}+\tilde{u}(i+1)w_{i,i+1}=-u(i)\lambda_{min}, \qquad (12)$$

where $w_{L,L+1}=0$ and $\tilde{u}(i)=u(i)-u(i-1)$, $i=1,2,\ldots,L$, in terms of the components of the left eigenvector $\underline{u}\equiv\underline{u}_{min}$ and $u(0)=0$. Then, keeping in mind that we are interested in situations when $\lambda_{min}(L)$ is a rapidly vanishing function of L, we neglect the right-hand side of Eq. (12) and derive an approximate expression for the left eigenvector from the first $L-1$ equations of Eq. (12). Using this result we obtain an estimate for λ_{min} from the last equation of Eq. (12):

$$\lambda_{min}\simeq\frac{\tilde{u}(L)}{u(L)}w_{L,L-1}=\frac{u(1)}{u(L)}w_{L,L-1}\prod_{j=1}^{L-1}\frac{w_{j,j-1}}{w_{j,j+1}}, \qquad (13)$$

which can be transformed into the final form by noticing that $u(1)/u(L)=p_{pr}(L)$ in Eq. (9):

$$\lambda_{min}\sim p_{pr}(L)w_{L,L-1}\prod_{j=1}^{L-1}\frac{w_{j,j-1}}{w_{j,j+1}}. \qquad (14)$$

The scaling properties of λ_{min} in Eq. (14) as well as the persistence probability $p_{pr}(L)$ in Eq. (9) now can easily be derived by establishing a correspondence of these quantities with the energy gap and surface magnetization, respectively, of the random transverse-field Ising model in one dimension defined by the Hamiltonian

$$H=-\sum_{i=1}^{L-1}J_i\sigma_i^x\sigma_{i+1}^x-\sum_{i=1}^L h_i\sigma_i^z. \qquad (15)$$

Here the σ_i^x, σ_i^z are Pauli matrices at site i and the J_i exchange couplings and the h_i transverse fields are random variables. The RTIM in Eq. (15) has received much attention recently [12–14,3,15,7] and as was shown in [7] there are simple expressions for the surface magnetization $m_s(L)$ as well as for the gap of the Hamiltonian $\Delta(L)$ of the RTIM in terms of the couplings and fields. Comparing those with our results in Eqs. (9) and (14), we can set up the correspondences

$$w_{i,i+1} \rightarrow J_i^2,$$

$$w_{i,i-1} \rightarrow h_i^2,$$

$$\delta_{RW} \rightarrow \delta, \qquad (16)$$

$$p_{pr}(L) \rightarrow m_s^2(L),$$

$$\lambda_{min}(L) \rightarrow \Delta^2(L),$$

$$\mu(\delta_{RW}) \rightarrow 1(2z(\delta)).$$

Consequently, similar relations hold for the average quantities when the transition probabilities (or, equivalently, the fields and the couplings) follow the same random modulation. In the following we use the correspondences in Eq. (16) to derive results.

(i) *At the critical point* with $\delta_{RW}=0$, which corresponds to Sinai's walk [1], the distribution of the leading eigenvalues of the FP operator is very broad and $\lambda_{min}(L)$ scales according to

$$\lambda_{min}(L) \sim \exp(-const \times L^{1/2}), \quad \delta_{RW}=0, \qquad (17)$$

similarly to the analogous result for the energy gap of the RTIM [7]. Note that this scaling relation (17), which is consistent with the known relation between relevant time and length scales [1], $L \sim (\ln t)^2$, can be most easily demonstrated by considering the probabilty distribution $P_L(\ln \lambda_{min})$, which is then expected to scale like

$$P_L(\ln \lambda_{min}) \sim L^{-1/2}\widetilde{p}(\ln \lambda_{min}/L^{1/2}), \qquad (18)$$

as we confirmed numerically.

(ii) The scaling behavior of the persistence probability (for zero drift $\delta_{RW}=0$) in Eq. (9) follows also directly from the analogous result for the surface magnetization $[m_s(L)]_{av}$ of the RTIM [7]. Here we just have to mention that $m_s(L)$ at the critical point is not self-averaging and its average value is dominated by the *rare events*, which are of $O(1)$. From this it follows that the same rare events determine the average of $m_s^2(L)$, thus the scaling behaviors of $[m_s(L)]_{av}$ and $[m_s^2(L)]_{av}$ are identical. Then, using the correspondences in Eq. (16), we have the exact result

$$[p_{pr}(L)]_{av} \propto L^{-\theta}, \quad \theta=1/2, \quad \delta_{RW}=0. \qquad (19)$$

(iii) In the noncritical situation with $\delta_{RW} \neq 0$ there is an average drift of the walk towards the site $i=L+1$ ($i=0$) for $\delta_{RW}<0$ ($\delta_{RW}>0$). In the latter case, analogously to the
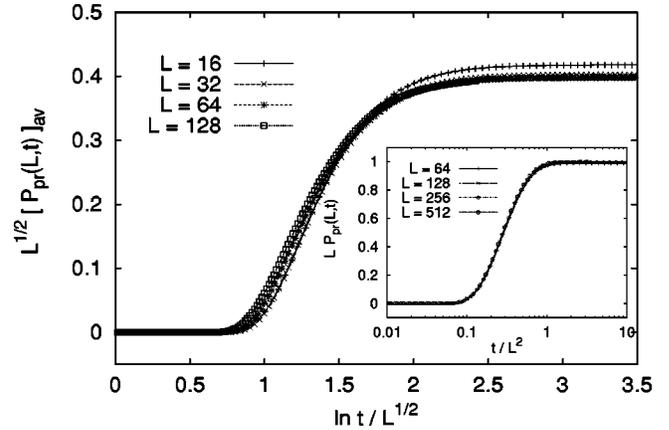
FIG. 1. Scaling plot of the time-dependent survival probability $P_{pr}(L,t)$ according to Eq. (23) for the asymmetric hopping model with a uniform distribution of hopping rates (averaged over 10^5 samples). The inset shows the corresponding scaling plot for the homogeneous case.

surface magnetization of the RTIM [7], the average persistence probability vanishes exponentially for large system sizes

$$[p_{pr}(L)]_{av} \sim \exp(-L/\xi), \quad \xi \sim \delta_{RW}^{-2}, \quad \delta_{RW}>0. \qquad (20)$$

If the average drift of the walk is towards the adsorbing site at $i=L+1$ and thus $\delta_{RW}<0$, there is a nonvanishing infinite system size limit of the persistence probability (similarly to the existence of a finite average surface magnetization of the RTIM [7])

$$\lim_{L\to\infty} [p_{pr}(L)]_{av} \sim (-\delta_{RW})^{\beta_{pr}}, \quad \delta_{RW}<0, \qquad (21)$$

with $\beta_{pr}=1$, which is approached via an exponential size dependence. The corresponding correlation length is again given by $\xi \sim (-\delta_{RW})^{-2}$, similarly to the case $\delta_{RW}>0$ in Eq. (20). Thus we can conclude that correlations defined on persistent walks are characterized by the average critical exponents

$$\theta=1/2, \quad \nu=2, \quad \beta_{pr}=1, \qquad (22)$$

which satisfy the scaling relation $\beta_{pr}=\theta\nu$.

(iv) The time-dependent persistence probability $P_{pr}(L,t)$ introduced above is simply given by $P_{pr}(L,t)=P_{1,L+1}(t) = \Sigma_k u_k(1)v_k(L+1)\exp(\lambda_k t)$. In the random, asymmetric case one expects the scaling relation

$$[P_{pr}(L,\ln t)]_{av} = b^{-\theta}[P_{pr}(L/b,\ln t/b^{1/2})]_{av} \qquad (23)$$

when lengths are rescaled by a factor $b>1$ and the relation in Eqs. (1) or (17) between time and length scales is used. Now taking $b=L$ in the limit $t\to\infty$, we recover the exact result in Eq. (19); on the other hand, with $b=(\ln t)^2$ we have in the large system limit an ultraslow decay $\lim_{L\to\infty}[P_{pr}(L,t)]_{av} \sim (\ln t)^{-1}$.

In the intermediate situation with $\ln t \sim L^{1/2}$ we have (with $b=L$) the finite size scaling form $[P_{pr}(L,t)]_{av} \sim L^{-1/2}\bar{p}(\ln t/L^{1/2})$, with $\lim_{y\to\infty}\bar{p}(y)=const$, in contrast to $P_{pr}^{hom}(L,t) \sim L^{-1}\bar{\bar{p}}(t/L^2)$ in the homogeneous case. In Fig. 1

we show the corresponding scaling plot for numerically generated results for finite systems that confirm this scaling picture.

(v) *Away from the critical point* we reach the region of anomalous diffusion, which is equivalent to the Griffiths-McCoy phase of the RTIM. The relevant energy scale $\lambda_{\min}(L)$ [$\Delta(L)$ for the RTIM] has a power-law scaling behavior

$$\lambda_{\min}(L) \sim L^{-1\mu(\delta_{\mathrm{RW}})}, \quad \Delta \sim L^{-z(\delta)} \qquad (24)$$

and according to Eq. (16) the two exponents μ and $1/z$ correspond to each other. At this point we use the result that the value of μ is known exactly from the time dependence of the average displacement of the walk in Eq. (2) in the form [2]

$$\left[\left(\frac{w_{\rightarrow}}{w_{\leftarrow}} \right)^{\mu} \right]_{\mathrm{av}} = 1. \qquad (25)$$

Essentially this follows from the observation that for any independent identically distributed random variables x the distribution $P(\lambda)$ of $\lambda = x_1 x_2 x_3 \cdots$, which is reminiscent of Eq. (14), has an algebraic singularity at $\lambda = 0$, $P(\lambda) \propto \lambda^{-1+\mu}$ with μ given by $[x^{\mu}]_{\mathrm{av}} = 1$; see [16].

Consequently, we obtain for the dynamical exponent z of the RTIM in the Griffiths-McCoy phase the implicit equation

$$\left[\left(\frac{J}{h} \right)^{1/z} \right]_{\mathrm{av}} = 1. \qquad (26)$$

Note that for any distribution of J and h one obtains immediately the result $1/z = 2\delta + O(\delta^2)$, with $\delta \ll 1$ as in Eq. (11), as has been observed earlier [12,13,7]. However, the exact result (26) is not restricted to $\delta \ll 1$, but is valid in the whole Griffiths-McCoy region. For example, for the uniform distribution

$$\pi(J) = \Theta(1-J)\Theta(J), \quad \rho(h) = h_0^{-1}\Theta(h_0-h)\Theta(h),$$

the dynamical exponent is given by the solution of the equation

$$z \ln(1-z^{-2}) = -\ln h_0 \quad (=-2\delta). \qquad (27)$$

The relation (27) is indeed satisfied by the numerical estimates for z reported in [13,15,7].

To summarize, in this paper we have revealed a fundamental relation between the anomalous diffusion of random walks in disordered environments and the slow dynamics, at criticality and in the Griffiths-McCoy region of the random transverse Ising chain. With this analogy at hand we were able to derive a number of exact results for both systems. Many applications of the above-mentioned analogy are obvious: There is an enormous number of *exact* results for various quantities of random walks in random one-dimensional environments and most probably many of them can be directly transferred to corresponding quantities of random quantum spin chains near the quantum critical point. It remains a subject of future research to study how far these relations carry over to higher dimensions.

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