Magnetic mirroring and cosmic ray pitch-angle diffusion

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Both quasilinear and nonlinear theories of cosmic ray scattering in fluctuating electric and magnetic fields are usually based on (renormalized) first-order perturbation theories, resulting in scattering parameters that depend functionally only on the variance and two-point correlation functions of the random fields. I show that these theories must fail to predict scattering of particles with pitch angles close to 90° since mirroring effects are not determined by these statistical quantities. This is shown by a comparison of Monte Carlo simulations in fields with the same covariance functions, but differing in the variability of the absolute value of the total magnetic field. Additionally, I report some numerical evidence that mirroring can be nondiffusive in some cases. [S1063-651X(98)14109-0]

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I. INTRODUCTION

It is well known that cosmic rays are scattered efficiently by the partially random electric and magnetic fields in the interplanetary and interstellar medium. Due to the comparatively small electric components of these fields, the fastest process is pitch-angle scattering, frequently described as a diffusive process in the quantity $\mu=\cos\phi$, where the pitch angle ϕ is the angle between the magnetic field and the particle's momentum vector. An important but still uncompleted task is to derive the cosmic ray transport parameters from the statistical properties of the random component of the fields. Usually, this is done perturbatively, decomposing the magnetic field into an ordered (e.g., homogeneous) component $\mathbf{B}_0$ and a fluctuating part $\delta\mathbf{B}$.

The simplest approach is quasilinear theory (QLT) [1], which describes pitch-angle diffusion as scattering by fluctuations in resonance with the helical motion the particles would follow in the ordered field. However, as noted by Wibberenz *et al.* [2], at least if one assumes a very simple geometry for $\delta\mathbf{B}$, the QLT predicts a considerably smaller mean free path for cosmic rays than is observed in the interplanetary medium. Many attempts [3–6] have been made in order to resolve this discrepancy, most of them emphasizing effects that are important in the region $\mu\approx 0$ where the QLT is inconsistent at least for static or slowly varying fields.

In particular, some authors introduced nonlinear modifications to the QLT taking magnetic mirroring into account [7–9]. A common feature of these theories and QLT is the prediction that parameters such as the pitch-angle diffusion coefficient depends on the covariance functions $\langle \delta B_i(\mathbf{x})\delta B_j(\mathbf{x}')\rangle$ only, where the angular brackets denote the ensemble average of the quantity enclosed.

Goldstein [10] questioned the validity of these approaches: Mirroring results from fluctuations in the field modulus $|\mathbf{B}|$ and adiabatic invariance of the magnetic moment. However, the fluctuations of $|\mathbf{B}|$ observed in the solar wind are much smaller than the values used in nonlinear theories and Monte Carlo simulations.

In this work I show that these mirroring effects cannot be adequately described by either quasilinear or nonlinear theories (including the ones cited above), which result in equations depending only on covariance functions $\Delta_{lm}(\mathbf{x},\mathbf{x}')=\langle \delta B_l(\mathbf{x})\delta B_m(\mathbf{x}')\rangle$ of the fluctuating fields. This is demonstrated by a comparison of particle propagation in three different types of random fields with the same covariance function, two of them with varying $|\mathbf{B}|$ and one where $|\mathbf{B}|$ is a constant in space. Pitch-angle diffusion close to $\mu=0$ is considerably reduced in the latter case.

Therefore, the covariance functions or, equivalently, correlation functions or power spectra provide only insufficient information about pitch-angle scattering. Additionally, I report some evidence for nondiffusive behavior in cases in which mirroring occurs.

II. NUMERICAL SIMULATIONS

In order to determine the pitch-angle diffusion coefficient $D_{\mu\mu}(\mu)$ by means of Monte Carlo simulations, I follow closely the approach of Kaiser, Birmingham, and Jones (KBJ) [11]. An ensemble of random fields, varying only in the z direction, is generated on a grid and the propagation of a number of particles is followed numerically by integration of the equations of motion. The particles are released at a starting value $\mu(t=0)=\mu_S$ and ''absorbed'' as soon as they reach a right or left boundary value μ_L or μ_R with $\mu_L<\mu_S<\mu_R$. If the propagation of an ensemble of particles is properly described by a Fokker-Planck equation, the evolution of the space- and gyrophase-averaged distribution function $f(\mu,t)$ should be governed by the diffusion equation

$$\frac{\partial}{\partial t}f(\mu,t)=\frac{\partial}{\partial\mu}\left(D_{\mu\mu}\frac{\partial}{\partial\mu}f(\mu,t)\right), \tag{1}$$

with the initial condition

$$f(\mu,0)=N\delta(\mu-\mu_S), \tag{2}$$

where N is the number of particles released. The time-integrated distribution function $F(\mu)=\int_0^\infty f(\mu,t)\,dt$ is finite, at least if the motion is really diffusive, since then all particles will have been absorbed sooner or later at the boundaries. Time integrating Eq. (1), one obtains

$$\frac{d}{d\mu}\left(D_{\mu\mu}\frac{dF(\mu)}{d\mu}\right)=-N\delta(\mu-\mu_S) \tag{3}$$

and hence

$$D_{\mu\mu}\frac{dF(\mu)}{d\mu}=\begin{cases}N_L\equiv\text{const}, & \mu<\mu_S\\ -N_R\equiv\text{const}, & \mu>\mu_S.\end{cases} \tag{4}$$

The constants N_L and N_R can be shown to be equal to the total number of particles absorbed at the left and the right boundary, respectively. From this equation, $D_{\mu\mu}$ can be determined by measuring, e.g., $dF/d\mu$ and N_L for an ensemble of realizations.

KBJ generated random fields $\delta\mathbf{B}$ with vanishing y and z components and an exponentially decaying correlation function $\Delta_{xx}(z-z'):=\langle\delta B_x(z)\delta B_x(z')\rangle$ characterized by a correlation length z_c,

$$\Delta_{xx}(z-z')=\langle\delta B^2\rangle\exp(-|z-z'|/z_c), \tag{5}$$

by simulating the following Gaussian Markov process on a grid. First, $\delta B_x(0)$ is drawn from a Gaussian distribution with mean zero and variance $\langle\delta B_x^2\rangle$. Then, knowing $\delta B_x(nh)$ at the grid point n, where h is the grid size, $\delta B_x[(n+1)h]$ at the next grid point is drawn from a Gaussian distribution with mean $\delta B_x(nh)\exp(-|h|/z_c)$ and variance $\delta B^2[1-\exp(-2|h|/z_c)]$ iteratively.

In order to simulate fields with exponentially decaying covariance functions that conserve the quantity $|\mathbf{B}_0+\delta\mathbf{B}|$ one needs two nonvanishing components. In contrast to KBJ's method, we construct fluctuating fields with $|B|\equiv\text{const}$ using the following algorithm. Introduce a phase angle ϕ on a spatial grid along the z axis with grid size h (draw ϕ_0 from a uniform random distribution from the interval $[0,2\pi)$) and compute ϕ_{i+1} on the grid point $i+1$ from the value on point i by $\phi_{i+1}=\phi_i+\Delta\phi_i$, where $\Delta\phi_i$ is drawn from a normal distribution with zero mean and variance $\sqrt{2h/z_c}$. If one now sets $\mathbf{B}_0=B_0\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector in the z direction and

$$\delta B_x(z)=\delta B\cos[\phi(z)],$$

$$\delta B_y(z)=\delta B\sin[\phi(z)], \tag{6}$$

and $\delta B_z(z)=0$, one obtains for the (ensemble-averaged) covariance functions

$$\Delta_{xx}(z-z')=\Delta_{yy}(z-z')=\frac{\delta B^2}{2}\exp\left(-\frac{|z-z_c|}{z_c}\right) \tag{7}$$

and $\Delta_{xy}(z-z')=0$ on the grid. The fields at points not on the grid are determined via linear interpolation of ϕ. These fields fulfill $|\mathbf{B}|=\sqrt{B_0^2+\delta B^2}\equiv\text{const}$ and $\nabla\cdot\mathbf{B}=0$. Note that the process defined by Eq. (6) is not Gaussian since δB_x and δB_y are bounded.

The fields investigated in this paper are constructed with either this algorithm (case A), or KBJ's algorithm (case C). In order to investigate the importance of fluctuations of the field modulus, I also performed calculations with the algorithm of case A, but drawing δB_x and δB_y from independent realizations (case B). This leaves the two-point correlation functions unchanged, but now $|\mathbf{B}|$ is no longer a constant such that (as in case C) mirroring effects should be expected. The angle Ψ between the mean field component $\mathbf{B}_0$ and the z axis was allowed to vary in the y-z plane. $\Psi=0$ corresponds to the standard slab model. For the case $\Psi\neq 0$, δB_y was set to zero such that $\delta\mathbf{B}\cdot\mathbf{B}_0=0$ and fluctuations of the total field modulus were only of second order in δB.

III. RESULTS

A. Nondiffusive behavior

For a comparison with classical results, some simulations were performed using KBJ's method to generate the fields on the grid. The results were in agreement with those published by KBJ.

However, for the study of the diffusive process it is useful to deviate slightly from this reference by defining the pitch angle as the angle between the momentum and the *local* magnetic field $\mathbf{B}(\mathbf{x})$ instead of the mean field $\mathbf{B}_0$. The reason is that we are mainly interested in the scattering through pitch angles of 90°: A (not too energetic) particle will move only a small fraction of one correlation length of the random field and therefore have no chance to perform any kind of averaging over $\mathbf{B}$. To lowest order, it performs a gyration around the local, not the mean, magnetic field. It is easy to see that this gyration in the local field corresponds to an oscillation of the pitch angle cosine μ with an amplitude of order $\delta B/B_0$ if one uses the standard definition. This behavior cannot be described by a diffusive process and, as the time-integrated distribution function cannot distinguish whether a particle is changing its pitch angle due to diffusion or this oscillation, it leads to undesired artificial contributions to the pitch-angle diffusion coefficient in our numerical method. Moreover, our local definition is closer to the frequently used concept of diffusion along a field line. Though the Fokker-Planck equation has originally been derived by QLT for the standard definition, I will therefore assume (and test) its validity for the locally defined pitch-angle. Both definitions coincide for $\delta B\to 0$. Note, however, that the results reported in this paper were changed only marginally if one used the standard definition and the local one was chosen only to eliminate the oscillatory component. A more detailed comparison shall be presented elsewhere. Figure 1 shows the time-integrated distribution function $F(\mu)$ for cases A and C, where $\delta B/B_0=0.1$, $\Omega z_c/v=1$ and $\Psi=0$, and case A was calculated using both the local and the standard definition of the pitch angle. It can be seen that it is considerably more difficult for the particles to obtain negative pitch angles in case A, where mirroring effects are minimal, than in case C with fields generated by KBJ's method. Due to the oscillatory artifacts, the transition region around $\mu=0$ is smeared out over an interval about $[-\delta B/B_0,\delta B/B_0]$ if one uses the standard definition of the pitch angle.

As noted above, QLT is inconsistent in the vicinity of $\mu=0$: On the one hand, it is assumed that the fields ''seen'' by the particles are uncorrelated; on the other hand, the particles perform to lowest order a periodic and localized motion giving rise to highly correlated fields. Therefore, though it is well known that there may be stochastic and diffusive mo-

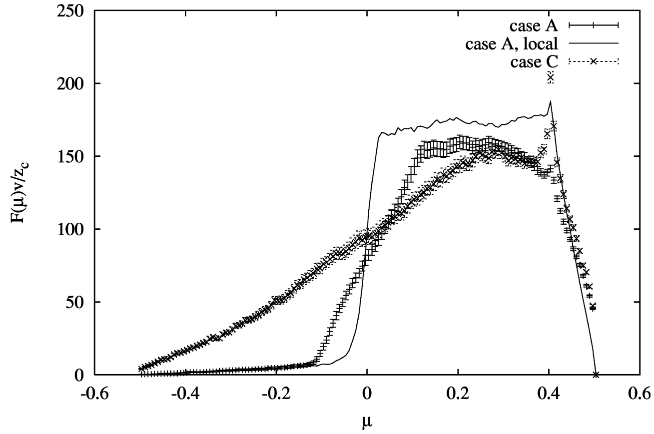

FIG. 1. Time-integrated distribution function $F(\mu)$ for case A (nonmirroring) and case C (mirroring), using the standard definition of the pitch angle. Also shown (full line) is case A with a locally defined pitch angle, free from the artificial contributions of oscillations in the local field close to $\mu=0$ and, therefore, showing a sharper transition in this region.

tion in deterministic fields (e.g., see, [12]), one may ask whether scattering through $\mu=0$ can be properly described as a diffusive process. The answer, of course, depends on the detailed formulation of the question, as it is clear that the short time behavior of the distribution is deterministic and the Fokker-Planck equation cannot resolve the time evolution over intervals smaller than the relaxation time of the particles. I present a case where the diffusion equation (1) leads to an inconsistent interpretation of the time-integrated distribution function.

The simulations in a standard slab geometry of the fields showed some evidence for a slight asymmetry of $D_{\mu\mu}$ around $\mu=0$ that might indicate nondiffusive behavior in this region. This becomes very prominent if the mean field is not parallel, but inclined with an angle Ψ with respect to the z axis (i.e., in an oblique slab model; in order to minimize mirroring I chose a random field with only one nonvanishing field component fulfilling $\delta\mathbf{B}\perp\mathbf{B}_0$): Fig. 2 shows $F(\mu)$ for

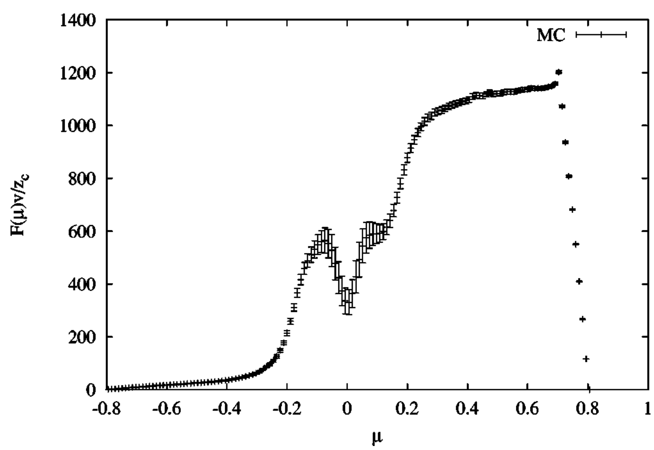

FIG. 2. Time-integrated distribution function $F(\mu)$ from Monte Carlo simulations for propagation in fields decribed by an oblique slab model with angle $\Psi=0.6, \Omega z_c/v=4$, and $\delta B/B_0=0.1$. F averaged over an ensemble of $R=8000$ realizations with $N=10$ particles.

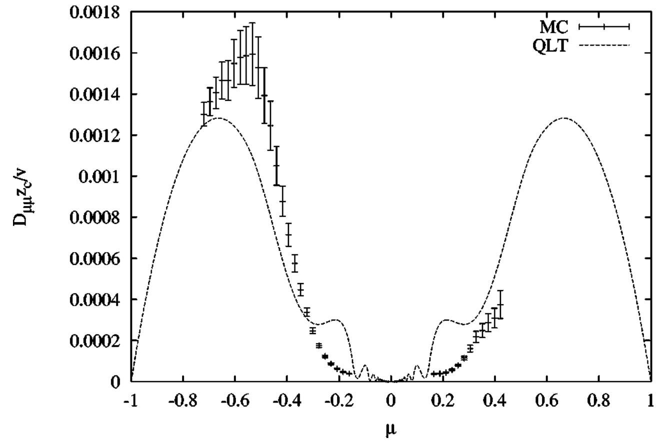

FIG. 3. Dependence of $D_{\mu\mu}$ in the numerical experiments on μ for propagation in the fields described in Fig. 2. MC, results of Monte Carlo simulations; QLT, prediction of the quasilinear theory. The region where $F'(\mu)<0$, i.e., where $D_{\mu\mu}$ would be negative, is excluded.

an ensemble with $\delta B/B_0=0.1$, $\Omega_0 z_c/v=4$, and $\Psi=0.6$, where $\Omega_0=eB_0/mc$ is the gyrofrequency in the mean field. According to Eq. (4), $F(\mu)$ should be a monotonically increasing function for $\mu<\mu_S$. However, it is decreasing in a region close to $\mu=0$, which would indicate a negative diffusion coefficient $D_{\mu\mu}$ if the process were really diffusive. As this is clearly unphysical, the description by the Fokker-Planck equation seems to be an inadequate approximation in this region. The shape of $F(\mu)$ at $\mu\approx 0$ remained almost unchanged when the initial value μ_S was moved from $\mu_S=0.7$ to $\mu_S=0.4$. Therefore, it is not likely that the reason for the inversion is an incomplete relaxation of the initial distribution.

For larger values of $|\mu|$, however, the QLT prediction was in reasonably good agreement with the outcome of the Monte Carlo simulations (Fig. 3). For the rest of the paper I will focus on the standard slab model with $\Psi=0$.

B. Mirroring and diffusion

According to QLT and its nonlinear modifications, $D_{\mu\mu}$ is a functional of the covariance functions $\Delta_{mn}(\mathbf{x})$, where $m,n\in\{x,y,z\}$. I now compare the propagation in fields constructed according to Eq. (6) (case A) with fields that do not conserve $|\mathbf{B}|$, but have the same covariance structure. These can be generated with the same method, but taking δB_x and δB_y from two independent realizations (case B).

For the slab-model discussed here, where $\delta\mathbf{B}$ varies only in the direction of the mean field (i.e., $\phi=0$), the functional dependence of $D_{\mu\mu}$ can be reduced to a dependence on the covariance function of the two circular polarization components $\Delta_{RR}(z)$ and $\Delta_{LL}(z)$, where $\delta B_{L,R}:=(\delta B_x \pm i\delta B_y)/\sqrt{2}$. For cases A and B as well as the fields simulated by KBJ, however, these are equal: $\Delta_{RR}(z)=\Delta_{LL}(z)=\delta B^2\exp(-|z|/z_c)/2$. Therefore a simulation in fields according to KBJ is also included in the comparison (case C).

Figure 4 shows exemplarily the results of simulations with $\delta B/B_0=0.1$ and $\Omega_0 z_c/v=1$. Close to $\mu=0$ the results of cases B and C are not equal, but of the same order of magnitude. Case A, however, where $|\mathbf{B}|$ is a constant and

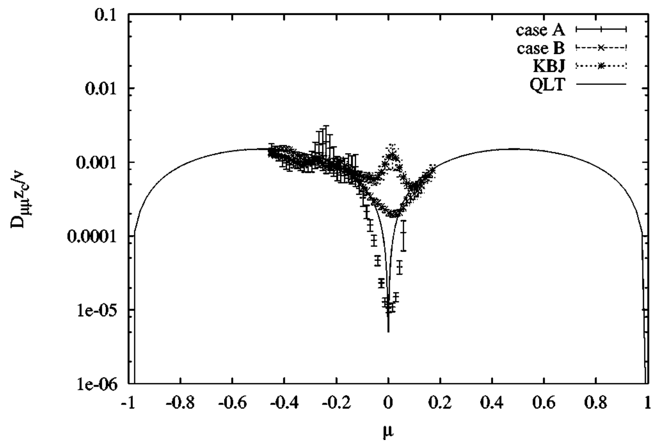

FIG. 4. Dependence of $D_{\mu\mu}$ on μ for propagation in fields with exponentially decaying correlation function, slab model with $\Omega z_c/v=1$, and $\delta B/B_0=0.1$. Case A, fields with constant $|\mathbf{B}|$ according to Eq. (6); case B, fields with nonconstant $|\mathbf{B}|$ with x and y components drawn from different realizations. KBJ, fields generated by the process described by KBJ; QLT, prediction of the quasilinear theory.

scattering by magnetic mirroring should be minimal, yields a pitch-angle diffusion coefficient that is considerably (about two orders of magnitudes) smaller than $D_{\mu\mu}$ in case C.

Results of simulations with different parameter sets showed that this effect is less pronounced for particles with higher rigidity (i.e., if $\Omega_0 z_c/v<1$) and more pronounced for particles with lower rigidity. This should be expected since only for sufficiently low velocities the fluctuations ''seen'' by the particles can be regarded as slowly varying such that the magnetic moment is an adiabatic invariant and mirroring can occur.

IV. CONCLUSIONS

I have shown that knowledge of the covariance functions or two-point correlation functions or, equivalently, of the power spectra of the random magnetic fields is, unlike assumed in most studies based on quasilinear theory and its nonlinear extensions, not sufficient to predict the transport properties of cosmic rays. This was demonstrated by simulating the propagation in random fields with identical covariance functions but differing in the variability of the total field modulus. As the latter quantity determines the relative importance of mirroring, the resulting pitch angle diffusion coefficient at pitch-angles about 90° could vary by orders of magnitude without changing the correlation functions.

Though there is evidence for nondiffusive behavior in some cases, a reinterpretation of the distribution functions in terms of modified theories based on, e.g., a more general master equation instead of the Fokker-Planck equation would not help as long as they use only information about the covariance functions Δ_{lm}. This is demonstrated in Fig. 1, showing the time-integrated distribution functions $F(\mu)$ for propagation in mirroring and nonmirroring fields: Obviously, the total number of particles with $\mu<0$ is considerably reduced in the latter case where scattering through $\mu=0$ is clearly hampered. If the dynamics would be specified completely by the covariance functions, both distributions would be identical or, if one takes a finite relaxation time into account, very similar.

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