Effects of noise on the phase dynamics of nonlinear oscillators

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(Received 15 December 1997; revised manuscript received 23 March 1998)

Various properties of human rhythmic movements have been successfully modeled using nonlinear oscillators. However, despite some extensions towards stochastical differential equations, these models do not comprise different statistical features that can be explained by nondynamical statistics. For instance, one observes certain lag one serial correlation functions for consecutive periods during periodic motion. This work aims at an extension of dynamical descriptions in terms of stochastically forced nonlinear oscillators such as $\ddot{\xi}+\omega_0^2\xi=n(\xi,\dot{\xi})+q(\xi,\dot{\xi})\Psi(t)$, where the nonlinear function $n(\xi,\dot{\xi})$ generates a limit cycle and $\Psi(t)$ denotes colored noise that is multiplied via $q(\xi,\dot{\xi})$. Nonlinear self-excited systems have been frequently investigated, particularly emphasizing stability properties and amplitude evolution. Thus, one can focus on the effects of noise on the frequency or phase dynamics that can be analyzed by use of time-dependent Fokker-Planck equations. It can be shown that noise multiplied via polynoms of arbitrary finite order cannot generate the desired period correlation but predominantly results in phase diffusion. The system is extended in terms of forced oscillators in order to find a minimal model producing the required error correction. [S1063-651X(98)09907-3]

PACS number(s): 05.40.+j, 87.10.+e

I. INTRODUCTION

Nonlinear oscillators have been frequently discussed in various fields, and their mathematical investigation has a rather long tradition. Depending on the explicit context one concentrates primarily on specific types of oscillators such as Van der Pol, Duffing, or Helmholtz oscillators. In the present paper we consider the case of randomly forced systems that, especially in case of self-excited systems, have been successively investigated since the early 1940s [1–3]. Most related works preferentially stress stability properties of the amplitude, or rather their changes under the impact of noise (for recent studies see, e.g., [4–9]). Thus, we concentrate on the effects of noise on frequency or phase dynamics that we analyze by using Fokker-Planck equations and Krylov-Bogoliubov approximations. In physics there exist several instances of nonlinear oscillators subjected to random excitation. Typically, such systems belong to the realm of macroscopic phenomena, since nonlinearities and statistics are involved. For example, in nonlinear optics we find extensive discussions about the influence of noise on optical multistability (e.g., [10–12]). One may also think of the large field to which the generalized (complex) Ginzburg-Landau equation applies (e.g., [13]).

Here, however, we focus on a fairly different system, namely, human movement. Several properties of rhythmical coordination patterns have been prosperously modeled in terms of nonlinear oscillators and many studies concentrated on stability of movement and its externally induced changes by means of phase transitions [14–17]. In the present paper we use related models as starting point for our investigation but emphasize that the entire discussion is by no means restricted to this somewhat specialized application of stochastically forced oscillators. Strictly speaking, different statistical features observed in rhythmic movements motivated the following work since, despite some extensions towards stochastical differential equations [18], they are not yet acommodated by dynamical models. For instance, one finds certain correlation functions for consecutive periods during rhythmic motion whenever a subject tries to voluntarily continue a periodic movement that was previously paced by a metronome [19]. Even without the external stimulus, the frequency of motion remains rather constant, and any errors are corrected immediately within the first subsequent period. Indeed, such a negative lag one correlation can be explained by nondynamical statistics [20], and the question arises, What kind of (lag one) correlation function can be modeled using dynamical systems? Hence, we aim at an extension of a dynamical description of human movement by means of additional random impacts. We therefore study systems with the form $\ddot{\xi}+\omega_0^2\xi=n(\xi,\dot{\xi})+q(\xi,\dot{\xi})\Psi(t)$. The nonlinearity n generates a stable limit cycle attractor and $\Psi(t)$ denotes colored noise that is multiplied via a finite polynom $q(\xi,\dot{\xi})$. As shown below one can estimate that continuous noise alone cannot produce the sought correlation functions at a desired order of magnitude. Besides additive noise, which has been extensively discussed in the literature (see, for instance, [21] and references therein) these estimates include the case of noise that is multiplied by polynoms of any arbitrary finite order. In order to find those kind of correlations we finally extend the system in terms of forced oscillators.

Before we go into the problem of stochastically forced nonlinear oscillators, however, we roughly summarize a simplified statistical model that can explain the generation of a negative lag one correlation between consecutive periods. One commonly looks at a series of periods $\{T_i\}$, $i=1\ldots N$, where mean period and covariances are given by $\overline{T}:=(1/N)\Sigma_{i=1}^{N}T_i$ and $\overline{\sigma_T^2(k)}:=\overline{T_iT_{i-k}}-\overline{T}^2$. In the context of timing and error correction the so-called lag one serial correlation function $\mu_T(1)$ is of predominant interest. It is defined as $\mu_T(1):=\sigma_T^2(1)/\sigma_T^2(0)$, which for large N can be

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written as $\mu_T(1) \approx (\Sigma_i T_i T_{i-1} - [\Sigma_i T_i]^2)/(\Sigma_i T_i^2 - [\Sigma_i T_i]^2)$. Following Wing and Kristofferson [20], a negative correlation can be directly modeled if the evolution is viewed as the result of a periodic process C_i and a transfer delay D_i. Each period T_i can then be written as $T_i := C_i + D_i - D_{i-1}$. The quantities C_i and D_i are considered to be *statistically independent*. Further, they are Gaussian processes and, except for the mean and variance, all cumulants of higher order vanish. These simple assumptions already lead to the wanted properties because one instantly obtains for the covariance matrices $\sigma_T^2(0) = 2\sigma_D^2(0) + \sigma_C^2(0)$ and $\sigma_T^2(1) = -\sigma_D^2(0)$, respectively. Thus, the lag one serial correlation function becomes negative, $\mu_T(1) = -\sigma_D^2(0)/\{2\sigma_D^2(0) + \sigma_C^2(0)\}$ and bounded by means of $0 \geqslant \mu_T(1) \geqslant -0.5$.

Certainly, this statistical approach is a strong one in that its requirements are minimal and in that the introduction of an ''internal clock'' C_i and a ''motor delay'' D_i is consistent with (neuro)physiological aspects of the system, at least to some extent. On the other hand, it disregards dynamical and corresponding stability properties of periodic human movement. These characteristics, however, have been successfully described in terms of nonlinear oscillators (cf. [22] and references therein). In the following sections we therefore present approximations of period correlation functions in case the underlying dynamics is described by various types of nonlinear oscillators.

II. PERIOD ESTIMATES FOR NONLINEAR OSCILLATORS

Fundamental systems for the description of rhythmic movement are stable limit cycle oscillators. Besides their harmonic parts these oscillators typically contain weak nonlinearities in the form of lower order polynoms. Here, we consider the basic equation

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} n(x,y), \tag{1}$$

which is reformulated by using the Van der Pol transformation; that is, we use the polar coordinates $x = r\cos\theta$, $y = -\omega_0 r\sin\theta$, and rescale time by $\tau = \omega_0 t$. With $\tilde{n}(r,\theta) := n(r\cos\theta, -\omega_0 r\sin\theta)$ we rewrite (1) as

$$\frac{d}{d\tau}\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{\tilde{n}(r,\theta)}{r\omega_0^2}\begin{pmatrix} r\sin\theta \\ \cos\theta \end{pmatrix}. \tag{2}$$

As mentioned previously we choose the nonlinearity n or $\tilde{n}$, respectively, in such a way that the resulting evolution describes a limit cycle. The corresponding period T of such a dynamical system (1) can be defined as

$$T = \int_0^T dt = \int_0^{2\pi} \dot{\theta}^{-1} d\theta = \frac{1}{\omega_0}\int_0^{2\pi}\left[1 - \frac{\tilde{n}(r,\theta)\cos\theta}{r\omega_0^2}\right]^{-1} d\theta$$
$$= \frac{2\pi}{\omega_0} + \frac{1}{\omega_0}\sum_{p=1}^{\infty}\underbrace{\int_0^{2\pi}\left[-\frac{\tilde{n}(r,\theta)\cos\theta}{r\omega_0^2}\right]^p d\theta}_{\Delta_T^{(p)}}, \tag{3}$$

where $\dot{\theta}$ denotes the derivative with respect to t. According to Eq. (3) the harmonic period $2\pi/\omega_0$ is corrected by terms $\Delta_T^{(p)}$ that depend on $n(x,y)$. If we assume that the nonlinearity n is polynomial, that is, $n(x,y) \propto x^m y^n$, each integral in Eq. (3) becomes

$$\Delta_T^{(p)} = \int_0^{2\pi}\left[-\frac{(-1)^n \cos^{m+1}\theta\,\sin^n\theta}{r^{1-m-n}\omega_0^{2-n}}\right]^p d\theta. \tag{4}$$

In particular, the first order correction for Eq. (3) can be expressed as

$$\Delta_T^{(1)} = \int_0^{2\pi}\frac{(-1)^{n+1}\cos^{m+1}\theta\sin^n\theta}{r^{1-m-n}\omega_0^{2-n}}\,d\theta$$

$$\propto \begin{cases} r^{m+n-1}\omega_0^{n-2} & \text{for } m \text{ odd, } n \text{ even} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

It is worthwhile to remark that in Eq. (5) the latter proportionality is only correct for an entirely decoupled system, i.e., for $dr/d\theta \equiv 0$. Without a principle loss of generality we now concentrate on Rayleigh, Van der Pol, and Duffing oscillators. In detail we write the nonlinearity as

$$n(x,y) \equiv \omega_0\left(\alpha - \frac{\beta}{3\omega_0^2}y^2 - \gamma x^2\right)y - \frac{\omega_0^2\eta}{3}x^3, \tag{6}$$

and refer to $\beta\cdots$ as Rayleigh, $\gamma\cdots$ as Van der Pol, and $\eta\cdots$ as Duffing component. It is well known that a Rayleigh oscillator such as $\ddot{x} + x - \dot{x} + \overline{\beta}\dot{x}^3$ also describes a Van der Pol oscillator $\ddot{y} + y - \dot{y} + 3\overline{\beta}y^2\dot{y}$ for the corresponding velocity $y=\dot{x}$. Thus, for the period estimate (3), symmetry properties reveal that both Rayleigh and Van der Pol terms do not change the period length in the first order approximation (5), whereas the Duffing term ($\propto x^3$) relates frequency and amplitude scaling at r^2.

Of course, these rough period estimates are only reliable in the case of stable oscillations. Since we want to discuss the systems' response to external perturbations, one has to investigate the transient regime or, in other words, relaxations onto the limit cycle. For the sake of simplicity, however, we restrict ourselves to the immediate vicinity of the stable limit cycle. There, perturbations are assumed to be reasonably small and, as a first estimate, one might average system (1) over a period T (see, e.g., [23,24]). Hence, Eq. (2) becomes

$$\frac{d}{d\tau}\begin{pmatrix} r \\ \theta \end{pmatrix} \approx \begin{pmatrix} \overline{n_0}(r) \\ \overline{\psi_0}(r) \end{pmatrix}, \tag{7}$$

where we introduce the abbreviations $\kappa := \beta + \gamma$ and

$$\overline{n_0}(r) = -\frac{d\overline{V_0}}{dr},$$

$$\overline{V_0}(r) := -\frac{1}{4}\left\{ \alpha - \frac{\kappa}{8} r^2 \right\} r^2,$$

and

$$\overline{\psi_0}(r) := 1 + \frac{\eta}{8} r^2. \tag{8}$$

Amplitude and frequency dynamics decouple and consequently system (7) can be integrated explicitly. Note that the simplified Eq. (7) is a rather rough approximation unless the nonlinearitiers are chosen properly; that is, the amplitude is considered to be nearly constant over a "cycle" (slowly varying amplitude approximation); for the nonlinearities in Eq. (6) we therefore assume that $|\alpha| \approx |\beta/3| \approx |\gamma|$ and $|\eta/3| \leqslant |\alpha|$ holds (cf. [23,25]). Especially, in the case of $\alpha = \beta/3\omega_0^2 = \gamma \equiv 1$ and $\eta \equiv 0$ one obtains the exact solution $x = \sin \omega_0 t$. Coming back to the discussion of Eqs. (7) and (8), respectively, we get with $r_0^2 := 4\alpha/\kappa$

$$r^2(\tau) = \frac{4\alpha}{\kappa - e^{-\alpha(\tau - c_1)}} \Rightarrow \frac{d\theta}{d\tau} = 1 + \frac{\eta r_0^2}{8}\left[1 - \frac{e^{-\alpha(\tau - c_1)}}{\kappa} \right]^{-1}, \tag{9}$$

with an integration constant c_1 given by $\exp\{\alpha c_1\} := \kappa[1 - r_0^2/r(\tau=0)^2]$. As shown in Fig. 1, the frequency $\dot{\theta}$ simply relaxes exponentially to a steady value $\propto (1 + \eta r_0^2/8)$. The relaxation is given by the gradient dynamics of $r(\tau)$ or $r(t)$, respectively. Therefore, one can only expect a positive correlation between consecutive periods. These estimates, however, only hold within a rather close vicinity of the limit cycle and, for larger perturbations, the transient regime has to be investigated numerically.

As shown in Fig. 2(c) the Rayleigh component stabilizes the oscillator at a certain velocity and in this way perturbations of the period length are eliminated rather quickly. Conversely, the Van der Pol term mainly affects the stabilization of the amplitude and it allows for slight oscillations around the basic frequency during the relaxation onto the limit cycle [see Fig. 3(c)]. This effect can even be amplified by adding a Duffing component, as shown in Fig. 4. In the case of a reasonably large Duffing coefficient η rather negative lag one correlations can be observed. The amplitude dependency $\propto r^2$, however, might result in a loss of stability so that the Duffing term should be handled with care [25].

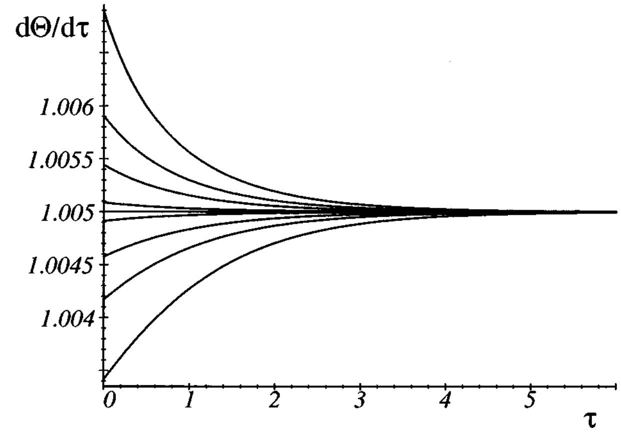

FIG. 1. Frequency relaxation. The solution $d\theta/d\tau$ given by Eq. (9) is plotted for different initial values $r(\tau=0)$; $\alpha = \beta/3 = \gamma = \eta/3 = \omega_0 = 1$.

III. STOCHASTICALLY FORCED OSCILLATORS

In real systems, perturbations occur continuously in an unpredictable fashion. We account for this by extending the nonlinear oscillator by means of external noise. Of course, we do not want to include any *a priori* knowledge concerning the sought (time-) correlation functions and, thus, we treat random dynamics in terms of Markov processes. In order to keep the deterministic properties of the oscillator $(x,y,\dots)^T = \boldsymbol{x}$, we write the system in form of a Langevin equation $\dot{\boldsymbol{\xi}}_x = \boldsymbol{N}(\boldsymbol{\xi}_x, t)$ where $\boldsymbol{\xi}_x$ denotes a random variable substituting $\boldsymbol{x}$. The nonlinear function $\boldsymbol{N}$ contains deterministic components resulting in stable oscillations as well as noise. The system is fully described by its time-dependent probability density $f(\boldsymbol{x},t)$, commonly defined as $f(\boldsymbol{x},t) := \langle \delta[\boldsymbol{x} - \boldsymbol{\xi}_x(t)] \rangle$. We compute $f(\boldsymbol{x},t)$ by integrating the corresponding Fokker-Planck equation $\dot{f}(\boldsymbol{x},t) = \mathcal{L}_{\mathrm{FP}} f(\boldsymbol{x},t)$, where $\mathcal{L}_{\mathrm{FP}}$ denotes the Fokker-Planck operator [21]. In fact, in this context we do not require a detailed intergration of the Fokker-Planck equation but rather look for *stochastically equivalent* systems; that is, (simpler) Langevin equations that obey an identical Fokker-Planck operator as the original dynamics.

A. Additive white noise

Aiming at stochastic extensions we start with the most simple case, namely, the addition of noise to our initial model (1). The additive white noise already allows for a basic understanding of various impacts of noise on periodic dynamics and, as we will see below, most phenomena of more complicated systems can be mapped onto this situation.

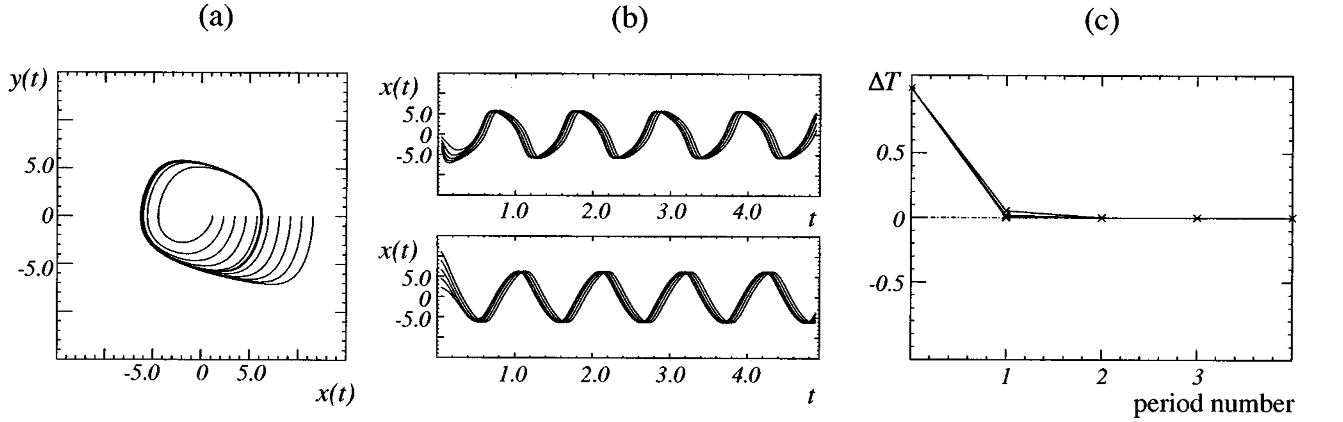

FIG. 2. Rayleigh oscillator. Equation (6) with $\alpha=\beta/3=1$ and $\gamma=\eta=0$ is integrated for several initial conditions $x(t=0)\in]0,2]$ and $\dot{x}(t=0)=0$. (a) shows the phase portrait $x(t)$ vs $y(t)$, (b) are the time series $x(t)$ and $y(t)$ each vs t, and (c) shows the deviation of the mean period length $\Delta T_i := T_i - \overline{T}$ vs the period number. The period length is implicitly defined as difference between consecutive roots, i.e., $x_t = 0 \wedge x_{t+T}=0$ where $\dot{x}_t$ and $\dot{x}_{t+T}$ must have the same sign. For sake of clarity, only the system's response on period increases is plotted. Note, that all quantities (including t and T) are considered to be dimensionless.

This special type of randomly excited oscillator has been discussed in the literature (cf., e.g., [21] and included references) but we recall it here because of its paradigmatic features. Explicitly, the nonlinear oscillator now reads

$$\frac{d}{dt}\begin{pmatrix}\xi_x\\ \xi_y\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\omega_0^2 & 0\end{pmatrix}\begin{pmatrix}\xi_x\\ \xi_y\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} n(\xi_x,\xi_y) + \omega_0^2\sqrt{2Q}\begin{pmatrix}0\\1\end{pmatrix}\Gamma(t), \tag{10}$$

where $\Gamma(t)$ is assumed to be δ-correlated Gaussian noise with vanishing mean $[\langle\Gamma(t)\rangle=0$ and $\langle\Gamma(t')\Gamma(t)\rangle=\delta(t'-t)]$. We achieve the corresponding Fokker-Planck operator as

$$\mathcal{L}_{\rm FP} = -y\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\left\{n(x,y) - \omega_0^2 x - \omega_0^4 Q\frac{\partial}{\partial y}\right\}. \tag{11}$$

The Van der Pol transformation is applied in order to distinguish between amplitude and frequency dynamics and Eq. (10) becomes a set of Stratonovich-Langevin equations

$$\frac{d}{d\tau}\begin{pmatrix}\xi_r\\ \xi_\theta\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} - \frac{1}{\xi_r}\left\{\frac{\tilde{n}(\xi_r,\xi_\theta)}{\omega_0^2} + \frac{\sqrt{2Q}}{\xi_r}\Gamma(\tau/\omega_0)\right\}\begin{pmatrix}\xi_r\sin\xi_\theta\\ \cos\xi_\theta\end{pmatrix}. \tag{12}$$

Recall that we have $\tilde{n}(r,\theta) := n(r\cos\theta, -\omega_0 r\sin\theta)$. Accordingly, the Fokker-Planck operator yields

$$\tilde{\mathcal{L}}_{\rm FP} = \frac{1}{\omega_0^2}\frac{\partial}{\partial r}\left\{\tilde{n}(r,\theta)\sin\theta - \frac{\omega_0^2 Q}{2r}(1+\cos 2\theta)\right\} + \frac{1}{r\omega_0^2}\frac{\partial}{\partial\theta}\left\{\tilde{n}(r,\theta)\cos\theta - r\omega_0^2 - \frac{\omega_0^2 Q}{r}\sin 2\theta\right\} + \frac{Q}{2}\left\{(1-\cos 2\theta)\frac{\partial^2}{\partial r^2} + (1+\cos 2\theta)\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}. \tag{13}$$

Analogous to the preceding section we insert the nonlinearities (6). Using abbreviations (8) we further average over a period which leads to

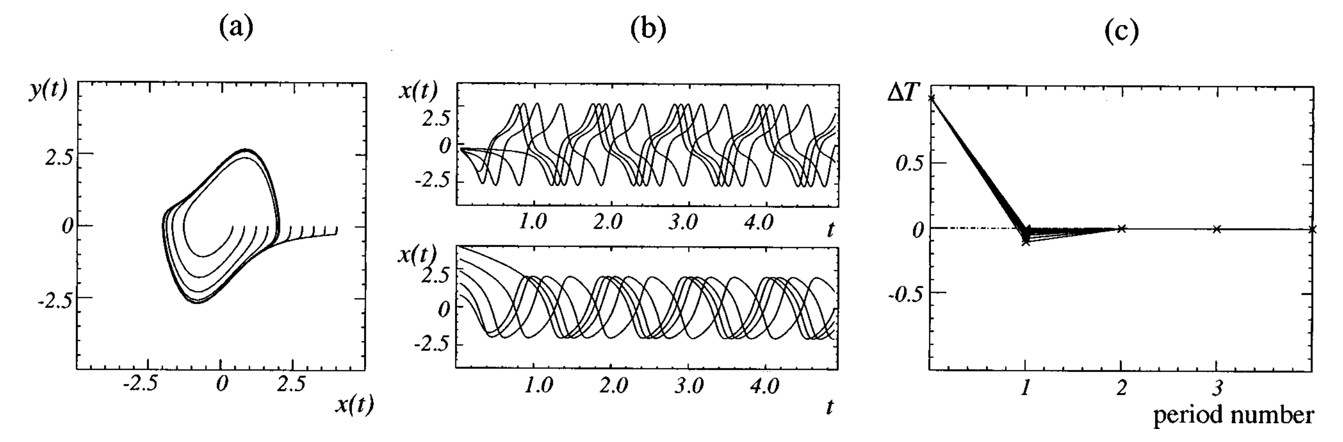

FIG. 3. Van der Pol oscillator. The system (6) with $\alpha=\gamma=1$ and $\beta=\eta=0$ is integrated for several initial conditions $x(t=0)\in]0,2]$ and $\dot{x}(t=0)=0$ (cf. Fig. 2). In contrast to the Rayleigh system, the Van der Pol oscillator shows a negative correlation between consecutive periods, since the relative period ΔT can become negative.

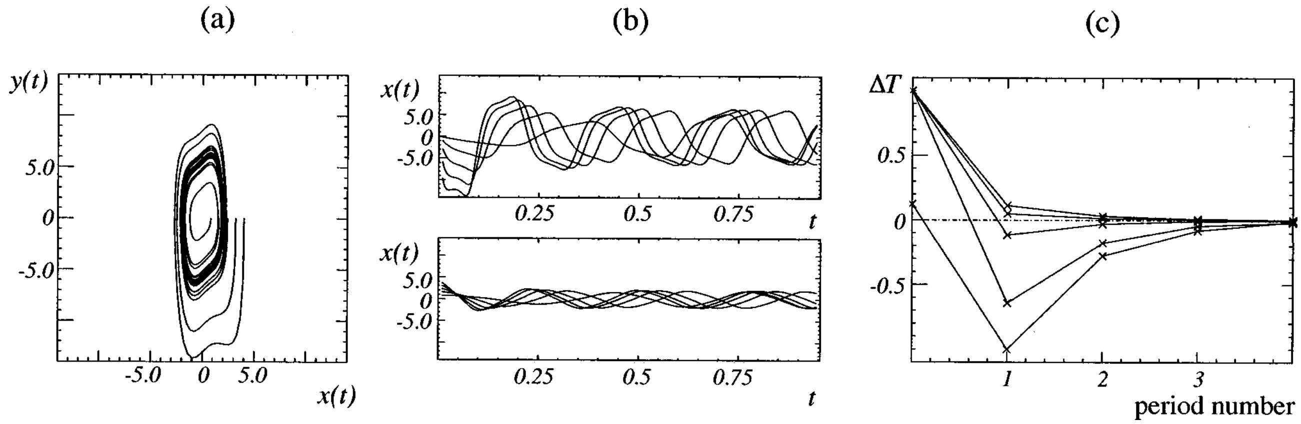

FIG. 4. Van der Pol-Duffing oscillator. Equation (6) with $\alpha=\gamma=1$, $\beta=0$ and $\eta/3=4$ is integrated for several initial conditions $x(t=0)\in\,]0,2]$ and $\dot{x}(t=0)=0$. Negative correlations are clearly shown in plot (c).

$$\tilde{\mathcal{L}}_{\rm FP}\approx\overline{\mathcal{L}}_{\rm FP}=-\frac{\partial}{\partial r}\left\{\overline{n}_0(r)+\frac{Q}{2r}\right\}-\overline{\psi}_0(r)\frac{\partial}{\partial\theta}+2\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}. \tag{14}$$

The averaging results in a decoupling of amplitude and frequency dynamics. Recall the previous discussion of the parameter values that guarantee the validity of the averaging and we further assume that Q is reasonably small. Thus for Eq. (12) we find a stochastically equivalent system with the form

$$\frac{d}{d\tau}\begin{pmatrix}\xi_r\\ \xi_\theta\end{pmatrix}=\begin{pmatrix}\overline{n}(\xi_r)\\ \overline{\psi}(\xi_r)\end{pmatrix}+\frac{\sqrt{Q}}{\xi_r}\begin{pmatrix}\xi_r\Gamma_r\\ \Gamma_\theta\end{pmatrix}. \tag{15}$$

In Eq. (15) we use Eq. (8) and we additionally abbreviate

$$\overline{n}(\xi_r)=-\frac{d\overline{V}}{d\xi_r},$$

$$\overline{V}(\xi_r):=\overline{V}_0(\xi_r)-\frac{Q}{2}\ln\xi_r,$$

and

$$\overline{\psi}(\xi_r):=\overline{\psi}_0(\xi_r)+\frac{Q}{2\xi_r^2}\ . \tag{16}$$

When we compare this form with the noiseless case we realize a diverging term $\propto\ln\xi_r$ which is added to the potential $\overline{V}_0$, resulting in a negligible probability to find the system at the origin (see Figs. 5 and 6, cf. [2,11,26,27], and see, for instance, [28,21,29] for details in the numerics of stochastic differential equations).

The terms Γ_r and Γ_θ in Eq. (15) are two independent (Gaussian) noise sources. With regard to the introduction of this paper one might be tempted to relate these two noise sources in some way to the two statistically independent values C_i and D_i in the Wing-Kristofferson model. Aside from a possible relation, however, the latter model posits that the resulting period is a sum of two random components. Such an additive form is not that obvious in case of our dynamical system, and we thus have to discuss Eq. (15) and its resulting period and frequency in more detail.

Let us first consider the case of a fixed amplitude $\xi_r\approx r_0$ and let us define a phase ϕ via $\xi_\theta=\overline{\psi}(r_0)\tau+\xi_\phi(\tau)$. Note that $d/dt\equiv\omega_0\ d/d\tau$ holds so that we obtain

$$\dot{\xi}_\phi=\frac{\omega_0\sqrt{Q}}{r_0}\Gamma_\theta(t)$$
$$\Rightarrow\hat{f}(\phi,\tau)\propto\int_{-\infty}^{\infty}\hat{f}(\phi',0)\exp\left\{-\frac{r_0^2(\phi-\phi')^2}{2\omega_0^2Qt}\right\}d\phi'. \tag{17}$$

Hence, a certain choice of initial conditions such as $\hat{f}(\phi,t=0)=\delta(\phi-\phi_0)$ yields directly

$$\hat{f}(\phi,t)=\frac{r_0}{\sqrt{2\pi\omega_0^2Qt}}\exp\left\{-\frac{r_0^2(\phi-\phi_0)^2}{2\omega_0^2Qt}\right\}$$
$$\Rightarrow\langle\phi\rangle=0\ \wedge\ \langle\phi^2\rangle=\frac{\omega_0^2Qt}{{r_0}^2}. \tag{18}$$

Consequently, the variance of the phase increases linearly in time, that is $\langle\phi^2\rangle\propto t$, which expresses a ''simple'' *diffusion* process of ϕ. Like the case of steady amplitudes, one can

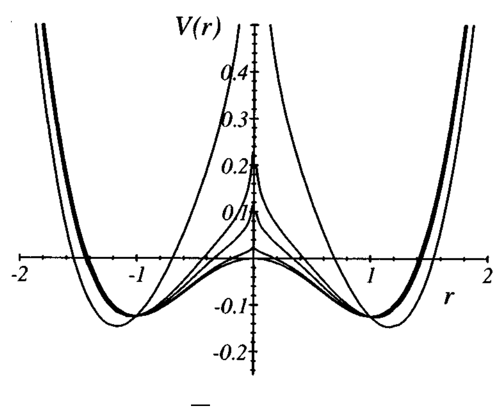

FIG. 5. Mean potential $\overline{V}(r)$ of the amplitude for different fluctuation strengths $Q=\{0,0.01,0.05,0.1,0.5\}$; the remaining parameters are $\alpha=\beta/3=\gamma=\omega_0=1$.

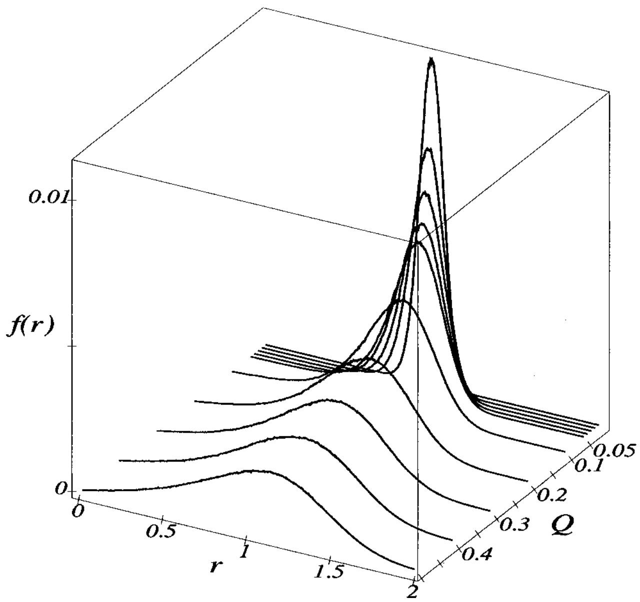

FIG. 6. Stationary probability distribution $f(r)$, $r^2 := \xi_x^2 + \xi_y^2$, for the original Langevin equation (10). The system is integrated 10^4 times over 10^4 periods. For every run the first 50 periods have been eliminated as transient regime, that is, the probability distribution is based on approximately 5×10^8 iterations per time series given a time step of $\Delta t = 10^{-3}$; $\alpha = \beta/3 = \gamma = \omega_0 = 1$ and $Q = \{0.01,\ldots,0.05,0.1,\ldots,0.5\}$. Obviously $f(r)$ reflects the potential $V(r)$ in Fig. 5 by means of $f(r) \propto \exp\{-V/Q\}$.

further estimate the impact of the amplitude dynamics on frequency and phase. With respect to the potential given in Eq. (16) for weak noise, we can approximate

$$\frac{d\xi_\theta}{d\tau} = \overline{\psi}(r_0 + \xi_{\delta r}) + \frac{\sqrt{Q}}{(r_0 + \xi_{\delta r})}\Gamma_\theta \approx \overline{\psi}(r_0) + \frac{\sqrt{Q}}{r_0}\Gamma_\theta + \left\{ \frac{\eta}{4} r_0 - \frac{Q}{r_0^3} - \frac{\sqrt{Q}}{r_0^2}\Gamma_\theta \right\} \xi_{\delta r}. \tag{19}$$

If the nonlinearities describing $\overline{n}(\xi_r)$ are also small the deviation $\xi_{\delta r}$ of the stationary amplitude r_0 can be computed from the linearized form of Eq. (15) given by

$$\frac{d\xi_{\delta r}}{d\tau} = -\lambda\ \xi_{\delta r} + \sqrt{Q}\Gamma_r$$

with

$$\lambda = \alpha + \frac{Q\kappa}{\alpha + \sqrt{\alpha^2 + Q\kappa}}. \tag{20}$$

Neglecting transient parts, the formal solution of Eq. (20) reads

$$\xi_{\delta r}(\tau) = \sqrt{Q}\int_{-\infty}^{\tau} e^{-\lambda(\tau - \tau')}\Gamma_r(\tau')\ d\tau' =: \sqrt{Q}\widetilde{\Gamma}_r(\tau) \tag{21}$$

that, inserted into Eq. (19), leads to

$$\frac{d\xi_\theta}{d\tau} \approx \overline{\psi}(r_0) + \frac{\sqrt{Q}}{r_0}\Gamma_\theta(\tau) + \frac{\eta r_0^4 - 4Q}{4r_0^3}\sqrt{Q}\widetilde{\Gamma}_r(\tau) = 1 + \frac{\eta}{8}r_0^2 + \frac{\sqrt{Q}}{r_0}\Gamma_\theta(\tau) + \frac{Q}{2r_0^2} + \frac{\eta r_0^4 - 4Q}{4r_0^3}\sqrt{Q}\widetilde{\Gamma}_r(\tau). \tag{22}$$

In Eq. (22) all the terms of the form $(\Gamma_r\Gamma_\theta)$ have been neglected because they vanish when calculating mean values. The three leading expressions on the right-hand side of Eq. (22) are of the same (first) order of magnitude, whereas the last two terms are of second or higher order. Therefore, the random amplitude $\xi_{\delta r}$ does not really influence the frequency dynamics and we should preferably write

$$\frac{d\xi_\theta}{d\tau} \approx \overline{\psi}_0(r_0) + \frac{\sqrt{Q}}{r_0}\Gamma_\theta(\tau). \tag{23}$$

This frequency evolution results in a similar estimate as in the case of a constant amplitude r_0 and we obtain phase diffusion. Only when the Duffing coefficient η is sufficiently large we may keep the form

$$\frac{d\xi_\theta}{d\tau} \approx \overline{\psi}_0(r_0) + \frac{\sqrt{Q}}{r_0}\Gamma_\theta(\tau) + \frac{\eta\sqrt{Q}}{4} r_0 \widetilde{\Gamma}_r(\tau). \tag{24}$$

Accordingly, we can compute the period as

$$T = T_0 + \frac{\eta r_0^2 - 4}{4r_0\omega_0}\sqrt{Q}\int_0^{2\pi}\Gamma_\theta d\theta - \frac{\eta\sqrt{Q}}{4\omega_0} r_0 \int_0^{2\pi}\widetilde{\Gamma}_r\ d\theta,$$

with

$$T_0 := \frac{2\pi}{\omega_0}\left(1 - \frac{\eta r_0^2}{8} + \left[\frac{\eta r_0^2}{8}\right]^2\right). \tag{25}$$

For nonergodic systems, the integrals over the noise remain random quantities, and the period can be written as

$$T = T_0 + \xi_{T_\theta} + \xi_{T_r}. \tag{26}$$

Indeed, this form is equivalent to the Wing-Kristofferson model since ξ_{T_θ} and ξ_{T_r} are two independent noise sources. It is worthwhile to remark that the existence of a ‘‘relevant’’ ξ_{T_r} requires a fairly large Duffing component $\propto \eta$ [see the last integral in Eq. (25)]. The influence of a random amplitude on the period length, however, is, as a second order correction, still very weak. The random force Γ_θ is much more important for the frequency dynamics and will predominantly lead to a plain *phase diffusion*, at least in the case of weak noise and weak nonlinearities. In that respect the system behaves like a harmonic oscillator and correlations between consecutive periods can be neglected and they themselves become random values, as shown in Fig. 7.

B. Multiplicative white noise

Instead of adding noise one can consider multiplicative random forces that might be viewed as locally dependent noise. For example, the strength of noise can become a func-

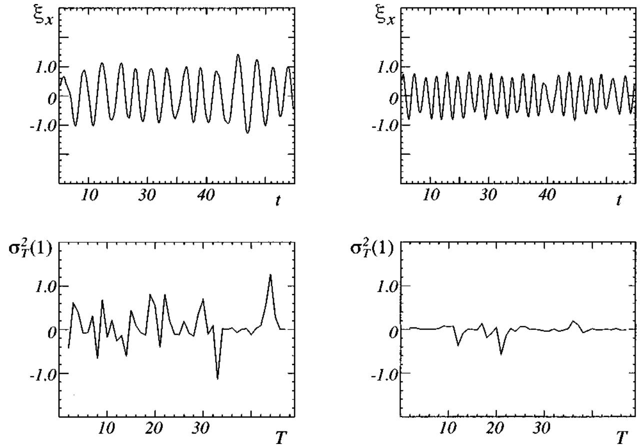

FIG. 7. Simulation of randomly forced oscillators. We chose $\omega_0=\alpha=\beta/3=\gamma=1$. On the left-hand side we took $\eta/3=0.2$ and on the right-hand side $\eta/3=5$. The random force was given by $Q=0.2$. The upper row shows a typical sequence of the time series $x(t)$, whereas the lower row represents the corresponding $\sigma_T^2(1)$. Here the period length T is determined as in Figs. 2–4 after smoothing the simulated time series with a Savitzky-Golay filter. $\sigma_T^2(1)$ remains rather random and the corresponding lag one correlation almost vanishes: for the left simulation we get $\overline{T}=5.932$, $\mu_T(1)=0.176$ and right $\overline{T}=3.839$, $\mu_T(1)=-0.073$; 10^5 periods have been considered to compute the mean values.

tion of the absolute elongation $|x|$. To analyze this case we introduce multiplicative white noise by means of

$$\frac{d}{dt}\begin{pmatrix}\xi_x\\ \xi_y\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\omega_0^2 & 0\end{pmatrix}\begin{pmatrix}\xi_x\\ \xi_y\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}n(\xi_x,\xi_y)+\omega_0^2\sqrt{2Q(\xi_x,\xi_y)}\begin{pmatrix}0\\ 1\end{pmatrix}\Gamma(t). \qquad (27)$$

For sake of simplicity, the function $Q(\xi_x,\xi_y)$ shall be given by $Q(\xi_x,\xi_y):=Q(\xi_x)=\Sigma_k Q_k \xi_x^k$; that is, a polynom of arbitrary order. In the literature one typically finds linear functions $Q(x)\propto x$ or low-order polynoms like $Q(x)\propto x^2$ (see, e.g., [30] and references therein). As mentioned above for $Q\propto x^{2n}$ the noise strength increases with increasing elongation. Generally, the corresponding Fokker-Planck operator can be written as

$$\mathcal{L}_{\rm FP}=-y\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\left\{n(x,y)-\omega_0^2 x-\omega_0^4 Q(x)\frac{\partial}{\partial y}\right\}. \qquad (28)$$

Analogous to the case of additive noise we discuss the averaged oscillator (6) after the Van der Pol transformation whose Fokker-Planck operator yields

$$\tilde{\mathcal{L}}_{\rm FP}\approx\overline{\mathcal{L}}_{\rm FP}=-\frac{\partial}{\partial r}\overline{n}_0(r)-\overline{\psi}_0(r)\frac{\partial}{\partial\theta}+\sum_k Q_k\left\{\frac{1}{2\pi}\int_0^{2\pi}d\theta'\,\mathcal{L}_{\rm FP}^{(k)}\right\}. \qquad (29)$$

The multiplication of Γ leads to corrections $\mathcal{L}_{\rm FP}^{(k)}$ that are defined by

$$\mathcal{L}_{\rm FP}^{(k)}:=\frac{(r\cos\theta)^k}{2r^2}\left\{1-2k+2k^2+(1-2k-2k^2)\cos 2\theta-[(1-2k)+(1+2k)\cos 2\theta]r\frac{\partial}{\partial r}-(1+k)\sin 2\theta\frac{\partial}{\partial\theta}+(1-\cos 2\theta)r^2\frac{\partial^2}{\partial^2 r}+(1+\cos 2\theta)\frac{\partial^2}{\partial^2\theta}\right\}. \qquad (30)$$

In order to average the last term in Eq. (30), we integrate over the angular variable θ. Thus for $k\neq 0$ we have to calculate integrals such as

$$\int_0^{2\pi}d\theta'\cos^k\theta',$$

$$\int_0^{2\pi}d\theta'\cos^k\theta'\cos 2\theta',$$

and

$$\int_0^{2\pi}d\theta'\cos^k\theta'\sin 2\theta'. \qquad (31)$$

Since $\cos^k$ is even, the last integral will always vanish and an instantaneous effect on the θ-‘‘dependent’’ part ($\propto\partial/\partial\theta$) of the Fokker-Planck operator does not occur. The remaining terms read with $\cos 2\theta=2\cos^2\theta-1$

$$\int_0^{2\pi}d\theta'\cos^k\theta'=\begin{cases}0 & \text{for } k \text{ odd}\\ 2\pi\prod_{j=0}^{k/2-1}\dfrac{k-2j-1}{k-2j} & \text{for } k \text{ even}.\end{cases} \qquad (32)$$

For even powers k these integrals do not vanish but for $k>0$ the products $\Pi\cdots$ are always smaller than unity; that is, they are of lower magnitude compared to the case of additive white noise ($k\equiv 0$). The dependency on the amplitude, however, changes essentially because for arbitrary even k one obtains

$$\overline{\mathcal{L}}_{\rm FP}^{(k)}=Q_k\Pi_k\left\{-\frac{\partial}{\partial r}r^{k-1}+\frac{1}{k+1}\frac{\partial^2}{\partial r^2}r^k+r^{k-2}\frac{\partial}{\partial\theta^2}\right\}. \qquad (33)$$

Here Π_k is defined as $\Pi_k:=\Pi_{j=0}^{k/2}(k-2j+1/k-2j+2)$. We again find a stochastically equivalent system

$$\frac{d\xi_r}{d\tau}=\hat{n}(\xi_r)+\left[2\sum_k{}' \,Q_k\frac{\Pi_k}{k+1}\xi_r^k\right]^{1/2}\Gamma_r$$

$$\wedge\,\frac{d\xi_\theta}{d\tau}=\hat{\psi}(\xi_r)+\frac{1}{\xi_r}\left[2\sum_k{}' \,Q_k\Pi_k\xi_r^k\right]^{\frac{1}{2}}\Gamma_\theta, \qquad (34)$$

with the additional abbreviations

$$\hat{n}(\xi_r) = -\frac{d\hat{V}}{d\xi_r}, \quad \hat{V}(\xi_r) := \overline{V}_0(\xi_r) - \frac{1}{2}\sum_k{}' \frac{Q_k}{k}\frac{k+2}{k+1}\Pi_k \xi_r^k. \tag{35}$$

Note that Σ' denotes a summation over even indices only. The deterministic part of the amplitude dynamics in Eq. (34) remains a gradient dynamics with the potential (35) that is quite similar to $\overline{V}$ [cf. definition (16)]. For the frequency dynamics $d\xi_\theta/d\tau$ in (34) we used

$$\hat{\psi}(\xi_r) := \overline{\psi}_0(\xi_r) - \frac{1}{2\xi_r^2}\left[\sum_k{}' Q_k(k-2)\Pi_k \xi_r^k\right] \times \left[\frac{\sum_k{}' Q_k \frac{\Pi_k}{k+1} k\xi_r^k}{\sum_k{}' Q_k \Pi_k \xi_r^k}\right]^{1/2}. \tag{36}$$

Concentrating on the discussion of the frequency dynamics in Eq. (34) we investigate the case of finite polynoms. Explicitly, we take polynoms up to the fourth order and define

$$\hat{\psi}_4(\xi_r) = \overline{\psi}_0(\xi_r) + \frac{1}{\xi_r^2}\left[\frac{Q_2\xi_r^2 + Q_4\xi_r^4}{8Q_0 + 6Q_2\xi_r^2 + 5Q_4\xi_r^4}\right]^{1/2} \times \left(Q_0 - \frac{5}{8}Q_4\xi_r^4\right). \tag{37}$$

For sake of simplicity, the term $[\cdots]^{1/2}$ will be dropped because it does not really influence the dependency on ξ_r and we approximate

$$\hat{\psi}_4(\xi_r) \approx \overline{\psi}_0(\xi_r) + \frac{Q_0}{2\xi_r^2} - \frac{5Q_4}{8}\xi_r^2 =: 1 + \frac{\hat{\eta}}{8}\xi_r^2 + \frac{Q_0}{2\xi_r^2}. \tag{38}$$

A fourth order multiplicative noise obviously corrects the Duffing coefficient $\eta \to \hat{\eta}$. Since the quadratic term $\propto Q_2$ does not affect the deterministic part of the frequency dynamics, we neglect it in the stochastic part as well (there it mainly acts as additive noise). Accordingly, the dynamics of the frequency can be approximated as

$$\frac{d\xi_\theta}{d\tau} \approx \hat{\psi}_4(\xi_r) + \frac{1}{\xi_r}\left[Q_0 + \frac{5}{8}Q_4\xi_r^4\right]^{1/2}\Gamma_\theta. \tag{39}$$

Following the discussion of Eq. (19) we expand the amplitude $\xi_r = (r_0 + \xi_{\delta r})$ and if we again focus on the vicinity of a stable limit cycle; that is, we assume the noise to be reasonably weak, the amplitude can be estimated by

$$\xi_r(\tau) = \tilde{r}_0 + \xi_{\delta r} \approx \tilde{r}_0 + \sqrt{Q_0}\,\tilde{\tilde{\Gamma}}_r(\tau). \tag{40}$$

$\tilde{\tilde{\Gamma}}$ is similarly defined as $\tilde{\Gamma}$ in (21) and since the explicit calculation of $\tilde{r}_0$ and $\tilde{\tilde{\Gamma}}$ exceeds the aims of the present paper and their explicit form does not change the forthcoming argument, we skip it here. The expression (40) can be inserted into Eq. (39) and an expansion of the factor $[\cdots]^{1/2}$ results in a comparable form like Eqs. (22) or (23), respectively. Besides the correction of the Duffing component that is due to an additional drift coefficient, the multiplication of noise can be directly reduced to additive noise. Apparently, polynoms of higher than fourth order can be treated equivalently since they only lead to corrections of even higher order compared to the considered case. As the dominant process, we always observe phase diffusion and correlations between consecutive periods that are more or less random.

C. Forcing via colored noise

So far we have shown that uncorrelated noise sources mainly result in phase diffusion and thus cannot be used to generate a certain period correlation. We now introduce further correlations within the noise itself in terms of colored noise sources. An immediate approach can be given by a time-dependent stiffness, $\omega_0 \to \omega_0 + \varepsilon\xi_\varpi(t)$, where ε is used as a smallness parameter. The stiffness of the oscillator may have stochastical properties such as

$$\omega = \omega_0[1 + \varepsilon\Psi(t)] \wedge \langle\Psi(t)\rangle = 0,$$

$$\langle\Psi(t)\Psi(t')\rangle = Q\, e^{-|t-t'|/\tau_c}, \tag{41}$$

which is well-known to be equivalent to the Ornstein-Uhlenbeck process [21]

$$\omega = \omega_0(1 + \varepsilon\xi_\varpi) \wedge \dot{\xi}_\varpi = -\frac{1}{\tau_c}\xi_\varpi + \sqrt{\frac{2Q}{\tau_c}}\Gamma(t). \tag{42}$$

In other words, colored noise can be expressed via an auxiliary dynamics that is forced by white noise. The harmonic oscillator including the stiffness dynamics (42) is known as the Kubo oscillator and is characterized by a vanishing amplitude [31,21]. Here, however, we consider the case of finite amplitudes generated by nonlinear oscillators with stable limit cycles such as

$$\ddot{x} + \omega_0^2(1 + \varepsilon\xi_\varpi)^2 x = n(x, \dot{x}) \tag{43}$$

that lead to a Fokker-Planck operator of the following form:

$$\mathcal{L}_{\rm FP} = -y\frac{\partial f}{\partial x} - \frac{\partial}{\partial y}\{n(x,y) - \omega_0^2(1+\varepsilon\varpi)^2 x\} + \frac{\partial}{\partial\varpi}\left\{\frac{\varpi}{\tau_c} + \frac{Q}{\tau_c}\frac{\partial}{\partial\varpi}\right\}. \tag{44}$$

Following our standard procedure, we apply the Van der Pol transformation and average over a period where we auxiliary define $\xi_\varpi(t) := \xi_\varpi(\tau)$. Note that the averaging requires the correlation length of the noise to be small, i.e., $\tau_c \ll 1/\omega_0$ (cf. [2]). For $n(x,y)$ given in Eq. (6) we achieve

$$\overline{\mathcal{L}}_{\rm FP} = -\frac{\partial}{\partial r}\overline{n}_0(r) - \left\{\overline{\psi}_0(r) + \varepsilon\varpi + \frac{\varepsilon^2}{2}\varpi^2\right\}\frac{\partial}{\partial\theta} + \frac{1}{\omega_0\tau_c}\frac{\partial}{\partial\varpi}\left\{\varpi + \frac{Q}{\omega_0}\frac{\partial}{\partial\varpi}\right\}, \tag{45}$$

and thus we find again a stochastically equivalent system

$$\frac{d}{d\tau}\begin{pmatrix}\xi_r\\ \xi_\theta\\ \xi_\varpi\end{pmatrix} = \left[\overline{n}(r),\ \overline{\psi}(r) + \varepsilon\xi_\varpi + \frac{\varepsilon^2}{2}\xi_\varpi,\ -\frac{\xi_\varpi}{\omega_0\tau_c}\right]^T + \frac{\sqrt{2Q}}{\omega_0\sqrt{\tau_c}}\begin{pmatrix}0\\ 0\\ \Gamma\end{pmatrix}. \tag{46}$$

If we consider the case $\xi_r \approx r_0$ and $\xi_\theta = \overline{\psi}(r_0)\tau + \xi_\phi$, this system can be reduced to

$$\frac{d}{dt}\begin{pmatrix}\xi_\phi\\ \xi_\varpi\end{pmatrix} \overset{\omega_0\varepsilon\ll 1}{\approx} \begin{pmatrix}\varepsilon\\ -1/\tau_c\end{pmatrix}\xi_\varpi + \sqrt{\frac{2Q}{\tau_c}}\begin{pmatrix}0\\ 1\end{pmatrix}\Gamma$$
$$\Rightarrow \ddot{\xi}_\phi + \frac{1}{\tau_c}\dot{\xi}_\phi = \omega_0\varepsilon\sqrt{\frac{2Q}{\tau_c}}\,\Gamma. \tag{47}$$

In contrast to the white noise case the phase dynamics is now a second order differential equation due to the exponential correlation in Ψ. Hence, we can distinguish different correlation times τ_c. The extreme limits lead to

$$\tau_c \to 0 \Rightarrow \dot{\xi}_\phi \approx \omega_0\varepsilon\sqrt{2Q\tau_c}\,\Gamma(t),$$
$$\tau_c \to \infty \Rightarrow \ddot{\xi}_\phi \approx \omega_0\varepsilon\sqrt{2Q/\tau_c}\,\Gamma(t). \tag{48}$$

Both situations describe pure diffusion processes for ξ_ϕ (see, e.g., [32] for a more general discussion). Summarizing we recognize that a stochastic forcing of the stiffness only results in phase diffusion so that correlations between periods are again negligible.

At last, we further extend the discussion to more general multiplicative colored noise sources. The basic equation reads

$$\frac{d}{dt}\begin{pmatrix}\xi_x\\ \xi_y\\ \xi_z\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ -\omega_0^2 & 0 & q(\xi_x,\xi_y)\\ 0 & 0 & -1/\tau_c\end{pmatrix}\begin{pmatrix}\xi_x\\ \xi_y\\ \xi_z\end{pmatrix} + \begin{pmatrix}0\\ 1\\ 0\end{pmatrix} n(\xi_x,\xi_y) + \sqrt{\frac{2Q}{\tau_c}}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\Gamma(t) \tag{49}$$

with a Fokker-Planck operator given by

$$\mathcal{L}_{\rm FP} = -y\frac{\partial f}{\partial x} - \frac{\partial}{\partial y}\{n(x,y) - \omega_0^2 x + zq(x,y)\} + \frac{\partial}{\partial z}\left\{\frac{z}{\tau_c} + \frac{Q}{\tau_c}\frac{\partial}{\partial z}\right\}. \tag{50}$$

Transforming the operator by means of the Van der Pol transformation and defining $\xi_z := \xi_\zeta(\tau)$ as well as $\tilde{q} := q(r\cos\theta, -\omega_0 r\sin\theta)$ we obtain for the averaged system (6)

$$\bar{\mathcal{L}}_{\rm FP} = -\frac{\partial}{\partial r}\bar{n}_0(r) - \bar{\psi}_0(r)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\zeta}\frac{\zeta}{\omega_0\tau_c} + \frac{Q}{\omega_0^2\tau_c}\frac{\partial^2}{\partial\zeta^2} + \underbrace{\frac{1}{2\pi\omega_0^2}\int_0^{2\pi} d\theta'\zeta\left[\frac{\partial}{\partial r}\sin\theta' + \frac{1}{r}\frac{\partial}{\partial\theta'}\cos\theta'\right]\tilde{q}(r,\theta')}_{=:\ \bar{\mathcal{L}}_{\rm FP}^{(q)}}. \tag{51}$$

Similar to the case of multiplicative white noise we discuss polynomial forms of multiplier q and take $q(\xi_x,\xi_y)$: $=\omega_0^2\Sigma_{kl}q_{kl}\xi_x^k\xi_y^l$. Then we achieve the corrections of Eq. (51) as

$$\overline{\mathcal{L}}_{\mathrm{FP}}^{(q)}=\sum_{kl}\frac{(-\omega_0)^l q_{kl}}{2\pi}\zeta r^{k+l-1}\int_0^{2\pi}d\theta'\bigg[l\ \cos^k\theta'\sin^{l-1}\theta' -\cos^k\theta'\sin^{l+1}\theta'+r\ \cos^k\theta'\sin^{l+1}\theta'\frac{\partial}{\partial r} +\cos^{k+1}\theta'\sin^l\theta'\frac{\partial}{\partial\theta'}\bigg]. \tag{52}$$

The last term in Eq. (52) corrects the drift coefficient of the frequency and is therefore the most important component. Obviously, this term remains finite only in case of odd exponents k and concurrently even exponents l. For the sake of simplicity we reduce ourselves to the case of $l=0$ and write $q(\xi_x,\xi_y)=q(\xi_x)$. Actually, we discuss the special case $q(\xi_x):=\omega_0^2(q_1\xi_x+q_3\xi_x^3)$, higher order polynoms will result in higher order corrections that can be neglected as shown below. Inserting that form into Eq. (52) leads to

$$\overline{\mathcal{L}}_{\mathrm{FP}}^{(q)}=\frac{\zeta}{2}\bigg\{\frac{3q_3}{4}r^2+q_1\bigg\}\frac{\partial}{\partial\theta}. \tag{53}$$

Note that for the Kubo system discussed above we have to replace $\Psi\to\Psi^2$. Given the operator (53) the frequency dynamics can be further reduced to

$$\frac{d\xi_\theta}{d\tau}=\overline{\psi}(\xi_r)+\frac{1}{2}\bigg(\frac{3q_3}{4}\xi_r^2+q_1\bigg)\Psi(\tau). \tag{54}$$

Even if ξ_r itself is given by a Langevin equation, it will only affect the dynamics of ξ_θ via the Duffing component $(\sim\overline{\psi}_0)$ or via $\xi_r^2\Psi$. Analogous to the discussion of multiplicative white noise, this effect is of higher order and can thus be neglected.

In conclusion, we see that neither Gaussian nor colored noise sources can generate the desired correlation function for consecutive periods. Fluctuations always produce some dominating phase diffusion that destroys any further correlation within frequency and period, respectively. It is worthwhile to remark that these claims are not restricted to the thusfar applied approximations. Of course, we achieve a decoupling of frequency and amplitude basically by use of a first order Krylov-Bogoliubov approximation by means of averaging. Higher order expansions, however, yield higher order corrections only and thus diffusion remains the dominant process. Moreover, the period of the oscillator is defined as integral over the frequency variable θ [cf. Def. (3)] and through that integration we already perform some kind of averaging along the interval $\theta\in[0,2\pi]$.

IV. FORCED OSCILLATIONS WITH NOISE

We have seen that diffusion is the prevailing effect resulting from various, essentially different, noise sources. Indeed, this fact expresses the absense of any ‘‘force’’ acting on the phase of a self-sustained or autonomous limit cycle oscillator. Thus, we finally extend the nonlinear oscillator by means of an external deterministic force. The force is assumed to be periodic in time and it can therefore bias the phase dynamics by means of phase locking. Without loss of generality we treat the case of a sinusoidal force oscillating with frequency Ω. The dynamical system becomes

$$\frac{d}{dt}\begin{pmatrix}\xi_x\\ \xi_y\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\omega_0^2 & 0\end{pmatrix}\begin{pmatrix}\xi_x\\ \xi_y\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}n(\xi_x,\xi_y) -2F_0\Omega^2\begin{pmatrix}0\\ 1\end{pmatrix}\sin\,\Omega t+\omega_0^2\sqrt{2Q}\begin{pmatrix}0\\ 1\end{pmatrix}\Gamma(t). \tag{55}$$

In contrast to our standard Van der Pol transformation, we project to polar coordinates regarding the forcing frequency Ω; that is, $\xi_x=\xi_r\cos(\tau+\xi_\phi)$, $\xi_y=-\Omega\xi_r\sin(\tau+\xi_\phi)$, and $\tau=\Omega t$. Equivalent to Eq. (14) the averaged Fokker-Planck operator (we average over $2\pi/\Omega$) becomes

$$\overline{\mathcal{L}}_{\mathrm{FP}}\approx-\frac{\partial}{\partial r}\{\overline{n}(r)-F_0\cos\,\phi\}-\frac{\partial}{\partial\phi}\bigg\{\frac{\omega_0^2-\Omega^2}{2\Omega^2}+\frac{\eta}{8}r^2 -\frac{F_0}{r}\sin\,\phi\bigg\}+\frac{Q}{2}\bigg\{\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\bigg\}. \tag{56}$$

As a stochastically equivalent system we find

$$\frac{d}{d\tau}\begin{pmatrix}\xi_r\\ \xi_\phi\end{pmatrix}=\begin{pmatrix}\overline{n}(\xi_r)\\ \overline{\chi}(\xi_r)\end{pmatrix}+\frac{F_0}{\xi_r}\begin{pmatrix}\xi_r\cos\,\xi_\phi\\ -\sin\,\xi_\phi\end{pmatrix}+\frac{\sqrt{Q}}{\xi_r}\begin{pmatrix}\xi_r\Gamma_r\\ \Gamma_\phi\end{pmatrix}, \tag{57}$$

where $\overline{n}$ is given in Eq. (16) and $\overline{\chi}$ is defined as

$$\overline{\chi}(\xi_r):=\frac{\omega_0^2-\Omega^2}{2\Omega^2}+\frac{\eta}{8}\xi_r^2+\frac{Q}{2\xi_r^2}. \tag{58}$$

Let us again concentrate on the phase dynamics. We assume that the oscillator is forced in resonance, i.e., $\omega_0\approx\Omega$. Further we neglect the Duffing component $(\eta\equiv0)$ as well as the term $Q/2\xi_r^2$ since they mainly result in a detuning that can be covered by the θ definition. In case of a weak forcing $(F_0\ll\overline{n})$ we can approximate the amplitude ξ_r by $\xi_r\approx r_0$. Thus, we reduce the problem to that of Brownian motion in a periodic potential $V_F=-(F_0/r_0)\cos\xi_\phi$ since we have

$$\frac{d\xi_\phi}{d\tau}=-\frac{F_0}{\xi_r}\sin\,\xi_\phi+\frac{\sqrt{Q}}{\xi_r}\Gamma_\phi\approx-\frac{F_0}{r_0}\sin\,\xi_\phi+\frac{\sqrt{Q}}{r_0}\Gamma_\phi. \tag{59}$$

For weak noise we can assume that the mean phase will always relax to a steady value with a fixed variance in contrast to Eq. (18), where the variance increases linearly in time. This follows directly if we linearize the potential and write the dynamics (59) as

$$\frac{d\xi_\phi}{d\tau}\approx-\frac{F_0}{r_0}\xi_\phi+\frac{\sqrt{Q}}{r_0}\Gamma_\phi=:-\frac{1}{\tau_F}\phi+\sqrt{\frac{2Q_F}{\tau_F}}\Gamma_\phi. \tag{60}$$

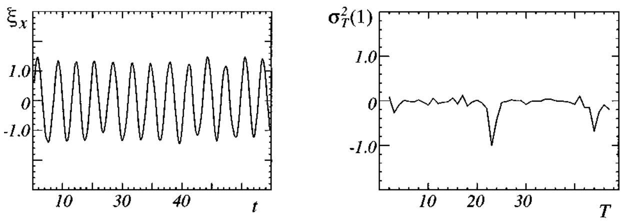

FIG. 8. Simulation of a driven oscillator under the impact of white noise. We chose $\omega_0=\alpha=\beta/3=\gamma=1$, and $\eta/3=0.2$. The noise strength is identical to Fig. 7 ($Q=0.2$). The forcing is determined by $F_0=1$ and $\Omega=1$, i.e., a strong external force in resonance with the harmonic part resulting in $\overline{T}\approx 2\pi$. Obviously, we achieve a rather negative covariance $\sigma_T^2(1)$ and, accordingly, the lag one correlation becomes $\mu_T(1)=-0.487$ (cf. Fig. 7).

For this Ornstein-Uhlenbeck process one can immediately compute the time dependent solution of the Fokker-Planck equation. Assuming that we have initial conditions like $\hat{f}(\phi,t=0)=\delta(\phi-\phi_0)$, we achieve

$$f(\phi,t)=\sqrt{\frac{1}{2\pi Q_F\tau_F}}[1-\mathrm{e}^{-2t/\tau_F}]^{1/2}$$
$$\times\exp\left\{-\frac{1}{2Q_F}\frac{(\phi-\phi_0\,\mathrm{e}^{-t/\tau_F})^2}{1-\mathrm{e}^{-2t/\tau_F}}\right\}$$
$$\Rightarrow\langle\phi\rangle=\phi_0\mathrm{e}^{-t/\tau_F}\wedge\langle\phi^2\rangle=\langle\phi\rangle^2+Q_F(1-\mathrm{e}^{-2t/\tau_F})$$
$$\overset{\phi_0=0}{\Rightarrow}\langle\phi\rangle=0\wedge\langle\phi^2\rangle=Q_F(1-\mathrm{e}^{-2t/\tau_F}). \tag{61}$$

The system's response on a decrease (or increase) of phase ($\langle\phi\rangle$) is an increase (or decrease) of frequency ($\langle\dot{\phi}\rangle\sim\langle\theta\rangle$). Note that in both cases, with and without external forcing the mean phase vanishes whereas the variances essentially differ [see Eq. (18)]. Depending on the mean relaxation time τ_F, the resulting correlation function of consecutive periods can become negative. Even for strong noise such random motions in periodic potentials have been extensively discussed in the literature. Recent studies mainly focus on stochastic resonance [33,34,29], so that we restrict ourselves to numerical experiments presented in Fig. 8. In that simulation, the mean lag one correlation function is rather close to the lower bound in the Wing-Kristofferson model [$\mu_T(1)\approx-0.5$]. As already noted, correlations become dependent on the relaxation time that itself depends on the forcing strength and via r_0 depends on both, the eigenfrequency ω_0 of the oscillator, and on the forcing frequency Ω. For the sake of legibility of the present paper we refer to forthcoming works that will show explicit dependencies in order to fit certain frequency and amplitude dependencies in the case of rhythmic human movement.

V. CONCLUSION

Aiming at a modeling of certain lag one serial correlation functions during a periodic dynamics we have shown that a negative correlation between consecutive periods during evolution along a limit cycle cannot be realized by introducing unspecific random forces. Additional white, as well as colored noise sources that both have been multiplied in terms of arbitrary finite polynoms, do not achieve the desired response because fluctuations acting on self-sustained nonlinear oscillators predominantly result in phase diffusion processes that are always superimposed on eventual (lower order) correlations.

Consequently, we extended the system to higher dimensions, here by means of a nonautonomous deterministic part. Alternatively one may also think of two or more coupled oscillators. In particular, periodically forced systems can allow for negative lag one correlation functions. In that case, the dynamics of the phase is reduced to Brownian motion in a periodic potential. Thus, significant properties like stability or relaxation times are well known and be approximated for actual values of serial correlations. Forthcoming works will show that such estimates will cover special amplitude and frequency dependencies of that type of correlation function.

In contrast to more traditional approaches (traditional in the field of human movement) that are based on nondynamical statistics of at least two independent noise sources, forced or coupled oscillators require only one additive Gaussian noise source generating the wanted correlation function. Correlations are therefore not a result of statistical properties only but a consequence of the deterministic interaction between two systems, the oscillator and the force.

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