Synchronization of coupled time-delay systems: Analytical estimations

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The synchronization threshold of coupled time-delay chaotic systems is estimated by two different analytical approaches. One of them is based on the Krasovskii-Lyapunov theory that represents an extension of the second Lyapunov method for delay differential equations. Another approach uses a perturbation theory of large delay time. The analytical expression relating synchronization threshold to the maximal Lyapunov exponent of uncoupled driving and response subsystems is derived. The analytical results are compared with the numerical simulations for two coupled Mackey-Glass systems. [S1063-651X(98)11809-3]

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I. INTRODUCTION

Cooperative behavior of coupled dynamical systems is an important field of nonlinear dynamics. Synchronization effects in systems with periodic behavior are widely used in engineering science. In recent years, chaotic synchronization has become an area of active research [1,2], especially in light of its potential application to secure communication [3–6]. This problem has aroused considerable interest in the construction of synchronized directionally coupled (sender-receiver or drive-response configurations) chaotic systems. The first examples of secure communication [3] were based on simple low-dimensional chaotic systems having only one positive Lyapunov exponent. However, it was later realized that such simple systems do not ensure a sufficient level of security [4]. To improve security, high-dimensional systems with multiple positive Lyapunov exponents (hyperchaotic systems) are preferable, but at the same time, it is desired to achieve the synchronization by transmitting just a single scalar variable. These opposite requirements complicate the problem essentially. Some ideas [5] on how to construct synchronized hyperchaotic systems for the case of coupled ordinary differential equations were proposed in Ref. [5]. Because these systems have a finite-dimensional phase space, the number of positive Lyapunov exponents is limited by the dimension of the phase space.

Recently chaotic time-delay systems have been suggested as good candidates for secure communication [6]. These infinite-dimensional systems are described by delay differential equations and can produce chaotic attractors with an arbitrarily large number of positive Lyapunov exponents. A typical example of this type is the Mackey-Glass system [7] in which the number of positive Lyapunov exponents increases linearly with increasing delay time [8].

In most publications the problem of synchronization is considered numerically either by direct solution of underlying dynamical equations or by calculating the maximal transverse Lyapunov exponent of the synchronization manifold. Due to the lack of analytical theory, the regular way of constructing synchronized hyperchaotic systems is still unknown.

In this paper, we consider the problem of synchronizing hyperchaotic systems described by the coupled delay differential equations. The time-delay systems represent a special, relatively simple case, of spatially extended systems described by partial differential equations. Thus the problem considered here sheds light on a more general problem of synchronizing spatiotemporal chaos. Here we develop two analytical approaches for estimating the synchronization threshold of coupled time-delay systems.

II. NUMERICAL EXAMPLE

We start our analysis with the specific example of two directionally coupled Mackey-Glass systems,

$$\dot{x}=f(x_\tau)-cx, \tag{1a}$$

$$\dot{y}=f(y_\tau)-cy+K(x-y), \tag{1b}$$

where $f(x_\tau)=ax_\tau/(1+x_\tau^b)$ and $x_\tau\equiv x(t-\tau)$. The term $K(x-y)$ in Eq. (1b) represents a dissipative coupling, where K is the coupling strength. At $K=0$, both the driving [Eq. (1a)] and response [Eq. (1b)] subsystems represent a standard Mackey-Glass delay differential equation [7]. Initially this equation has been introduced as a model of blood generation for patients with leukemia. Later this model became popular in chaos theory as a model for producing high-dimensional chaos to test various methods of chaotic time-series analysis, controlling chaos, etc. The electronic analog of this system has been proposed in Ref. [9].

Usually (e.g., [8]) the parameters a, b, and c are fixed at $a=0.2$, $b=10$, and $c=0.1$, and the delay time τ is varied. The number of parameters in Eqs. (1) can be reduced by dividing these equations by c and changing the time scale $tc\rightarrow t$. The parameters τ, a, and k are transformed as follows: $\tau c\rightarrow\tau$, $a/c\rightarrow a$, and $K/c\rightarrow K$. As a result, the given set of parameters becomes $a=2$, $b=10$, and $c=1$, and τ is ten times smaller than that of Ref. [8].

The changes in the qualitative behavior of the driving attractor as the parameter τ is varied are as follows [8]. The instability occurs at $\tau=\tau_1=0.471$. For $0.471<\tau<1.33$, there is a stable limit cycle attractor. A period-doubling bifurcation sequence is observed at $1.33<\tau<1.68$. For $\tau>1.68$, numerical simulations show chaotic attractors at

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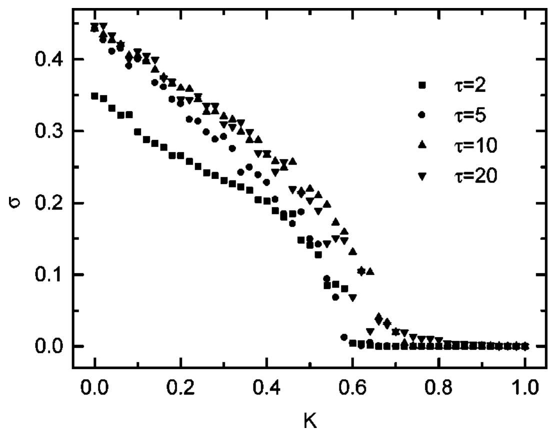

FIG. 1. The rms deviation σ as a function of coupling strength K for different values of delay time τ.

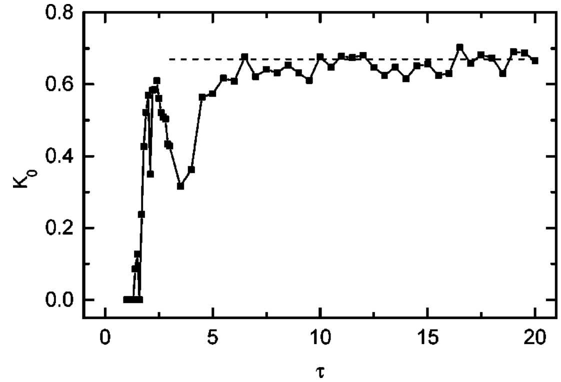

FIG. 2. Synchronization threshold K_0 as a function of delay time τ. The dashed line corresponds to analytical estimation [Eq. (18)] obtained in a singular perturbation limit $\tau\to\infty$.

most parameter values. The number of positive Lyapunov exponents and information dimension of the strange attractor increase linearly with an increase of τ. Specifically, at $\tau=10$ there are five positive Lyapunov exponents and the information dimension is of the order of ten.

To identify synchronization in Eqs. (1) we introduce the rms deviation $\sigma=\sqrt{\langle(y-x)^2\rangle}$, where $\langle\;\rangle$ indicates the time average. This parameter is finite for an unsynchronized state and vanishes for a synchronized state. In Fig. 1, the σ vs K dependence, obtained by the numerical integration of Eqs. (1), is presented. Here and below we use the second-order Runge-Kutta method with the step size $h=0.01$. With an increase of τ, the synchronization threshold K_0 increases and then saturates to a finite value equal approximately to 0.7.

Similar results are obtained from the linear theory based on a calculation of the maximal transverse Lyapunov exponent $\lambda(K)$ of the identity synchronization manifold $y=x$. Small deviations $\Delta=y-x$ from the identity manifold are governed by a variational equation

$$\dot{\Delta}=f'(x_\tau)\Delta_\tau-(K+1)\Delta, \tag{2}$$

where f' denotes the derivative of the function f. If one is interested in the solutions of this equation at times $t\geq 0$, it is necessary to define the initial $\Delta(t)$ for the entire interval $[-\tau,0]$, $\Delta(\vartheta)=\Delta_{\rm in}(\vartheta)$, $-\tau\leq\vartheta\leq 0$, where $\Delta_{\rm in}(\vartheta)$ is a given continuous initial function in a suitable function space $\mathcal{C}$. The state of delay system at time t can be described by an extended state vector $\mathbf{\Delta}_t\in\mathcal{C}$ constructed in the interval $[t-\tau,t]$ according to the prescription $\mathbf{\Delta}_t(\vartheta)=\Delta(t+\vartheta)$, $-\tau\leq\vartheta\leq 0$. The norm of this vector is $\|\mathbf{\Delta}_t\|=\{\int_{-\tau}^{0}\Delta^2(t+\vartheta)d\vartheta\}^{1/2}$. By analogy with the systems described by ordinary differential equations, we define the maximal transverse Lyapunov exponent as

$$\lambda(K)=\lim_{t\to\infty}\frac{1}{t}\ln\frac{\{\int_{-\tau}^{0}\Delta^2(t+\vartheta)d\vartheta\}^{1/2}}{\{\int_{-\tau}^{0}\Delta^2(\vartheta)d\vartheta\}^{1/2}}. \tag{3}$$

For $K=0$, this exponent coincides with the maximal Lyapunov exponent λ_0 of uncoupled driving and response subsystems, $\lambda_0=\lambda(0)$. The synchronization threshold corresponds to the coupling strength $K=K_0$ at which the transverse Lyapunov exponent vanishes, $\lambda(K_0)=0$. The dependence of the synchronization threshold K_0 on delay time τ is shown in Fig. 2. These results are in a good agreement with the behavior of the rms deviation σ presented in Fig. 1.

Note that synchronization is achieved (by transmitting a single scalar variable) even in the case of large τ when uncoupled subsystems have multiple positive Lyapunov exponents. Intuitively, one can expect that synchronization is more difficult to achieve when the driving and response subsystems have a large number of positive Lyapunov exponents, since more unstable directions are needed to be stabilized. The above example demonstrates that this intuition is incorrect: for large τ the synchronization threshold K_0 is independent of τ but the number of positive Lyapunov exponents of the Mackey-Glass system increases as τ is increased. Thus the number of positive Lyapunov exponents is not the appropriate parameter to define the condition of synchronization for coupled time-delay systems. In Sec. IV we show that the parameter responsible for the synchronization condition is the product of the maximal Lyapunov exponent λ_0 and delay time τ.

III. KRASOVSKII-LYAPUNOV THEORY

The analytical estimation of the synchronization threshold for coupled time-delay systems can be obtained in the framework of the Krasovskii-Lyapunov theory [10]. This theory represents an extension of the second Lyapunov method for the case of delay differential equations. Consider a rather general form of the identical, one-way coupled scalar time-delay systems,

$$\dot{x}=F(x,x_\tau,p_0), \tag{4a}$$

$$\dot{y}=F(y,y_\tau,p_0+K(y-x)). \tag{4b}$$

Here we suppose that the system has a parameter p available for external perturbations. For the driving system [Eq. (4a)], this parameter is fixed to a value $p=p_0$. For the response system [Eq. (4b)], it is varied proportionally to the deviation $y-x$, $p=p_0+K(y-x)$ so that in the synchronized state $y=x$ it takes the same value $p=p_0$ as for the driving system.

Small deviations $\Delta=y-x$ are governed by a linear delay differential equation

$$\dot{\Delta} = -r(t)\Delta + s(t)\Delta_\tau, \tag{5}$$

where $-r(t) = (\partial_x + K\partial_p)F(x,x_\tau,p_0)$, $s(t) = \partial_{x_\tau}F(x,x_\tau,p_0)$. The symbols ∂_x, ∂_{x_τ}, and ∂_p denote the partial derivatives of the function $F(x,x_\tau,p)$ with respect to the variables x, x_τ, and parameter p, respectively. The driving and response subsystems, described by Eqs. (4), are synchronized if the origin of Eq. (5) is stable. Following the Krasovskii theory, we introduce a positively defined functional (similar to a Lyapunov function in the case of ordinary differential equations)

$$V(t) = \tfrac{1}{2}\Delta^2 + \mu \int_{-\tau}^{0} \Delta^2(t+\vartheta)\,d\vartheta$$

to estimate a sufficient condition of stability. Here $\mu > 0$ is an arbitrary positive parameter. The origin of Eq. (5) is stable if the derivative of the functional $V(t)$ along the trajectory of Eq. (5),

$$\dot{V}(t) = -r(t)\Delta^2 + s(t)\Delta\Delta_\tau + \mu\Delta^2 - \mu\Delta_\tau^2,$$

is negative. The right-hand side of this equation is a negatively defined function of the variables Δ and Δ_τ if $s^2(t) + 4\mu[\mu - r(t)] < 0$ or $r(t) > s^2(t)/4\mu + \mu \equiv \varphi(s,\mu)$. For any s, $\varphi(s,\mu)$ as a function of μ has an absolute minimum $\varphi_{\min} = |s|$ at $\mu = |s|/2$. Thus $\varphi(s,\mu) \geqslant |s|$ for any s and $\mu > 0$, and we obtain the stability condition of Eq. (5) in the form

$$r(t) > |s(t)|. \tag{6}$$

If this inequality holds for all $t > 0$, the identity synchronization manifold of Eqs. (4) is asymptotically stable. Generally, this condition requires a knowledge of the solution $x(t)$, since the parameters $r(t)$ and $s(t)$ are the functions of this variable. However, in some cases one can do without any knowledge of the explicit solution $x(t)$. Let us demonstrate this for the case of coupled Mackey-Glass systems considered in Sec. II. Here we have $F(x,x_\tau,p) = f(x_\tau) - x - p$ with $p_0 = 0$ and we obtain $r(t) = 1 + K = \text{const}$, $|s(t)| = |f'(x_\tau)|$. The stability condition [Eq. (6)] takes the form $K > \max|f'(x_\tau)| - 1$, where the maximum is defined on the trajectory of the driving system. This maximum can be replaced by the absolute maximum of the function $|f'(x_\tau)|$ obtained at $x_\tau = [(b+1)/(b-1)]^{1/b}$. As a result, we obtain the analytical criterion of synchronization, $K > a(b-1)^2/4b - 1 \equiv \tilde{K}_0$. For given values of parameters $a = 2$, $b = 10$, this leads to an estimate of the synchronization threshold $\tilde{K}_0 = 3.05$. The result differs significantly from the value $K_0 \approx 0.7$ numerically obtained in Sec. II, because the second Lyapunov method gives a sufficient, but not a necessary condition of stability. The method assures synchronization at $K > \tilde{K}_0$, but does not forbid a possibility of synchronization at $K < \tilde{K}_0$. Nevertheless, this theory gives the analytical proof for the statement that the number of positive Lyapunov exponents is not responsible for the synchronization condition, since the analytical threshold $\tilde{K}_0$ is independent of τ. Below, we consider another analytical approach that gives a more accurate estimate of the synchronization threshold.

IV. SYNCHRONIZATION THRESHOLD FOR LARGE DELAYS

Here we consider the case of a large delay time $\tau \to \infty$. To make our description more transparent, we restrict ourselves to the specific example of coupled Mackey-Glass systems [Eqs. (1)]. The solution of Eq. (1a) can be presented as a successive mapping of the functions $x_{n-1}(\theta) \to x_n(\theta)$ defined on the unity interval $0 \leqslant \theta \leqslant 1$, so that for any time $t = \tau(n-1+\theta)$, the output $x(t)$ is expressed as $x(t) = x_n(\theta)$ with the integer $n = \text{floor}(t/\tau) + 1$ and $\theta = t/\tau - n + 1$, where floor(t/τ) denotes the largest integer not larger than t/τ. The mapping $x_{n-1}(\theta) \to x_n(\theta)$ is governed by an ordinary differential equation

$$\frac{1}{\tau}\dot{x}_n = f(x_{n-1}) - x_n, \tag{7}$$

with the boundary condition $x_n(0) = x_{n-1}(1)$. Here $\dot{x}_n$ denotes the derivative of $x_n(\theta)$ with respect to θ. The initial function $x_0(\theta)$, $\theta \in [0,1]$ corresponds to the initial condition of delay differential Eq. (1a).

In a similar manner, one can rewrite Eqs. (1b) and (2). Hereafter, we use only the variational Eq. (2). In the mapping notion, it takes the form

$$\frac{1}{\tau}\dot{\Delta}_n = f'(x_{n-1})\Delta_{n-1} - (K+1)\Delta_n \tag{8}$$

with the boundary condition $\Delta_n(0) = \Delta_{n-1}(1)$. Equation (3) defining the maximal transverse Lyapunov exponent can be rewritten as

$$\lambda\tau = \lim_{n\to\infty} \frac{1}{n} \ln \frac{\left\{ \int_0^1 \Delta_n^2(\theta)\,d\theta \right\}^{1/2}}{\left\{ \int_0^1 \Delta_0^2(\theta)\,d\theta \right\}^{1/2}}. \tag{9}$$

Equations (7) and (8) can be easily integrated, since they are linear with respect to x_n and Δ_n, respectively. Taking into account the boundary conditions, we obtain the mapping expressions in the integral form,

$$x_n(\theta) = \tau \int_0^\theta f[x_{n-1}(\theta')] e^{\tau(\theta'-\theta)}\,d\theta' + x_{n-1}(1)e^{-\tau\theta}, \tag{10}$$

$$\Delta_n(\theta) = \tau \int_0^\theta f'[x_{n-1}(\theta')]\Delta_{n-1}(\theta') e^{\tau(K+1)(\theta'-\theta)}\,d\theta' + \Delta_{n-1}(1)e^{-\tau(K+1)\theta}. \tag{11}$$

These expressions are exact. Now we take advantage of the large delays. For large τ, the exponents in the integrals of Eqs. (10) and (11) have sharp peaks at the upper limit $\theta' = \theta$. Moving in front of the integrals the corresponding functions evaluated at the point $\theta' = \theta$ and integrating the exponents, we derive approximate expressions

$$x_n(\theta) = f[x_{n-1}(\theta)](1 - e^{-\tau\theta}) + x_{n-1}(1)e^{-\tau\theta}, \tag{12}$$

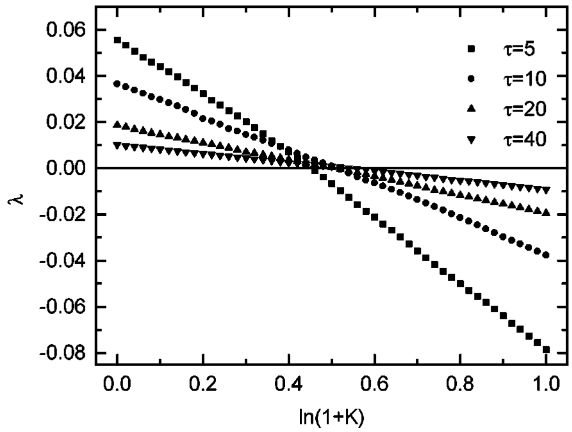

FIG. 3. Maximal transverse Lyapunov exponent λ vs ln(1+K) for different values of delay time τ.

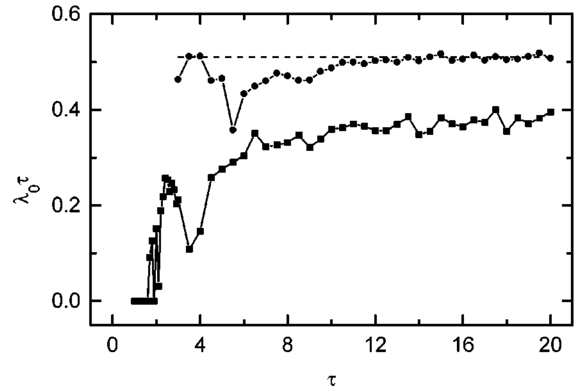

FIG. 4. The dependence of the product $\lambda_0\tau$ on delay time τ. The squares correspond to the exact values obtained from Eqs. (9), (10), and (11). The circles correspond to approximate Eqs. (9), (12), and (13). The dashed line corresponds to Eqs. (17), (14), and (15) or Eqs. (19) and (14) obtained in a singular limit $\tau\to\infty$.

$$\Delta_n(\theta)=\frac{1}{K+1}f'[x_{n-1}(\theta)]\Delta_{n-1}(\theta)(1-e^{-\tau(K+1)\theta})+\Delta_{n-1}(1)e^{-\tau(K+1)\theta}. \tag{13}$$

Iterating these maps for arbitrary initial functions $x_0(\theta)$, $\Delta_0(\theta)$, $\theta\in[0,1]$ we obtain continuous-time solutions $x(t)$ and $\Delta(t)$ for any $t>0$. For large τ, there is a thin layer of time $\theta\propto 1/\tau$ near the boundary $\theta=0$ in which $x_n(\theta)$ and $\Delta_n(\theta)$ change rapidly, and the boundary conditions $x_n(0)=x_{n-1}(1)$ and $\Delta_n(0)=\Delta_{n-1}(1)$ are satisfied. In a singular limit $\tau\to\infty$, the thickness of this layer vanishes and the solutions $x(t)$, $\Delta(t)$ become discontinuous at the points $t=n\tau$. In this limit, Eqs. (12) and (13) reduce to

$$x_n(\theta)=f[x_{n-1}(\theta)], \tag{14}$$

$$\Delta_n(\theta)=\frac{1}{K+1}f'[x_{n-1}(\theta)]\Delta_{n-1}(\theta). \tag{15}$$

Formally, these equations can be derived from Eqs. (7) and (8) by setting derivatives $\dot{x}_n$ and $\dot{\Delta}_n$ equal to zero. Using these equations one can find a relationship between the transverse Lyapunov exponent $\lambda(K)$ and the maximal Lyapunov exponent $\lambda_0=\lambda(0)$ of uncoupled driving and response subsystems. From Eq. (15) it follows

$$|\Delta_n(\theta)|=\frac{|\Delta_0|}{(K+1)^n}\exp\left(\sum_{i=0}^{n-1}\ln|f'[x_i(\theta)]|\right).$$

Substituting this in Eq. (9), we derive the desired relationship

$$\lambda(K)=\lambda_0-\frac{1}{\tau}\ln(K+1), \tag{16}$$

where

$$\lambda_0=\lim_{n\to\infty}\frac{1}{n\tau}\times\ln\frac{\left\{\int_0^1\exp\left(2\sum_{i=0}^{n-1}\ln|f'[x_i(\theta)]|\right)\Delta_0^2(\theta)d\theta\right\}^{1/2}}{\left\{\int_0^1\Delta_0^2(\theta)d\theta\right\}^{1/2}}. \tag{17}$$

To confirm this relationship numerically, we calculated the λ vs ln(K+1) dependence for various values of delay time τ using exact Eqs. (9), (10), and (11) (Fig. 3). For large τ, this dependence becomes a straight line with the slope equal to $-1/\tau$.

Equation (16) leads to a simple expression for the synchronization threshold

$$K_0=\exp(\lambda_0\tau)-1, \tag{18}$$

which shows that the synchronization condition is determined by the maximal Lyapunov exponent λ_0 of uncoupled driving and response subsystems rather than the whole spectrum of positive Lyapunov exponents. Besides, this expression explains why the synchronization threshold K_0 saturates to a finite value when τ is increased. The reason is that for large τ, the maximal Lyapunov exponent λ_0 decays as $1/\tau$ and the product $\lambda_0\tau$ saturates to a finite value. Figure 4 shows the dependence of this product on τ calculated from the exact Eqs. (9), (10), and (11), the approximate Eqs. (9), (12), and (13), and in a singular limit defined by Eqs. (17), (14), and (15).

The last estimate based on Eq. (17) can be simplified. According to Eq. (14), different points of function $x_n(\theta)$ transform independently. Thus for a fixed value of θ, Eq. (14) can be considered as a conventional one-dimensional map that transforms the points rather than functions. The Lyapunov exponent

$$\tilde{\lambda}_0 = \lim_{n\to\infty} \frac{1}{n\tau} \sum_{i=0}^{n-1} \ln|f'[x_i(\theta)]| \tag{19}$$

of this map is independent of initial condition $x_0(\theta)$, and hence of θ. For large n, the sum $\Sigma_{i=0}^{n-1} \ln|f'[x_i(\theta)]|$ in Eq. (17) can be replaced by $\tilde{\lambda}_0 n\tau$. This leads to the equality $\lambda_0 = \tilde{\lambda}_0$. Thus in a singular limit $\tau\to\infty$, the maximal Lyapunov exponent λ_0 can be calculated from Eq. (19) as the Lyapunov exponent of the one-dimensional map (14). We checked this statement numerically by calculating the product $\lambda_0\tau$ from Eq. (17) for different initial functions $x_0(\theta)$, and the product $\tilde{\lambda}_0\tau$ from Eq. (19) for different initial conditions x_0. In all cases the result was the same $\lambda_0\tau \approx \tilde{\lambda}_0\tau \approx 0.51$. Substituting this into Eq. (18) we obtain the estimate of the synchronization threshold $K_0 \approx 0.67$. For comparison with the numerical results of Sec. II, this value is shown in Fig. 2 by a dashed line. A good quantitative agreement is obtained for $\tau > 5$. Thus the analytical relations [Eqs. (16) and (18)], obtained in a singular limit $\tau\to\infty$, are valid starting from $\tau\approx 5$.

V. CONCLUSIONS

We have developed two analytical approaches to define the synchronization condition for one-way coupled time-delay systems and illustrated them for the specific example of coupled Mackey-Glass equations. Our analysis shows that the synchronization is possible (by transmitting a single scalar variable) for an arbitrarily large number of positive Lyapunov exponents of uncoupled driving and response subsystems. Moreover, the synchronization condition is determined by the maximal Lyapunov exponent, rather than the whole spectrum of positive Lyapunov exponents. In the limit of a large delay time, we have derived an analytical expression relating the synchronization threshold to the maximal Lyapunov exponent of uncoupled subsystems. This result is useful for constructing synchronized time-delay systems with a possible application to secure communication. Another possible application of this result is related to an experimental diagnosis of time-delay systems. Using the idea of system synchronization with its prerecorded history [2] and the relationship between the maximal Lyapunov exponent and the synchronization threshold, one can experimentally measure the maximal Lyapunov exponent in the following way. The system's output must be collected in memory and used to synchronize the current and past states of the system. Varying the coupling strength, one can determine the synchronization threshold and reconstruct the maximal Lyapunov exponent.

Note that the approach based on the transverse Lyapunov exponent calculated along the chaotic trajectory (Sec. IV) gives just a necessary synchronization condition and does not guarantee a high-quality synchronization [11]. If there are transversally unstable periodic orbits or fixed points embedded in a chaotic set of synchronized motions, even very small disturbances from noise or inaccuracies from parameter mismatch can cause synchronization to break down and lead to substantial amplitude excursion from the synchronized state. These brief desynchronization events may be undesirable in some applications. By contrast, the approach based on the second Lyapunov method (Sec. III) gives a sufficient but not a necessary condition of synchronization [12]. It assures that all transverse perturbations decay without transient growth for all time. Although this method gives a rather rough estimate of the synchronization threshold, it guarantees a high-quality synchronization that is stable to perturbations caused by noise or slight parameter mismatch.

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