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Macroscopic equations of motion for two-phase flow in porous media

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The usual macroscopic equations of motion for two-phase immiscible displacement in porous media are known to be physically incomplete because they do not contain the surface tension and surface areas governing capillary phenomena. Therefore, a more general system of macroscopic equations is derived here that incorporates the spatiotemporal variation of interfacial energies. These equations are based on the theory of mixtures in macroscopic continuum mechanics. They include wetting phenomena through surface tensions instead of the traditional use of capillary pressure functions. Relative permeabilities can be identified in this approach that exhibit a complex dependence on the state variables. A capillary pressure function can be identified in equilibrium that shows the qualitative saturation dependence known from experiment. In addition, the proposed equations include a description of the spatiotemporal changes of residual saturations during immiscible displacement. [S1063-651X(98)12507-2]

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I. INTRODUCTION

An aquifer or a petroleum reservoir may be seen as mixture of porous rocks and soil with various gases and liquids, usually residing at depths that prohibit precise measurements on the constituents [1,2]. Despite many years of research and the fact that the microscopic laws of fluid dynamics are well known, the prediction of macroscopic multiphase flow in porous media has not met with success (see [3–5] for recent overviews).

Microscopic and macroscopic descriptions of multiphase fluid flow in porous media differ considerably from each other and both have their own characteristic problems [1,2,4]. A microscopic description fails because it is generally impossible and not interesting to know the detailed microstructure and flow patterns on the pore scale. In contrast herewith the generally accepted macroscopic models are incomplete. Owing to its widespread importance in applied problems, the transition from the microscale to the macroscale for two-phase flow in porous media has attracted considerable interest from physicists in recent years [6–22]. Relatively little or no attention has been focused, however, on the continuum limit of the proposed microscopic models. Effective macroscopic equations of motion used by engineers are based on generalizations of Darcy's law, although every practitioner knows that they are incomplete [23,1]. Microscopic models by physicists often attempt to predict the relative permeabilities and the capillary pressure functions for simple microstructures, although it is clear that they are highly nonunique.

Detailed and rigorous derivations of macroscopic equations from microscopic ones is usually the last step in understanding a physical phenomenon. Experimental observations and tests of phenomenological macroscopic models usually precede the microscopic understanding. It is therefore of interest to improve the current macroscopic description that is known to fail even for simple laboratory systems [24].

Given the need for more predictive macroscopic theories, this paper proposes a different set of macroscopic equations. Loosely speaking, the traditional theory includes only mass and momentum balance but fails to incorporate the energy balance. Obviously the interfacial energy depends on the interfacial area and hence the set of macroscopic observables describing the state of the mixture must be enlarged to include surface areas. Relative permeabilities and capillary pressures must then be identified within the proposed framework. I shall begin the discussion by presenting briefly the traditional formulation and discussing its problems in Sec. II. Afterward, the macroscopic balance laws for mass, momentum, and energy (Sec. III) will be combined with constraints and constitutive relations to introduce macroscopic equations of motion (Sec. IV) that differ from the conventional equations. My presentation will conclude with a discussion of the consequences, the identification of relative permeabilities and capillary pressures, and a brief summary in Sec. VI.

II. PROBLEMS WITH EXISTING MACROSCOPIC EQUATIONS

Consider simultaneous flow of two incompressible and immiscible fluid phases denoted generically as water (subscript $\mathcal{W}$) and oil (subscript $\mathcal{O}$) inside a porous medium. The accepted and widely used macroscopic equations of motion [1,4,18,25] are based on mass conservation

$$\phi \frac{\partial S_{\mathcal{W}}}{\partial t} = -\nabla \cdot \overline{\mathbf{v}}_{\mathcal{W}}, \tag{1}$$

$$\phi \frac{\partial S_{\mathcal{O}}}{\partial t} = -\nabla \cdot \overline{\mathbf{v}}_{\mathcal{O}},$$

$$S_{\mathcal{W}} + S_{\mathcal{O}} = 1 \tag{2}$$

and the well known generalization of Darcy's law [1,25]

$$\overline{\mathbf{v}}_{\mathcal{W}} = -\left[\frac{\mathbf{K}k^{r}_{\mathcal{W}}}{\mu_{\mathcal{W}}}(\boldsymbol{\nabla}P_{\mathcal{W}} - \rho_{\mathcal{W}}g\boldsymbol{\nabla}Z)\right],$$
$$\overline{\mathbf{v}}_{\mathcal{O}} = -\left[\frac{\mathbf{K}k^{r}_{\mathcal{O}}}{\mu_{\mathcal{O}}}(\boldsymbol{\nabla}P_{\mathcal{O}} - \rho_{\mathcal{O}}g\boldsymbol{\nabla}Z)\right], \tag{3}$$

all of which are assumed to be valid on length scales large compared to a typical pore diameter. The variables in these equations are the macroscopic pressure fields of water and oil, denoted as $P_{\mathcal{W}}$ and $P_{\mathcal{O}}$, the water and oil saturations $S_{\mathcal{W}}$ and $S_{\mathcal{O}}$, and the macroscopic (Darcy) velocity fields $\overline{\mathbf{v}}_{\mathcal{W}}$ and $\overline{\mathbf{v}}_{\mathcal{O}}$. The saturation is defined as the ratio of water (oil) volume to pore space volume. Saturations are averages over a macroscopic region much larger than the pore size, but much smaller than the system size. Their arguments are the macroscopic space and time variables $(\mathbf{x},t)$. The porosity ϕ is the ratio of pore volume to total volume of the medium. The fluid viscosities and densities are denoted by μ and ρ. The acceleration of gravity is g and the function $Z(\mathbf{x})$ is the depth function. In the equations above $\mathbf{K}$ stands for the absolute (single-phase flow) permeability tensor, $k^{r}_{\mathcal{W}}$ is the relative permeability for water, and $k^{r}_{\mathcal{O}}$ is the relative permeability for oil. The equations above are closed by defining the capillary pressure as

$$P_c(\mathbf{x},t) = P_{\mathcal{O}}(\mathbf{x},t) - P_{\mathcal{W}}(\mathbf{x},t) \tag{4}$$

and by postulating for it the constitutive relationship

$$P_c(\mathbf{x},t) = P_c(S_n(\mathbf{x},t)), \tag{5}$$

where

$$S_n(\mathbf{x},t) = \frac{S_{\mathcal{W}}(\mathbf{x},t) - S_{\mathcal{W}i}}{1 - S_{\mathcal{W}i} - S_{\mathcal{O}r}} \tag{6}$$

is a normalized saturation. The water saturation obeys $S_{\mathcal{W}i} < S_{\mathcal{W}} < 1 - S_{\mathcal{O}r}$, where the two numbers $0 \leqslant S_{\mathcal{W}i}$, $S_{\mathcal{O}r} \leqslant 1$ are two parameters representing the irreducible water saturation $S_{\mathcal{W}i}$ and the residual oil saturation $S_{\mathcal{O}r}$. The residual oil saturation gives the amount of oil remaining in a porous medium after a water flood. The normalized saturation S_n varies between 0 and 1 as $S_{\mathcal{W}}$ varies between $S_{\mathcal{W}i}$ and $1 - S_{\mathcal{O}r}$.

Next it is argued [26,1] that each fluid flows in flow channels given by the solid matrix and the presence of the other fluid. One postulates that to each saturation there corresponds a unique configuration of flow channels. This picture is then formalized into the constitutive assumption that the relative permeabilities

$$k^{r}_{\mathcal{W}}(\mathbf{x},t) = k^{r}_{\mathcal{W}}(S_n(\mathbf{x},t)), \tag{7}$$

$$k^{r}_{\mathcal{O}}(\mathbf{x},t) = k^{r}_{\mathcal{O}}(S_n(\mathbf{x},t)) \tag{8}$$

are functions of saturation alone.

Although these equations are widely accepted and almost universally applied in numerical reservoir simulation [27] or aquifer modeling [28], they must be considered physically incomplete. Even if the picture of flow channels is accepted it is clear that the resulting permeability depends not only on saturation (i.e., volume fraction) but also on the surface area with other fluid or solid phases. This experimental fact is well known from observations of single-phase flow in porous media [26,1].

The second basic problem with the macroscopic equations of motion (1)–(8) arises from the experimental observation that the parameters $S_{\mathcal{O}r}$ and $S_{\mathcal{W}i}$ are neither constant nor known in advance for a given experiment. Instead they vary with space and time and depend on the flow conditions in the experiment [29–37]. More precisely, the residual oil saturation depends on the microscopic capillary number $\mathrm{Ca} = \mu_{\mathcal{W}}v/\sigma_{\mathcal{O}\mathcal{W}}$, where v is a typical flow velocity and $\sigma_{\mathcal{O}\mathcal{W}}$ is the surface tension between the two fluids. The dependence $S_{\mathcal{O}r}(\mathrm{Ca})$ is called a capillary desaturation curve. The measured values of $S_{\mathcal{O}r}$ are known to depend also on the duration of the flood. Capillary desaturation experiments contradict clearly to the assumption that the functions $k^{r}_{\mathcal{W}}(S_{\mathcal{W}})$, $k^{r}_{\mathcal{O}}(S_{\mathcal{W}})$, and $P_c(S_{\mathcal{W}})$ depend only on saturation. Instead they depend also on velocity [38,39] and pressure. Hence they depend on the solution and cannot be considered to be constitutive relations characterizing the system. Therefore, the system of equations of motion must be considered to be incomplete.

III. MACROSCOPIC CONSERVATION LAWS

The basic idea of this paper is to include the energy balance involving the macroscopic interfacial energy into the equations of motion. As the interfacial tensions do not appear explicitly in Eqs. (1)–(8) it is clear that this requires an enlarged description of the macroscopic state. While previously the macroscopic state of the fluids was described by saturations, pressures, and velocities it must now include macroscopic interfacial areas per unit volume. The necessity to include interfacial areas in a macroscopic description is also stressed by volume averaging approaches [40–42]. The main difference between volume averaging techniques and the approach of the present paper is that the present approach is based purely on macroscopic mixture theory and makes no reference to the underlying microscopic equations.

To discuss mass, energy, and momentum balance laws in a unified manner consider first a general conserved quantity Ψ defined per unit mass. Let $\boldsymbol{\tau}$ denote the flux of this conserved quantity across a surface of a material control region $\mathbb{G}(t)$ in $\mathbb{R}^3$. The general multiphase conservation law in the theory of mixtures reads in differential form [43,44]

$$\frac{D^{\alpha}}{Dt}(\phi_{\alpha}\rho_{\alpha}\Psi_{\alpha}) + \phi_{\alpha}\rho_{\alpha}\Psi_{\alpha}\boldsymbol{\nabla}\cdot\mathbf{v}_{\alpha} - \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_{\alpha} - \phi_{\alpha}\rho_{\alpha}g_{\alpha} = e_{\alpha}. \tag{9}$$

Here the subscript α refers to the different phases of the mixture,

$$D^{\alpha}/Dt = \partial/\partial t + \mathbf{v}_{\alpha}\cdot\boldsymbol{\nabla} \tag{10}$$

denotes the material derivative, $\mathbf{v}_{\alpha}$ is the velocity, ρ_{α} is the mass density, and ϕ_{α} is the volume fraction of phase α. Of course

$$\sum_{\alpha}\phi_{\alpha} = 1 \tag{11}$$

and introducing the volume fraction of the void space, i.e., the porosity,

$$\phi = \sum_{\alpha(\neq \mathrm{solid})} \phi_\alpha, \tag{12}$$

allows one to define saturations S_α of the fluid phases through

$$\phi_\alpha = \phi S_\alpha. \tag{13}$$

In Eq. (9) g_α denotes the external supply of Ψ. e_α represents the interphase transfer of the conserved quantity Ψ into phase α from all other phases, that is,

$$e_\alpha = \sum_\beta e_{\alpha\beta}, \tag{14}$$

where $e_{\alpha\beta}$ is the transfer of Ψ from phase β into phase α. Of course global conservation of Ψ would require that the summation of Eq. (9) gives zero, $\Sigma_\alpha e_\alpha = 0$, but this will not be required here. This general formalism can now be applied to mass, momentum, and energy conservation.

For mass conservation $\Psi_\alpha = 1$, $\boldsymbol{\tau}_\alpha = 0$, $g_\alpha = 0$, and $e_\alpha = M_\alpha$ is the transfer of mass from all other phases into phase α. Inserting into Eq. (9) yields

$$\frac{\partial}{\partial t}(\phi_\alpha \rho_\alpha) + \boldsymbol{\nabla} \cdot (\phi_\alpha \rho_\alpha \mathbf{v}_\alpha) = M_\alpha, \tag{15}$$

as usual.

For momentum conservation $\Psi_\alpha = \mathbf{v}_\alpha$ is the velocity, $\boldsymbol{\tau}_\alpha = \Sigma_\alpha$ is the stress tensor, $g_\alpha = \mathbf{b}_\alpha$ is the body force, and $e_\alpha = \mathbf{m}_\alpha$ is the momentum transfer from all other phases into phase α. Inserting into Eq. (9) and using mass balance now yields

$$\phi_\alpha \rho_\alpha \frac{D^\alpha \mathbf{v}_\alpha}{Dt} - \boldsymbol{\nabla} \cdot \Sigma_\alpha - \phi_\alpha \rho_\alpha \mathbf{b}_\alpha = \mathbf{m}_\alpha - \mathbf{v}_\alpha M_\alpha, \tag{16}$$

where the divergence $\boldsymbol{\nabla} \cdot \Sigma_\alpha$ of the stress tensor is a vector whose ith coordinate is $\Sigma_j \partial(\Sigma_\alpha)_{ij}/\partial x_j$.

For energy conservation only the kinetic energy and the interfacial energies are considered, while thermal effects will be ignored. Hence, in this case

$$\Psi_\alpha = \frac{1}{2} \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha + \frac{1}{2} \sum_\beta \frac{\sigma_{\alpha\beta} A_{\alpha\beta}}{\phi_\alpha \rho_\alpha + \phi_\beta \rho_\beta}, \tag{17}$$

where $\sigma_{\alpha\beta}$ is the surface tension between phases α and β and $A_{\alpha\beta}$ is the specific internal surface of the $\alpha\beta$ interface. The denominator in the interfacial energy terms (giving the $\alpha\beta$ density) ensures the same normalization with respect to mass and volume as for the kinetic energy term. The factor $\frac{1}{2}$ in front of it attributes the interfacial energy between phases α and β equally to the two phases. This assumption will be relaxed in a future work [45]. Proceeding with this assumption, the energy flux becomes

$$\boldsymbol{\tau}_\alpha = \Sigma_\alpha \cdot \mathbf{v}_\alpha - \mathbf{q}_\alpha, \tag{18}$$

where $\Sigma_\alpha \cdot \mathbf{v}_\alpha$ is the work done by stress and $\mathbf{q}_\alpha$ is the flux of surface energy. The external energy supply is similarly split,

$$g_\alpha = \mathbf{b}_\alpha \cdot \mathbf{v}_\alpha + h_\alpha, \tag{19}$$

into the work $\mathbf{b}_\alpha \cdot \mathbf{v}_\alpha$ done by body forces and the external supply h_α of interfacial energy. Finally, the energy exchange from all phases into phase α is $e_\alpha = E_\alpha$.

Inserting Eqs. (17)–(19) into the general balance law (9) gives the energy balance for immiscible displacement as

$$\frac{1}{2}\left(\boldsymbol{\nabla} \cdot \mathbf{v}_\alpha + \frac{D^\alpha}{Dt}\right)\left(\phi_\alpha \rho_\alpha \sum_\beta \frac{\sigma_{\alpha\beta} A_{\alpha\beta}}{\phi_\alpha \rho_\alpha + \phi_\beta \rho_\beta}\right) + \boldsymbol{\nabla} \cdot \mathbf{q}_\alpha - \Sigma_\alpha : \boldsymbol{\nabla} \mathbf{v}_\alpha - \phi_\alpha \rho_\alpha h_\alpha = E_\alpha - \mathbf{m}_\alpha \cdot \mathbf{v}_\alpha + M_\alpha \frac{\mathbf{v}_\alpha^2}{2}, \tag{20}$$

where

$$\Sigma_\alpha : \boldsymbol{\nabla} \cdot \mathbf{v}_\alpha = \sum_{i=1}^{d} \sum_{j=1}^{d} (\Sigma_\alpha)_{ij} \frac{\partial(\mathbf{v}_\alpha)_j}{\partial x_i}. \tag{21}$$

Equations (17) and (20) embody the idea to include interfacial areas and energies into the macroscopic energy balance. This is the first step beyond the traditional approach described, e.g., in [1,26,43].

The second step beyond the traditional theory consists in splitting each fluid phase into a connected (percolating) and a disconnected (nonpercolating) subphase. The idea here is that such a division is necessary to handle the dynamics of residual phase saturations. During an immiscible displacement process a finite fraction of each fluid phase becomes trapped in the pores. Once trapped the droplets are immobile and fixed through large local capillary forces. Only at very high flow rates can such trapped droplets be mobilized. Therefore, the second main idea of the present theory is to treat these droplets as an individual phase interacting with the connected parent phase mainly via mass exchange.

Consider now the simplest case of two immiscible fluids (called oil and water as before) flowing inside a rigid porous medium. Dividing each of the two fluid phases into two subphases, one has a total of five phases. Each fluid superphase is divided into a percolating (connected) and a trapped (disconnected) subphase. Here percolating means that each point in the subset occupied by the percolating subphase can be connected to the exterior boundaries of the sample by a path within the subphase. The disconnected subphase is the complement within the chosen superphase. The connected (percolating) subphase can also be defined as the region inside of which an external applied pressure gradient can propagate. Therefore, the phase index α above can assume five values summarized in Table I.

IV. CONSTITUTIVE RELATIONS AND CONSTRAINTS

This section complements the basic balance laws with geometrical and physical constraints and with constitutive relations. The fluids are assumed to be incompressible

$$\rho_1(\mathbf{x},t) = \rho_2(\mathbf{x},t) = \rho_{\mathcal{W}} = \text{const}, \tag{22}$$

$$\rho_3(\mathbf{x},t) = \rho_4(\mathbf{x},t) = \rho_{\mathcal{O}} = \text{const} \tag{23}$$

TABLE I. Overview of the subphases and their indices for immiscible displacement of two fluids inside a rigid porous medium.

Phase index α	Phase description
1	connected (percolating) water phase
2	disconnected (trapped) water phase
3	connected (percolating) oil phase
4	disconnected (trapped) oil phase
$\mathcal{R}$	rigid porous rock phase

and their stress tensors are approximated as

$$\Sigma_\alpha = -\phi_\alpha P_\alpha \mathbf{1} \tag{24}$$

for all $\alpha \in \{1,2,3,4\}$, where $\mathbf{1}$ is the unit tensor. In Eq. (24) it is assumed that all shear stresses in the fluid are negligible and that P_α are the true pressures in the fluids. It will be assumed that the only body force is gravity, $\mathbf{b}_\alpha = g\boldsymbol{\nabla} Z$, where g is the acceleration of gravity and $Z(\mathbf{x})$ is the depth function. In addition, inertial effects will be neglected, $D^\alpha \mathbf{v}_\alpha / Dt = 0$, for all phases. All of these assumptions are also made in the derivation of the traditional equations described in Sec. III [1,26,43].

It will be assumed that there are no external sources or sinks of surface energy, $g_\alpha = 0$. Next it is assumed that $\mathbf{q}_\alpha = \mathbf{0}$, i.e., there is no diffusive flux of surface energy during the time scale of observation. A crucial assumption is that the exchange of interfacial energy between the phases is dominated by separation and coalescence of disconnected subphase droplets from/with their connected parent phase, i.e., through mass transfer, and that all other mechanisms are negligible. This amounts to setting $E = 0$.

The porous matrix as well as the disconnected (trapped) subphases are immobile and hence

$$\mathbf{v}_{\mathcal{R}} = \mathbf{v}_2 = \mathbf{v}_4 = \mathbf{0}. \tag{25}$$

The interfacial area between a disconnected subphase and its superphase vanishes and the same holds for the contact between the two disconnected subphases. Hence

$$A_{12} = A_{34} = A_{24} = 0. \tag{26}$$

Of course $A_{\alpha\alpha} = 0$ and symmetry $A_{\alpha\beta} = A_{\beta\alpha}$ holds.

The disconnected fluid phases have macroscopic pressures $P_2(\mathbf{x},t)$ and $P_4(\mathbf{x},t)$ obtained by averaging the pressure in the disconnected drops and droplets. The resulting functions $P_2(\mathbf{x},t)$ and $P_4(\mathbf{x},t)$, however, are generally not continuous and hence not differentiable.

The wetting properties are assumed to be constant and they are specified by the requirements that the individual interface areas with the rock are constant, i.e.,

$$A_{1\mathcal{R}}(\mathbf{x},t) = A_{1\mathcal{R}} = \text{const}, \tag{27}$$

$$A_{2\mathcal{R}}(\mathbf{x},t) = A_{2\mathcal{R}} = \text{const}, \tag{28}$$

$$A_{3\mathcal{R}}(\mathbf{x},t) = A_{3\mathcal{R}} = \text{const}, \tag{29}$$

$$A_{4\mathcal{R}}(\mathbf{x},t) = A_{4\mathcal{R}} = \text{const}. \tag{30}$$

Of course $A = A_{1\mathcal{R}} + A_{2\mathcal{R}} + A_{3\mathcal{R}} + A_{4\mathcal{R}}$ is the total specific internal surface of the porous medium. The ratio

$$W_{\mathcal{W}} = \frac{A_{1\mathcal{R}} + A_{2\mathcal{R}}}{A} \tag{31}$$

varies from 1 to 0 as the medium varies from water wet to oil wet.

Finally, it remains to specify the momentum exchanges $\mathbf{m}_\alpha = \Sigma_\beta \mathbf{m}_{\alpha\beta}$ and the mass exchanges $M_\alpha = \Sigma_\beta M_{\alpha\beta}$. In both cases $\mathbf{m}_{\alpha\alpha} = 0$ and $M_{\alpha\alpha} = 0$ will be used. The momentum exchange is conventionally modeled [43] through Stokes drag in the form

$$\mathbf{m}_{\alpha\beta} = R_{\alpha\beta}(\mathbf{v}_\beta - \mathbf{v}_\alpha) = \frac{\phi_\alpha}{\Lambda_{\alpha\beta}} (\mathbf{v}_\beta - \mathbf{v}_\alpha), \tag{32}$$

where the mobility $\Lambda_{\alpha\beta} = k_{\alpha\beta}/\mu_{\alpha\beta}$ is expressed as usual in terms of the permeability $k_{\alpha\beta}$ and the viscosity $\mu_{\alpha\beta}$. The mobilities $\Lambda_{\alpha\beta}$ can now be modeled through

$$\Lambda_{\alpha\beta} = \frac{k_{\alpha\beta}}{\mu_{\alpha\beta}} = \frac{\phi_\alpha^3}{\mu_\alpha A_{\alpha\beta}^2}, \tag{33}$$

which is analogous to the Carman-Kozeny relation for absolute permeabilities. This assumption reflects the traditional view that each fluid flows in its own channel. If there is no common interface, then it is not possible to exchange momentum; hence

$$\mathbf{m}_{12} = \mathbf{m}_{34} = \mathbf{m}_{24} = \mathbf{0} \tag{34}$$

completes the specification of the momentum exchange.

The rigid porous matrix has a volume fraction $1 - \phi = \text{const}$ that is assumed to be spatially and temporally constant. The rigid matrix is dynamically inert [see Eq. (25)] and hence only the balance laws for the fluid phases will appear below [46].

Chemical reactions between oil, water, and rock are excluded and hence

$$M_{13} = M_{14} = M_{23} = M_{24} = M_{1\mathcal{R}} = M_{2\mathcal{R}} = M_{3\mathcal{R}} = M_{4\mathcal{R}} = 0. \tag{35}$$

Mass transfer occurs within each fluid phase between its connected and the disconnected subphases. These nonvanishing mass transfers are

$$M_{\mathcal{W}} = M_{12} = -M_{21}, \tag{36}$$

$$M_{\mathcal{O}} = M_{34} = -M_{43}. \tag{37}$$

In general, the mass transfer may be a function of the velocities, saturations, and interface areas of the other phases. A precise form for the dependence $M_{\mathcal{W}} = M_{\mathcal{W}}(\mathbf{v}_1, A_{13}, S_1 S_3)$ is not needed here. Any concrete assumption for such a dependence would have to be checked by experiment.

V. RESULTS

Combining the balance laws with the constitutive equations gives the macroscopic equations of motion of the two pore fluids [47]. Mass conservation yields four equations

$$\rho_{\mathcal{W}}\phi\,\frac{\partial}{\partial t}\,S_1+\rho_{\mathcal{W}}\phi\boldsymbol{\nabla}\cdot(S_1\mathbf{v}_1)=M_{\mathcal{W}},\tag{38}$$

$$\rho_{\mathcal{W}}\phi\,\frac{\partial}{\partial t}\,S_2=-M_{\mathcal{W}},\tag{39}$$

$$\rho_{\mathcal{O}}\phi\,\frac{\partial}{\partial t}\,S_3+\rho_{\mathcal{O}}\phi\boldsymbol{\nabla}\cdot(S_3\mathbf{v}_3)=M_{\mathcal{O}},\tag{40}$$

$$\rho_{\mathcal{O}}\phi\,\frac{\partial}{\partial t}\,S_4=-M_{\mathcal{O}},\tag{41}$$

while momentum conservation gives only two equations

$$\boldsymbol{\nabla}(\phi_1P_1)-\phi_1\rho_{\mathcal{W}}g\boldsymbol{\nabla}Z=R_{13}\mathbf{v}_3-(R_{13}+R_{14}+R_{1\mathcal{R}}+M_{\mathcal{W}})\mathbf{v}_1,\tag{42}$$

$$\boldsymbol{\nabla}(\phi_3P_3)-\phi_3\rho_{\mathcal{O}}g\boldsymbol{\nabla}Z=R_{31}\mathbf{v}_1-(R_{31}+R_{32}+R_{3\mathcal{R}}+M_{\mathcal{O}})\mathbf{v}_3\tag{43}$$

because P_2 and P_4 are not differentiable. Energy conservation gives again four equations

$$\begin{aligned}\frac{1}{2}\left(\boldsymbol{\nabla}\cdot\mathbf{v}_1+\frac{D^1}{Dt}\right)&\left(\frac{\phi_1\rho_{\mathcal{W}}\sigma_{\mathcal{W}\mathcal{O}}A_{13}}{\rho_{\mathcal{W}}\phi_1+\rho_{\mathcal{O}}\phi_3}+\frac{\phi_1\rho_{\mathcal{W}}\sigma_{\mathcal{W}\mathcal{O}}A_{14}}{\rho_{\mathcal{W}}\phi_1+\rho_{\mathcal{O}}\phi_4}\right.\\&\left.+\frac{\phi_1\rho_{\mathcal{W}}\sigma_{\mathcal{W}\mathcal{R}}A_{1\mathcal{R}}}{\rho_{\mathcal{W}}\phi_1+\rho_{\mathcal{R}}(1-\phi)}\right)\\&=-\phi_1P_1\boldsymbol{\nabla}\cdot\mathbf{v}_1-R_{13}\mathbf{v}_1\cdot\mathbf{v}_3+(R_{13}+R_{14}+R_{1\mathcal{R}})\mathbf{v}_1^2\\&\quad+M_{\mathcal{W}}\,\frac{\mathbf{v}_1^2}{2},\end{aligned}\tag{44}$$

$$\frac{\partial}{\partial t}\left(\frac{\phi_2\rho_{\mathcal{W}}\sigma_{\mathcal{W}\mathcal{O}}A_{23}}{\rho_{\mathcal{W}}\phi_2+\rho_{\mathcal{O}}\phi_3}+\frac{\phi_2\rho_{\mathcal{W}}\sigma_{\mathcal{W}\mathcal{R}}A_{2\mathcal{R}}}{\rho_{\mathcal{W}}\phi_2+\rho_{\mathcal{R}}(1-\phi)}\right)=0,\tag{45}$$

$$\begin{aligned}\frac{1}{2}\left(\boldsymbol{\nabla}\cdot\mathbf{v}_3+\frac{D^3}{Dt}\right)&\left(\frac{\phi_3\rho_{\mathcal{O}}\sigma_{\mathcal{W}\mathcal{O}}A_{13}}{\rho_{\mathcal{W}}\phi_1+\rho_{\mathcal{O}}\phi_3}+\frac{\phi_3\rho_{\mathcal{O}}\sigma_{\mathcal{W}\mathcal{O}}A_{23}}{\rho_{\mathcal{W}}\phi_2+\rho_{\mathcal{O}}\phi_3}\right.\\&\left.+\frac{\phi_3\rho_{\mathcal{O}}\sigma_{\mathcal{O}\mathcal{R}}A_{3\mathcal{R}}}{\rho_{\mathcal{O}}\phi_3+\rho_{\mathcal{R}}(1-\phi)}\right)\\&=-\phi_3P_3\boldsymbol{\nabla}\cdot\mathbf{v}_3-R_{31}\mathbf{v}_1\cdot\mathbf{v}_3+(R_{31}+R_{32}+R_{3\mathcal{R}})\mathbf{v}_3^2\\&\quad+M_{\mathcal{O}}\,\frac{\mathbf{v}_3^2}{2},\end{aligned}\tag{46}$$

$$\frac{\partial}{\partial t}\left(\frac{\phi_4\rho_{\mathcal{O}}\sigma_{\mathcal{W}\mathcal{O}}A_{14}}{\rho_{\mathcal{W}}\phi_1+\rho_{\mathcal{O}}\phi_4}+\frac{\phi_4\rho_{\mathcal{O}}\sigma_{\mathcal{O}\mathcal{R}}A_{4\mathcal{R}}}{\rho_{\mathcal{O}}\phi_4+\rho_{\mathcal{R}}(1-\phi)}\right)=0.\tag{47}$$

The unknown fields in these equations are the four saturations S_α ($\alpha=1,2,3,4$) of the fluids related to ϕ_α by Eq. (13), the two pressures P_1 and P_3, the six components of the two velocities $\mathbf{v}_1$ and $\mathbf{v}_3$ of the connected fluids, and the three surface areas A_{13}, A_{14}, and A_{23}. Note that all $A_{\alpha,\mathcal{R}}$, $\alpha=1,2,3,4$, are constant by virtue of Eqs. (27)–(30). This is a total of 15 unknowns for which the condition $S_1+S_2+S_3+S_4=1$ [see Eq. (11)], together with Eqs. (38)–(47), forms a set of 15 equations.

A detailed analysis of these equations will be presented elsewhere [45]. Here only the simplest comparison with the traditional equations (1)–(8) will be discussed. In particular the question arises whether the present equations allow one to determine relative permeabilities and capillary pressure functions.

The capillary pressure, defined in Eq. (4), is an equilibrium property. In the traditional equations it is related to the saturation profile in complete gravitational equilibrium [26], i.e., when $\mathbf{v}_\alpha=\mathbf{0}$ and when all terms involving time derivatives are set to zero. Proceeding in this way, one finds that the resulting equations can be solved in the direction perpendicular to gravity to give

$$P_c(\mathbf{x},\infty)=P_3-P_1=\frac{c_3}{\phi S_3(\mathbf{x},\infty)}-\frac{c_1}{\phi S_1(\mathbf{x},\infty)},\tag{48}$$

where c_3 and c_1 are integration constants. Using $S_{\mathcal{W}}=S_1+S_2$ and $1-S_{\mathcal{W}}=S_3+S_4$ shows that in the directions perpendicular to gravity

$$P_c(\mathbf{x},\infty)=P_3-P_1=\frac{c_3}{\phi(1-S_4-S_{\mathcal{W}})}-\frac{c_1}{\phi(S_{\mathcal{W}}-S_2)}.\tag{49}$$

If it is assumed that S_2 and S_4 are spatially constant throughout the sample then it is tempting to identify them with $S_{\mathcal{W}i}$ and $S_{\mathcal{O}r}$ because this would imply the assumption (5) of the traditional formulation. Moreover, the capillary pressure would indeed have the familiar S-shaped form as a function of saturation. It should be noticed, however, that Eq. (49) holds only without flow, while assumption (5) is assumed to hold always. Therefore, in the present formulation the capillary pressure depends strongly on the flow regime. This prediction agrees with what is known from theory and experiment [38,39]. Notice also that in Eq. (49) the quantities S_2 and S_4 are state variables and hence the capillary pressure includes the spatiotemporal variability of the residual saturations.

The generalization of Darcy's law follows from the equations of momentum balance (42) and (43) by solving these equations for $\mathbf{v}_1$ and $\mathbf{v}_3$. This yields a generalized form of Darcy's law for two phase flow in the form

$$\begin{aligned}\mathbf{v}_1=k^r_{\mathcal{W}\mathcal{W}}\,\frac{k}{\mu_{\mathcal{W}}}\,&[\boldsymbol{\nabla}(\phi_1P_1)-\phi_1\rho_{\mathcal{W}}\mathbf{g}]\\&+k^r_{\mathcal{W}\mathcal{O}}\,\frac{k}{\mu_{\mathcal{O}}}\,[\boldsymbol{\nabla}(\phi_3P_3)-\phi_3\rho_{\mathcal{O}}\mathbf{g}],\end{aligned}\tag{50}$$

$$\begin{aligned}\mathbf{v}_3=k^r_{\mathcal{O}\mathcal{W}}\,\frac{k}{\mu_{\mathcal{W}}}\,&[\boldsymbol{\nabla}(\phi_1P_1)-\phi_1\rho_{\mathcal{W}}\mathbf{g}]\\&+k^r_{\mathcal{O}\mathcal{O}}\,\frac{k}{\mu_{\mathcal{O}}}\,[\boldsymbol{\nabla}(\phi_3P_3)-\phi_3\rho_{\mathcal{O}}\mathbf{g}],\end{aligned}\tag{51}$$

where the relative permabilities $k^r_{\alpha\beta}$ are defined by these equations. The $k^r_{\alpha\beta}$ are related to the resitivities $R_{\alpha\beta}$, which in turn are given by Eqs. (32) and (33).

This result differs from the traditional generalization of Darcy's law given in Eqs. (3) in three respects. First, Eqs. (50) and (51) includes off-diagonal coupling terms $k^r_{\mathcal{WO}}$ and $k^r_{\mathcal{OW}}$ that are missing in Eqs. (3). Second, Eqs. (50) and (51) include a driving force $\phi P_\alpha \boldsymbol{\nabla} S_\alpha$ resulting from the term $\boldsymbol{\nabla}(\phi_\alpha P_\alpha)$. The driving force is proportional to saturation gradients. Third, and perhaps most importantly, the relative permeabilities defined through Eqs. (50) and (51) depend not only on saturation as assumed in Eqs. (7) and (8) but on

$$k^r_{\alpha\beta} = k^r_{\alpha\beta}(R_{13}, R_{31}, R_{14}, R_{23}, R_{1\mathcal{R}}, R_{3\mathcal{R}}, M_{\mathcal{W}}, M_{\mathcal{O}}), \tag{52}$$

$$k^r_{\alpha\beta} = k^r_{\alpha\beta}(S_1, S_2, S_3, S_4, A_{13}, A_{14}, A_{32}, \partial S_2/\partial t, \partial S_4/\partial t, A_{1\mathcal{R}}, A_{3\mathcal{R}}) \tag{53}$$

by virtue of Eqs. (42), (43) and (32), (33), (39) and (41), respectively. This expresses a dependence on saturation, changes of saturation, disconnected (residual) saturations, interfacial area, and wetting properties ($A_{1\mathcal{R}}$ and $A_{3\mathcal{R}}$). Thus the present theory gives a much more detailed picture of relative permeabilities than the traditional assumptions (7) and (8).

VI. CONCLUSION

In conclusion, the present paper has discussed the problems associated with the macroscopic equations of motion for two-phase immiscible displacement in porous media. After discussion of the traditional formulation an alternative theory has been developed based on mixture theory. The present equations differ from the traditional ones in the following main aspects.

(i) The set of macroscopic state variables is enlarged to include specific surface area of the fluid phases.

(ii) The set of macroscopic balance laws is enlarged to include interfacial energy balance.

(iii) The theory allows a macroscopic quantification of the wetting properties through the surface areas.

(iv) The dynamics of the disconnected fluid phases is included. This allows one to describe the spatiotemporal behavior of residual saturations.

(v) The theory predicts a dynamic capillary pressure that, at least in complete gravitational equilibrium, shows the correct qualitative behavior as a function of saturation.

(vi) The relative permeabilities are found to depend on macroscopic state variables other than saturation. In particular they depend on wetting properties and the internal surfaces areas.

All of these results are consistent with experiment. More analysis of the equations is necessary to compare their solutions with experiment.

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[46] Because the rock matrix is assumed to be rigid and impermeable one has $R_{\mathcal{R}1}=R_{\mathcal{R}2}=R_{\mathcal{R}3}=R_{\mathcal{R}4}=\infty$, which implies dissipation. Because of this and Eq. (25) the balance laws for the matrix phase would either be trivial or ill defined.
[47] Balance equations for the rock matrix do not arise (see Ref. [46]).