Mode fluctuation distribution for spectra of superconducting microwave billiards

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High-resolution eigenvalue spectra of several two- and three-dimensional superconducting microwave cavities have been measured in the frequency range below 20 GHz and analyzed using a statistical measure that is given by the distribution of the normalized mode fluctuations. For chaotic systems the limit distribution is conjectured to show a universal Gaussian, whereas integrable systems should exhibit a non-Gaussian limit distribution. For the investigated Bunimovich stadium and the three-dimensional Sinai billiard we find that the distribution is in good agreement with this prediction. We study members of the family of limaçon billiards, having mixed dynamics. It turns out that in this case the number of approximately 1000 eigenvalues for each billiard does not allow us to observe significant deviations from a Gaussian, whereas an also measured circular billiard with regular dynamics shows the expected difference from a Gaussian. [S1063-651X(98)06008-5]

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I. INTRODUCTION

One of the main research lines in quantum chaos is to investigate the statistics of energy levels of quantum systems whose classical counterpart is chaotic. A very popular class of systems are Euclidean billiards, which are classically given by the free motion inside a domain $\Omega \subset \mathbf{R}^2$ with elastic reflections at the boundary $\partial\Omega$. The corresponding quantum billiard is described by the stationary Schrödinger equation ($\hbar = 2m = 1$)

$$\Delta \psi_n(\mathbf{q}) + k_n^2 \psi_n(\mathbf{q}) = 0 \quad \text{for} \quad \mathbf{q} \in \Omega \tag{1}$$

with Dirichlet boundary conditions $\psi_n(\mathbf{q}) = 0$ for $\mathbf{q} \in \partial\Omega$.

It has been conjectured that the energy level statistics of integrable systems can be described by a Poissonian random process [1], whereas classically strongly chaotic systems should obey the statistics of random matrix ensembles such as the Gaussian orthogonal ensemble or the Gaussian unitary ensemble [2–4]. This implies, for example, that the nearest-neighbor level spacing distribution is expected to show level repulsion for chaotic systems, in contrast to integrable systems, which are expected to show level attraction. Surprisingly, this means that the statistics of classically chaotic systems are much more rigid than those of integrable systems.

These conjectures have been tested successfully for several systems. However, there are exceptions for both integrable and chaotic systems. For example, the so-called arithmetic systems, which are strongly chaotic, are found to have a level spacing distribution showing level attraction similar to the Poissonian distribution [5–10]. Therefore, a different statistics, the distribution of the normalized mode fluctuations, has been proposed [11,12] as a possible signature of quantum chaos. This statistics was investigated in the integrable case for the eigenvalues of the Laplacian on a torus [13] and later in [14] the unnormalized fluctuations, possessing no limit distribution, have been studied for regular and chaotic billiards. For chaotic systems the limit distribution of the normalized mode fluctuations is conjectured [11,12] to show a universal Gaussian, whereas integrable systems should exhibit a non-Gaussian limit distribution. This conjecture was tested successfully for several regular and chaotic billiard systems in [9,12,15,16].

By using two-dimensional microwave cavities quantum billiards can be simulated experimentally [17–20]. This is possible because of the equivalence of the stationary Schrödinger equation for quantum billiards and the corresponding Helmholtz equation for electromagnetic resonators in two dimensions. In three dimensions the electromagnetic Helmholtz equation is vectorial and cannot be reduced to an effective scalar form. Thus it is structurally different from the scalar Schrödinger equation. Nevertheless, the applicability of the statistical concepts developed in the theory of quantum chaos and random matrix theory is also given for such three-dimensional systems [20–22]. Therefore, experiments with superconducting microwave resonators provide in general eigenvalue spectra of very high resolution for which an analysis of the distribution of the normalized mode fluctuations is interesting.

The paper is organized as follows. In Sec. II the mode fluctuation distribution is introduced. The experimental setup and the measurement of the eigenfrequencies using superconducting microwave resonators are described in Sec. III. In Sec. IV the analysis of the mode fluctuation distribution using the experimental data is carried out.

II. MODE FLUCTUATION DISTRIBUTION

The analysis of the eigenvalue spectrum starts with the spectral staircase function

$$N(k) = \#\{n \mid k_n \leqslant k\}, \tag{2}$$

which counts the number of energy levels below a given energy k. The mean behavior of $N(k)$ is given by the generalized Weyl law [23], which reads for two-dimensional billiards with Dirichlet boundary conditions

$$\overline{N}(k) = \frac{\mathcal{A}}{4\pi} k^2 - \frac{\mathcal{L}}{4\pi} k + \mathcal{C}, \tag{3}$$

where $\mathcal{A}$ is the area of the billiard, $\mathcal{L}$ is the length of the boundary, and $\mathcal{C}$ takes curvature and corner contributions into account. For three-dimensional electrodynamical billiards we have

$$\bar{N}(k) = \frac{\mathcal{V}}{3\pi^2}k^3 - \left(\frac{2}{3\pi^2}\int\frac{d\sigma}{\mathcal{R}} - \frac{1}{12\pi^2}\int da\,\frac{(\pi-\omega)(\pi-5\omega)}{\omega}\right)k + \text{const}, \tag{4}$$

where $\mathcal{V}$ is the volume of the billiard, $\mathcal{R}$ is the mean radius of the curvature over the surface σ, and ω is the dihedral angle along the edges a [24,25].

In order to obtain a spectrum that is independent of the system specific constants, one considers the unfolded spectrum $\{x_n := \bar{N}(k_n)\}$ [26,27]. Consequently, the unfolded energy spectrum has a mean level spacing of unity. The counting function for the unfolded spectrum will be denoted for simplicity again by $N(x)$. Thus the fluctuating part of the spectral staircase function is given by

$$N_{\rm fluc}(x) := N(x) - \bar{N}(x) = N(x) - x. \tag{5}$$

In the following we will assume that all spectra have been unfolded and that the systems are completely desymmetrized.

The normalized mode fluctuations are given by

$$W(x) := \frac{N_{\rm fluc}(x)}{\sqrt{D(x)}}, \tag{6}$$

where $D(x)$ is the variance

$$D(x) := \frac{\Xi(c)}{(c-1)x}\int_x^{cx}[N_{\rm fluc}(y)]^2 dy, \tag{7}$$

with $c>1$, and $\Xi(c)$ is a correction necessary for integrable systems to obtain for $W(x)$ a variance of one; see [16] for details. The conjecture put forth in [11,12] can be formulated as follows (see [16]) For bound conservative and scaling systems the quantity $W(x)$ [Eq. (6)] possesses a limit distribution for $x\to\infty$. This distribution is absolutely continuous with respect to the Lebesgue measure on the real line, with a density $P(W)$ defined by

$$\lim_{T\to\infty}\frac{1}{(c-1)T}\int_T^{cT} g(W(x))\rho(x/T)dx = \int_{-\infty}^{\infty} g(W)P(W)dW, \tag{8}$$

where $g(x)$ is a bounded continuous function and $\rho(t)\geqslant 0$ is a continuous density on $[1,c]$ with $[1/(c-1)]\int_1^c \rho(t)dt = 1$.

Furthermore, the limit distribution has zero mean and unit variance

$$\int_{-\infty}^{\infty} WP(W)dW = 0, \quad \int_{-\infty}^{\infty} W^2P(W)dW = 1. \tag{9}$$

Now the basic conjecture of [11,12] reads as follows. If the corresponding classical system is strongly chaotic, having only isolated and unstable periodic orbits, then $P(W)$ is universally a Gaussian,

$$P(W) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}W^2\right). \tag{10}$$

In contrast, a classically integrable system leads to a non-Gaussian density $P(W)$. For large classes of integrable systems it has been proved that the limit distribution is not Gaussian; see [16,28–30] and references therein.

For chaotic systems this conjecture has been tested numerically in [9,12,15] and experimentally in [31]. A review and a detailed comparison between chaotic and nonchaotic systems is given in [16]. A Gaussian distribution for chaotic systems corresponds to maximum randomness for the fluctuations, whereas the mode fluctuation distribution for integrable systems is less random [12].

Berry's semiclassical analysis [32] gives results for the asymptotic behavior of the saturation behavior of the spectral rigidity. In a similar way one can determine the asymptotic behavior of $D(x)$. For generic integrable billiards one has

$$D(x) \sim \text{const}\times\sqrt{x}. \tag{11}$$

For generic classically chaotic systems with antiunitary symmetry (e.g., time-reversal symmetry) one obtains

$$D(x) \sim \frac{1}{2\pi^2}\ln x + \text{const}. \tag{12}$$

As discussed in [16], one can extend the conjecture to such chaotic systems, for which $N_{\rm fluc}(k)$ is modulated by a long-range oscillation $N_{\rm long}(k)$. In this case one has to include the additional term in the unfolding process $\{x_n := \bar{N}(k_n) + N_{\rm long}(k_n)\}$. This procedure has been used, for instance, in the case of the truncated hyperbola billiard, where a prominent contribution to $N(k)$ is given by families of closed nonperiodic orbits running into a boundary point where the curvature is discontinuous [33]; see [12,16] for the result of $P(W)$ for this system.

The same is also necessary for the stadium billiard, where a family of marginally stable orbits, the bouncing ball orbits (BBOs) gives rise to a strong modulation [19,34]. Taking the contribution of the BBOs into account, one observes excellent agreement of $P(W)$ with the Gaussian normal distribution [16].

III. EXPERIMENT

For a precise test of the distribution of the normalized mode fluctuations an accurate measurement of the resonances of all investigated microwave billiards is necessary. Since 1991, we have experimentally studied several two- and three-dimensional systems using superconducting microwave resonators of niobium. In Fig. 1 the shapes of some measured billiards with their dimensions are shown. Altogether five desymmetrized two-dimensional systems are investigated, a $\gamma=1.8$ Bunimovich stadium billiard [19,35,36], a circular billiard (not desymmetrized), and three members

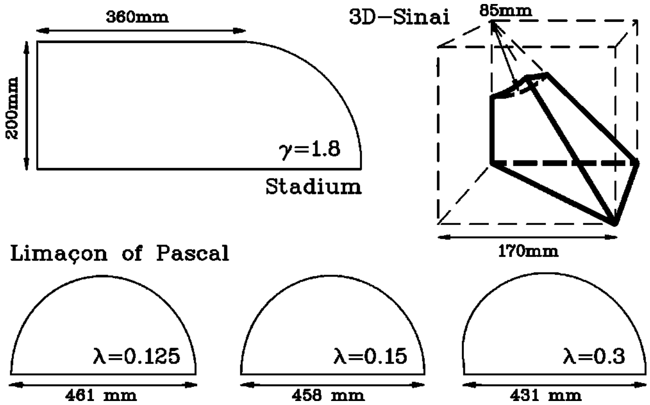

FIG. 1. Investigated billiard systems. In the upper part on the left side a Bunimovich stadium billiard ($\gamma=1.8$) and on the right a 3D Sinai billiard are illustrated. In the lower part three billiards of the limaçon family with different parameters λ are shown changing the chaoticity from nearly regular ($\lambda=0.125$) to chaotic ($\lambda=0.3$).

of the family of limaçon billiards, which have first been theoretically studied as billiards in [37]. Their boundary is defined as the quadratic conformal map of the unit disk onto the complex w plane: $w=z+\lambda z^2$, where $\lambda\in[0,1/2]$ controls the chaoticity of the system. The billiards of the limaçon family are also called Pascalian snails; their shape has already been mentioned by Dürer in 1525 [38]. We also analyzed a desymmetrized three-dimensional Sinai billiard [22,39–41].

The measurements were carried out in a liquid helium bath cryostat. The billiards were excited up to a frequency of 20 GHz, the upper limit of the Hewlett-Packard 8510B vector network analyzer used, using four capacitively coupling dipole antennas sitting in small holes on the niobium surface and penetrating up to a maximum of 1 mm into the cavity to avoid disturbances of the electromagnetic field inside the resonators. Using one antenna for the excitation and either another or the same one for the detection of the microwave signal, we were able to measure the transmission or the reflection spectrum of the resonators. The spectra were taken in 10-kHz steps and the measured resonances have quality factors of up to $Q\approx10^7$ and signal-to-noise ratios of up to approximately 70 dB, which made it easy to separate the resonances from each other and from the background even in the higher-frequency range, where the level density strongly increases. Especially in the case of the circular billiard with mainly twofold degenerate resonances, the advantage of using superconducting cavities is obvious. Due to mechanical imperfections, the degenerate modes show a very weak splitting, but nevertheless one is able to resolve all resonances. As a consequence, all the important characteristics such as eigenfrequencies and widths could be extracted with a very high accuracy [42,43]. A detailed analysis of the original spectra yields a total number of 1000–1200 resonances for the two-dimensional (2D) billiards (about 660 resonances for the circular billiard) and nearly 1900 resonances for the 3D billiard. A detailed comparison with numerical data confirms that the measured spectra are almost complete. These eigenvalue sequences $\{k_1,k_2,\dots,k_n\}$ (with $k=2\pi/c_0 f$ and c_0 the speed of light) form the basis of the present test of the mode fluctuation distribution.

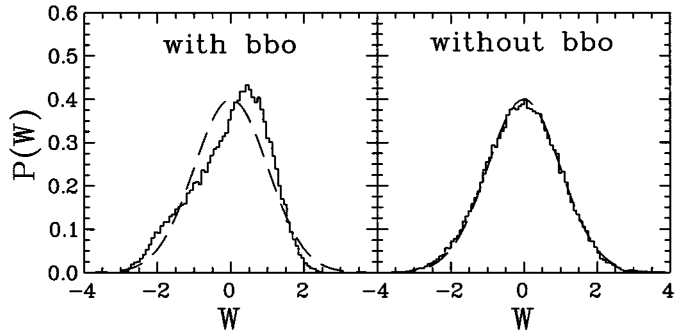

FIG. 2. Mode fluctuation distribution $P(W)$ for the $\gamma=1.8$ Bunimovich stadium billiard. On the left side the distribution with bouncing ball orbits is shown, which is asymmetric with bias to positive W. On the right side the distribution without BBOs is displayed, which is in very good agreement with the Gaussian (dashed curve) expected for chaotic systems [Eq. (10)].

IV. APPLICATION TO THE MEASURED EIGENFREQUENCY SPECTRA

In this section we study the mode fluctuation distribution discussed in Sec. II for the experimentally investigated systems presented in Sec. III.

A. 2D Bunimovich stadium billiard

First we discuss the results for the $\gamma=1.8$ stadium billiard. As proved in [35], the stadium billiard is strongly chaotic (i.e., ergodic, mixing, K system). Ergodicity, however, does not prevent a system from having a family of marginally stable periodic orbits, as long as they are of measure zero in phase space. In the stadium billiard such a family is given by the BBOs, which have perpendicular reflections at the straight line segments. Their contribution dominates the fluctuating part of the spectral staircase function [19]. Therefore, the above-stated conjecture is in its basic form not applicable (see, however, the refinement given in [16] and the discussion below). After unfolding the spectrum and calculating the distribution of the normalized mode fluctuation $W(x)$ according to Eqs. (6) and (7), a density $P(W)$ results as shown in the left part of Fig. 2. Obviously it is not a Gaussian; the distribution is shifted to positive $W(x)$ due to the existence of the BBOs. In fact, in [44] it is shown using the results of [34] that in this case $W(x)$ is a bounded function, such that the limit distribution $P(W)$ is not Gaussian.

Since $P(W)$ is expected to show Gaussian behavior only if the corresponding classical system is strongly chaotic excluding stable or neutrally stable motion, the contribution of the BBOs has to be taken into account in the unfolding procedure [16]; see Sec. II. Finally, the predicted Gaussian distribution of $P(W)$ is obtained, as can be seen in the right part of Fig. 2. To have a measure how good the distribution $P(W)$ agrees with a Gaussian we use the Kolmogorov-Smirnov test, which gives a significance level computed from the maximum value of the absolute difference between the two cumulative distributions.

For the case with BBOs one obtains a significance level of 57.2% and for the case with extracted BBOs 75.2%. The influence of the BBOs that are visible in the distributions in Fig. 2 is also reflected in the value of the significance level.

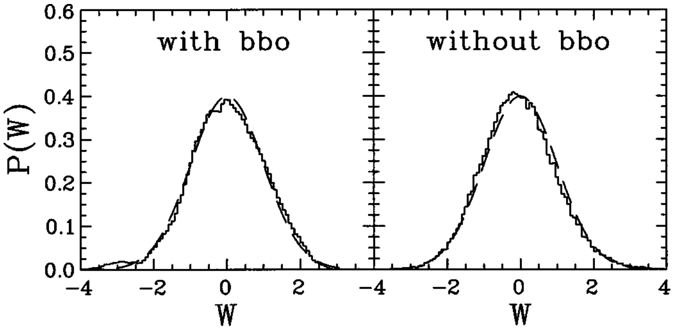

FIG. 3. Mode fluctuation distribution $P(W)$ for the 3D Sinai billiard. On the left side the distribution with BBOs is shown, where at $W\approx-3$ a significant deviation from the Gaussian occurs. On the right side the distribution without BBOs is displayed, which shows the predicted Gaussian behavior. Again the dashed curve is the expected Gaussian for chaotic systems [Eq. (10)].

To determine the normalized mode fluctuations [Eq. (6)] we have calculated the fluctuating part $N_{\rm fluc}(x)$ of the unfolded experimental spectra according to Eq. (5). For the variance $D(x)$, [Eq. (7)] we have used the ansatz $D(x)=\sqrt{\text{const}\times\sqrt{x}}$ for BBO contribution included and $D(x)=1/(2\pi^2)\ \ln(x)+\text{const}$ after their extraction, see Sec. II. Then $W(x)$ was calculated for 10^5–10^6 randomly distributed values of x in the interval $[0,x_{\rm max}]$, with $x_{\rm max}$ being the upper unfolded eigenenergy. The constant term in $D(x)$ was chosen according to Eq. (9) to give a unit variance for the obtained distribution $P(W)$. These results using the experimentally determined set of energy levels confirm our previous analysis using the numerically computed eigenspectrum [16].

B. 3D Sinai billiard

Another investigated system where the bouncing ball orbits play an important role is the 3D Sinai billiard, which is also proved to be strongly chaotic [39]. In contrast to the 2D Bunimovich stadium billiard, the 3D Sinai billiard possesses families of BBOs of dimensions 2 and 3. In Fig. 3 the mode fluctuation distributions with and without BBOs are displayed. In the distribution with BBOs (left part of Fig. 3) only slight deviations from the Gaussian occur; in particular, a significant peak at $W\approx-3$ appears. Thus the influence of the BBOs is less visible in the considered energy range as in the case of the distribution for the $\gamma=1.8$ stadium billiard, which might be due to the superpositions of different BBOs. This confirms our previous results of [22]. Taking into account the contribution of the BBO modes, one obtains the right part of Fig. 3. Calculating the significance level for the first case (with BBOs), one obtains a value of 78.5% and for the second case (with extracted BBOs) 75.2%.

C. Mixed 2D limaçon billiards

In this section we test the conjecture stated in Sec. II for the billiards of the limaçon family. We have investigated three billiards of different chaoticity with parameters $\lambda=0.125$, 0.15, and 0.3. Investigations of the classical Poincaré surface of section for these configurations have shown [45–47] that the fraction of the chaotic phase space is 55% ($\lambda=0.125$), 66% ($\lambda=0.15$), and nearly 100% ($\lambda=0.3$). The

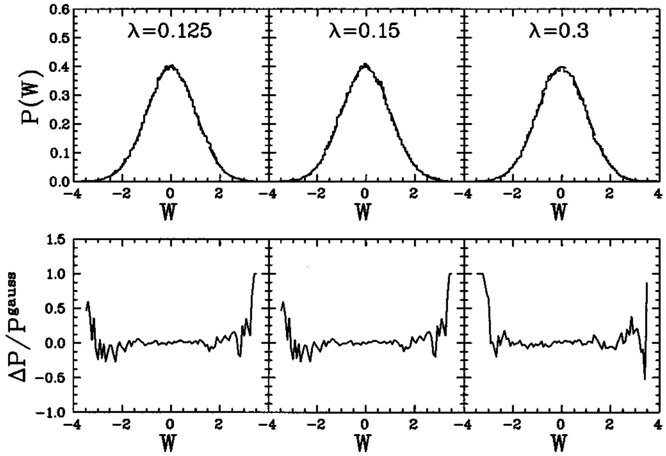

FIG. 4. Mode fluctuations distributions $P(W)$ for the three investigated billiards of the limaçon family in the upper part. All three distributions follow the predicted Gaussian for chaotic systems, which is represented by the dashed curve [Eq. (10)]. In the lower part the differences between the histogram and a Gaussian is plotted.

quantum mechanical counterpart of these three billiards exhibit the same behavior concerning their chaoticity [48,49]; see also [47].

Therefore, they are very suitable to study the conjecture. Note that the classical dynamics of the $\lambda=0.3$ billiard is not completely ergodic since small stability islands still exist [50]; see also [51] for analytical results around $\lambda=0.25$. However, the corresponding islands of stability are so small that they do not affect the energy spectrum in the range under consideration.

As can be seen in Fig. 4, the billiard with $\lambda=0.3$ shows indeed the predicted Gaussian distribution $P(W)$. For the calculation of $P(W)$ we followed the same procedure described for the stadium billiard with an ansatz for the variance $D(x)=1/(2\pi^2)\ \ln(x)+\text{const}$. On the other hand, the two billiards with $\lambda=0.125$ and 0.15 belong to the class of mixed systems. Their classical counterparts possess, aside from isolated and unstable periodic orbits, also stable orbits [37,47]. From this one would expect that, according to this, the distribution $P(W)$ should show non-Gaussian behavior. However, the histograms in Fig. 4, obtained by assuming $D(x)\propto\sqrt{x}$, which gives a good description for $D(x)$ in the energy interval considered, allow no significant distinction between $P(W)$ and the Gaussian distribution; see also the lower part of Fig. 4, where the difference ΔP between a Gaussian and the calculated distribution $P(W)$ is shown. These characteristics are also expressed in the significance levels obtained from the Kolmogorov-Smirnov test, which lie around 85%. Presumably one needs a large number of energy levels to be able to see significant deviations.

D. 2D circular billiard

Finally, we have studied the mode fluctuation distribution for the circular billiard. Since this system is integrable, one would expect a deviation of the mode fluctuation distribution from a Gaussian due to the existence of neutrally stable periodic orbits. To obtain the distribution $P(W)$ we use the ansatz for the variance $D(x)=\text{const}\times\sqrt{x}$. In Fig. 5 the dis-

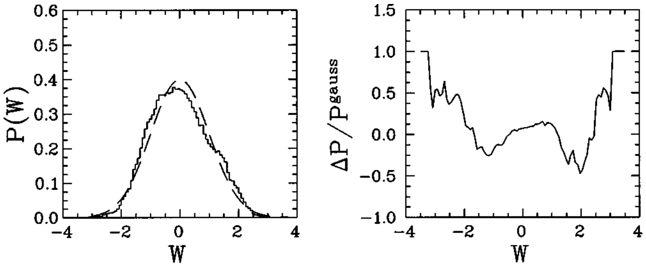

FIG. 5. Mode fluctuation distribution $P(W)$ for the circular billiard on the left side. The histogram shows the expected deviation from a Gaussian for chaotic systems, which is represented by the dashed curve [Eq. (10)]. On the right side one can see the difference between the Gaussian and the histogram.

tribution $P(W)$ is shown, which clearly deviates from the Gaussian. For the circular billiard one obtains a significance level of 67.1%. This result is in contrast to the result of [31], where no difference between the mode fluctuation distribution for the numerical obtained eigenmodes of a regular system and a Gaussian for chaotic systems could be found, although nearly the same number of eigenfrequencies (≈ 660) is used.

V. CONCLUSION

In this paper we have studied the distribution of the normalized mode fluctuations for experimentally obtained high-resolution eigenvalue spectra of several two- and three-dimensional superconducting microwave billiards. The conjecture for this statistical measure states that the distribution $P(W)$ of the normalized mode fluctuations for a given eigenvalue spectrum leads to a Gaussian when the corresponding classical system is strongly chaotic, having only unstable and isolated periodic orbits.

This has been successfully tested using the eigenvalues of a $\gamma = 1.8$ stadium billiard and a 3D Sinai billiard. Both systems are strongly chaotic, but possess families of bouncing ball orbits, whose contribution has been subtracted for the determination of $P(W)$. The same result has been obtained with a $\lambda = 0.3$ billiard of the limaçon family. Two other investigated billiards of the limaçon family, the $\lambda = 0.125$ and 0.15 billiard, belong to the class of mixed systems. The fact that no visible difference from a Gaussian occurs in the distribution $P(W)$ might be due to the finite number of eigenvalues in the given energy range we used to calculate the distribution and the chaoticity of the systems (55% and 66%, respectively), which are closer to chaotic than to regular dynamics. A variation of the included number of modes $(500,600,\ldots,1100)$ in the mode fluctuation distribution shows no significant change in the results. In the case of the circular billiard we get a clear deviation from the Gaussian, as one would expect. This is in contrast to the result of [31], where no such difference for a regular system could be found.

Therefore, characterizing the chaoticity using a small number of energy levels with the help of the conjecture stated in [12] is very difficult. For regular and chaotic systems, respectively, the reachable number of eigenvalues obtained by experiments explained in Sec. III is sufficient to have satisfactory results for the mode fluctuation distribution. However, for special regular (rectangular billiards [31]) or mixed systems the needed number of eigenvalues, approximately 10^4–10^6, can only hardly be achieved experimentally, so that a practical usage of this conjecture to obtain information about the chaoticity of such systems is limited.

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