Generalized force model of traffic dynamics

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Floating car data of car-following behavior in cities were compared to existing microsimulation models, after their parameters had been calibrated to the experimental data. With these parameter values, additional simulations have been carried out, e.g., of a moving car which approaches a stopped car. It turned out that, in order to manage such kinds of situations without producing accidents, improved traffic models are needed. Good results were obtained with the proposed generalized force model. [S1063-651X(98)15006-7]

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I. INTRODUCTION

During the last five years, gas-kinetic [1,2], fluid-dynamic [1–3], and other models have been developed, aimed at an understanding of stop-and-go traffic. The topic is related to the fields of nonlinear dynamics [3], phase transitions [4], and stochastic processes [5]. In addition, microscopic traffic models were proposed for the description of interacting driver-vehicle units. They can be classified into cellular automata models [5], which are discrete in space and time, and continuous models [1,6,7]. The latter are required for detailed studies of car-following behavior and traffic instabilities, which are necessary for an investigation of the consequences of technical optimization measures (e.g., of autopilots for an automatic control of vehicle acceleration and braking).

Therefore, a research group of the Bosch GmbH has recently recorded follow-the-leader data by means of a floating car α which measured the vehicle speed v_α, the netto distance s_α to the car in front, the acceleration a_α of it, and the relative velocity Δv_α. By a correlation analysis it was demonstrated that, among all possible combinations of subsets of these four quantities, s_α, Δv_α, and v_α are the most significant variables for the description of vehicle dynamics [8]. In Sec. III, we will find plausible reasons for this.

The follow-the-leader data, if plotted in the s_α-Δv_α plane, show the characteristic oscillation of vehicle motion around states with relative velocity zero (cf. Fig. 1), which was already reported by Hoefs [9]. Except for the previously mentioned significance analysis, the data were also used for calibrating existing microsimulation models. With the resulting optimal sets of parameter values, the models were simulated for the observed situation. That is, the first vehicle was moved according to its measured velocity, and the following vehicle was simulated according to the respective model under consideration, starting with the same initial velocity and distance as the floating car. The average relative quadratic deviation D between the simulated and actually measured distances [cf. Eq. (10)] was used as a measure of the goodness of fit of the respective model [8].

In order to obtain improved results, we have developed a generalized force model, in which each term and each parameter has a clear meaning. Moreover, by parameter calibration it turns out that all model parameters have the right order of magnitude. Therefore, it can be easily said what the parameter values will look like, if the speed limit, the acceleration capability, the average vehicle length, the visibility, or the reaction time is modified (e.g., due to technical measures). In addition, this model achieves a better fit at a reduced number of model parameters than previous models. Finally, the generalized force model manages to cope successfully with particular situations like vehicles approaching standing cars, in which other models produce accidents.

II. DISCUSSION OF PREVIOUS MODELS

The first microscopic traffic models were developed in the 1960s. Many of them are special cases of the follow-the-leader model proposed by Gazis, Herman, and Rothery [6]. This assumes that the dynamics of a vehicle α with velocity $v_\alpha(t)$ at place $x_\alpha(t)$ is given by the equation of motion

$$\frac{dx_\alpha(t)}{dt} = v_\alpha(t) \tag{1}$$

and the acceleration equation

$$\frac{dv_\alpha(t+T)}{dt} = \kappa(t+T)[v_{\alpha-1}(t) - v_\alpha(t)] \ . \tag{2}$$

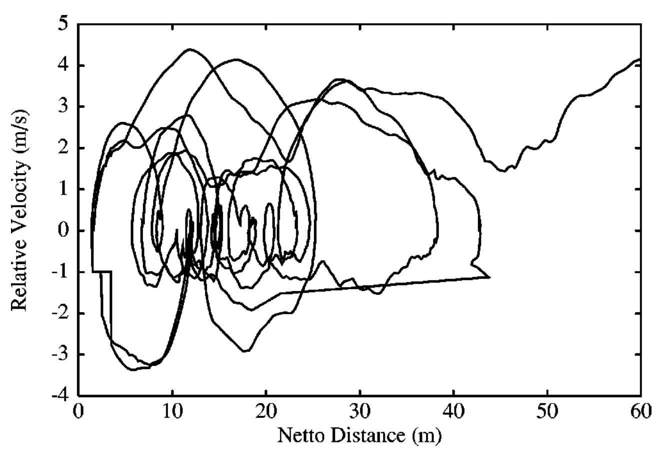

FIG. 1. The follow-the-leader data show the oscillatory nature of the relative motion of vehicles.

*World Wide Web address: http://www.theo2.physik.uni-stuttgart.de/helbing.html

According to this, a driver adapts to the velocity $v_{\alpha-1}(t)$ of the car in front, but this is delayed by the adaptation time T≈1.3 s. The deceleration is proportional to the relative velocity

$$\Delta v_\alpha = v_\alpha - v_{\alpha-1}, \tag{3}$$

where the proportionality factor κ reflects the sensitivity to the stimulus Δv_α. The sensitivity was assumed to depend on the vehicle velocity, and on the brutto distance

$$S_\alpha = x_{\alpha-1} - x_\alpha, \tag{4}$$

in the following way:

$$\kappa(t+T) = \kappa_0 \frac{[v_\alpha(t+T)]^m}{[S_\alpha(t)]^l}. \tag{5}$$

This choice allowed one to fit all equilibrium velocity-density relations of the form

$$V_e(\rho) = v_0 \left[1 - \left(\frac{\rho}{\rho_{\max}} \right)^{l-1} \right]^{1/(1-m)} \tag{6}$$

by appropriate specification of the exponents l and m (v_0 is the maximum velocity, ρ the spatial vehicle density, and $\rho_{\max}$ the maximum vehicle density). The best fit was reached for fractional exponents $m \approx 0.8$ and $l \approx 2.8$, so that this model has no obvious interpretation. Apart from that, the model does not allow one to distinguish drivers with different preferred velocities, and it cannot describe the acceleration of a single vehicle correctly.

Only a few years ago, Bando *et al.* proposed a very charming microscopic traffic model. Despite its simplicity and its few parameters, their optimal velocity model (OVM) described many properties of real traffic flows [7] and is easily interpretable. It is based on the acceleration equation

$$\frac{dv_\alpha(t)}{dt} = \kappa[V(s_\alpha) - v_\alpha(t)], \tag{7}$$

so that the vehicles adapt to a distance-dependent optimal velocity

$$V(s_\alpha) = V_1 + V_2 \tanh(C_1 s_\alpha - C_2) \tag{8}$$

with a certain relaxation time $\tau = 1/\kappa$. Here

$$s_\alpha = x_{\alpha-1} - x_\alpha - l_{\alpha-1} = S_\alpha - l_{\alpha-1} \tag{9}$$

denotes the netto distance, where l_α means the length of vehicle α. Like the follow-the-leader models, the optimal velocity model is able to describe the formation of stop-and-go waves and emergent traffic jams, but it overcomes the aforementioned problems.

We carried out a calibration of the optimal velocity model with respect to the empirical follow-the-leader data, which we obtained from Bleile *et al.* The optimization procedure was based on the evolutionary Boltzmann strategy [10], and the optimization criterion was the average relative quadratic deviation

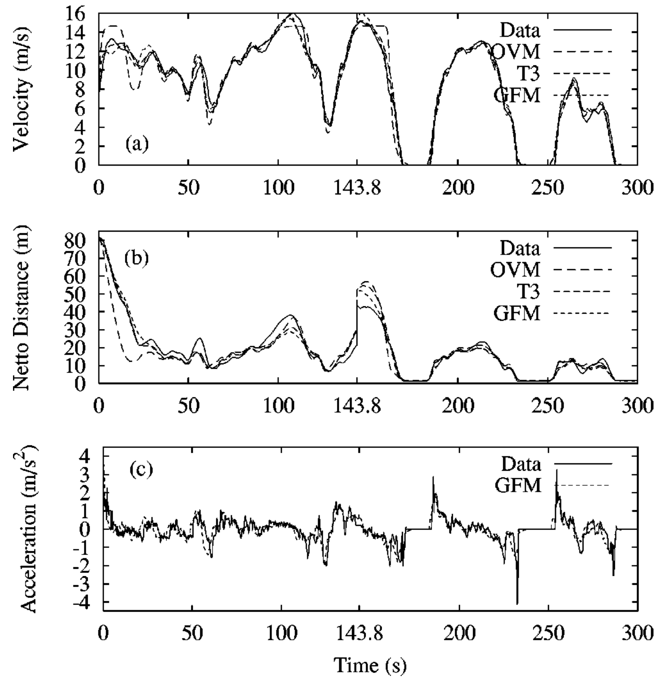

FIG. 2. The time-dependent velocity $v_\alpha(t)$ (a), distance $s_\alpha(t)$ (b), and acceleration $dv_\alpha(t)/dt$ (c) according to the optimal velocity model (OVM), the general force model (GFM), and the T3 model in comparison with follow-the-leader data of city traffic. According to (a), most of the models compare well with the measured velocities. Only the OVM shows a significant overshooting, indicating too large accelerations. Fitting of the vehicle distances is a much harder task, as shown in (b), but the T3 model and the GFM perform well. In (c), one can see that the empirical accelerations and decelerations are usually limited to the range between -3 and $+4$ m/s^2, which is met by the GFM. Note that the vehicles had to stop three times due to red traffic lights (during the periods between 169.9 and 184 s, 233.5 and 253.5 s, and after $t=288$ s). At a time $t=143.8$ s, the vehicle in front was turning right, so that the second car was following another vehicle, afterwards.

$$D = \frac{1}{N} \sum_{t=1}^{N} \left(\frac{s_\alpha(t) - s_\alpha^{\rm m}(t)}{s_\alpha^{\rm m}(t)} \right)^2 \tag{10}$$

of the simulated distance $s_\alpha(t)$ from the measured vehicle distance $s_\alpha^{\rm m}(t)$. The resulting optimal parameter values for city traffic in Stuttgart are $\kappa = 0.85\ \mathrm{s}^{-1}$, $V_1 = 6.75$ m/s, $V_2 = 7.91$ m/s, $C_1 = 0.13\ \mathrm{m}^{-1}$, and $C_2 = 1.57$.

A comparison with the data shows that the extremely short relaxation time $\tau = 1/\kappa = 1.17$ s results in values of the acceleration which are too high, which leads to an overshooting of the vehicle velocity [cf. Fig. 2(a)]. The unrealistically high accelerations also become obvious in Fig. 3, since empirical accelerations are limited to 4 m/s^2 [cf. Fig. 2(c)]. A similar problem occurs with the deceleration behavior, if a standing car (e.g., at the end of a traffic jam or in front of a red traffic light) is approached from a large distance by an initially freely moving car. It turns out that the moving vehicle reacts too late to the vehicle at rest. The values of deceleration are unrealistic large, but still not sufficient to avoid an accident (cf. Fig. 4).

These problems are solved by the *T3 model*

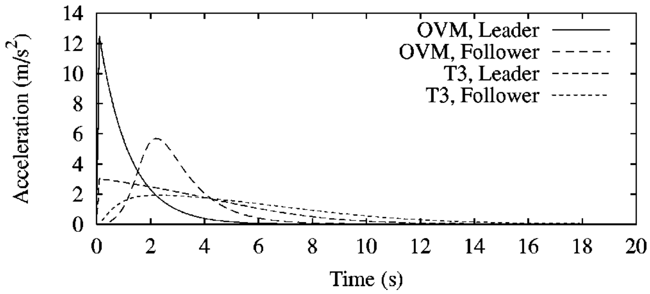

FIG. 3. Acceleration of an unobstructed vehicle and of a following vehicle according to the optimal velocity model (OVM) and the $T3$ model. Initially, both vehicles are at rest.

$$\frac{dv_\alpha}{dt}=\frac{1+b_1v_\alpha+b_2s_\alpha+b_3v_\alpha s_\alpha+b_4v_{\alpha-1}+b_5v_\alpha v_{\alpha-1}}{c_0+c_1v_\alpha+c_2s_\alpha+c_3v_\alpha s_\alpha+c_4v_{\alpha-1}+c_5v_\alpha v_{\alpha-1}} \tag{11}$$

proposed by Bleile *et al.* [8]. b_k and c_k are model parameters. The regression model (11), which is based on a rational function, describes all aspects of vehicle dynamics in cities realistically (cf. Figs. 2–4), but at the cost of additional parameters. Whereas the optimal velocity model needs only five model parameters, the T3 model contains 11 parameters. If the model equations are scaled to dimensionless equations (by scaling space or velocity and time by characteristic model quantities), the number of parameters is reduced by 2. In Sec. II, we will propose an alternative model which reaches about the same goodness of fit as the T3 model, but with a considerably smaller number of parameters.

III. GENERALIZED FORCE MODEL

Motivated by the success of so-called social force models in the description of behavioral changes [11,1], especially of pedestrian dynamics [12,1], we developed a related model for the dynamics of interacting vehicles. In setting up an equation of motion by specifying the effective acceleration and deceleration forces, the approach is analogous to the molecular-dynamics method which is used for the simulation

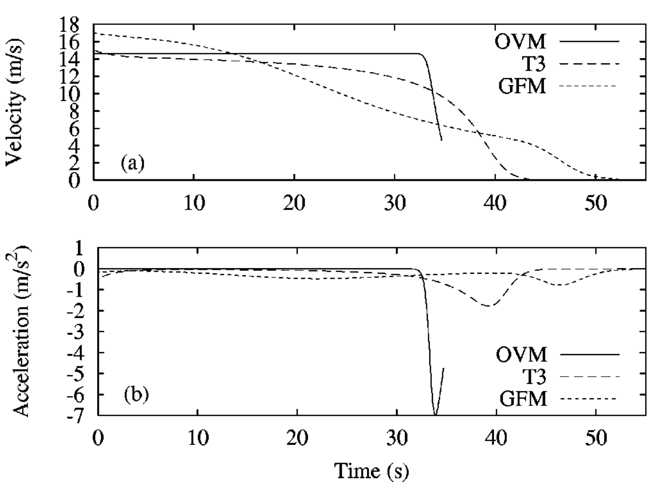

FIG. 4. Time-dependent velocity $v_\alpha(t)$ (a) and acceleration $dv_\alpha(t)/dt$ (b) for a vehicle which approaches a standing vehicle according to the different simulation models discussed. The optimal velocity model produces an accident at time $t=34.7$ s.

of many-particle systems [13], e.g., of driven granular media [14].

Besides various methodological similarities in the theoretical treatment of traffic and granular flows, there are also phenomenological analogies like the formation of density waves [15]. In both cases, the interactions are dissipative, i.e., they do not conserve kinetic energy. However, there are also differences. For granular media, the interaction forces are short ranged and belong to collision processes, where particles touch and temporarily deform each other. Vehicle interactions are long ranged and correspond to deceleration maneuvers. They are usually not related to collisions (i.e., accidents), since the drivers try to keep a safe distance s from each other which can be considerably larger than the vehicle length [cf. Eq. (16)]. Therefore, the effective space requirements of vehicles are much larger than their actual size. Moreover, vehicular interactions do not conserve momentum (in contrast to granular ones).

Another difference between granular and traffic dynamics is that the laws of granular interactions are very well known, pretty much like the basic laws of physics, whereas the laws of driver-vehicle dynamics (if they exist at all) are still to be established. The particular challenge of modeling vehicle dynamics is its dependence on factors like perceptions, psychological motivations and reactions, or social behaviors. Thus, in contrast to physical processes, driver behavior cannot be expected to be describable by a few natural constants.

According to the social force concept, the amount and direction of a behavioral change (here the temporal change of velocity, i.e., the acceleration) is given by a sum of generalized forces. These reflect the different motivations which an individual feels at the same time, e.g., in response to their respective environment. Since these forces do not fulfill Newton's laws like *actio = reactio*, they are called *generalized forces.* Alternatively, they are named *behavioral* or *social forces*, since they mostly correspond to social interactions. The success of this approach in describing traffic dynamics is based on the fact that driver reactions to typical traffic situations are more or less automatic and determined by the optimal behavioral strategy (which is the results of an initial learning process). A detailed motivation, description, and discussion of the social force concept was given in Refs. [11,1,12].

The driver behavior is mainly given by the motivation to reach a certain desired velocity v_α^0 (which will be reflected by an acceleration force f_α^0), and by the motivation to keep a safe distance from other cars β (which will be described by repulsive interaction forces $f_{\alpha,\beta}$):

$$\frac{dv_\alpha}{dt}=f_\alpha^0(v_\alpha)+\sum_{\beta(\neq\alpha)}f_{\alpha,\beta}(x_\alpha,v_\alpha;x_\beta,v_\beta)+\xi_\alpha(t)\ . \tag{12}$$

The fluctuating force $\xi_\alpha(t)$ may be used to include individual variations of driver behavior, but in our present investigations it was set to zero. If we assume that the acceleration force is proportional to the difference between the desired and actual velocity, and suppose that the most important interaction concerns the car $(\alpha-1)$ in front, we end up with

$$\frac{dv_\alpha}{dt}=\frac{v_\alpha^0-v_\alpha}{\tau_\alpha}+f_{\alpha,\alpha-1}(x_\alpha,v_\alpha;x_{\alpha-1},v_{\alpha-1})\ . \tag{13}$$

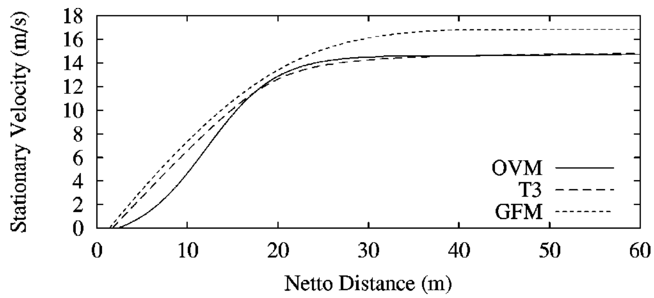

FIG. 5. Velocity-distance relations of the different traffic models in the stationary case, if all vehicles have identical parameters.

The acceleration time τ_α is a third of the time which a freely accelerating vehicle needs to reach 95% of the desired velocity.

Now we have to specify the interaction force $f_{\alpha,\alpha-1}$. For

$$f_{\alpha,\alpha-1}=\frac{V(s_\alpha)-v_\alpha^0}{\tau_\alpha}, \tag{14}$$

and $\tau_\alpha=1/\kappa$, we would again obtain the optimal velocity model. We extend this relation by a complementary term which should guarantee early enough and sufficient braking in cases of large relative velocities Δv_α. This term should increase with growing velocity difference Δv_α, but it should be only effective if the velocity of the following vehicle is larger than that of the leading vehicle, i.e., if the Heaviside function $\Theta(\Delta v_\alpha)$ is equal to 1. Moreover, the additional deceleration term should increase with decreasing distance s_α, but vanish for large distance $s_\alpha\to\infty$. The braking time τ'_α belonging to this term should be smaller than τ_α, since deceleration capabilities of vehicles are greater than acceleration capabilities. We chose the following formula which meets the above conditions:

$$f_{\alpha,\alpha-1}=\frac{V(s_\alpha)-v_\alpha^0}{\tau_\alpha}-\frac{\Delta v_\alpha\Theta(\Delta v_\alpha)}{\tau'_\alpha}\; e^{-[s_\alpha-s(v_\alpha)]/R'_\alpha}. \tag{15}$$

This formula takes into account that vehicles prefer to keep a certain velocity-dependent safe distance

$$s(v_\alpha)=d_\alpha+T_\alpha v_\alpha, \tag{16}$$

where d_α is the minimal vehicle distance, and T_α is the safe time headway (i.e., about the reaction time). R'_α can be interpreted as range of the braking interaction.

We can further reduce the number of parameters (and the numerical effort), if we replace the previous $V(s_\alpha)$ function (8) by

$$V_\alpha(s_\alpha,v_\alpha)=v_\alpha^0\{1-e^{-[s_\alpha-s(v_\alpha)]/R_\alpha}\}. \tag{17}$$

In the case of identical model parameters of all vehicles, the corresponding equilibrium velocity-distance relation results from the implicit condition $v_\alpha=V_\alpha(s_\alpha,v_\alpha)$, and is depicted in Fig. 5.

The traffic model defined by Eqs. (13), (15), and (17) will, in the following, be called the *generalized force model* of traffic dynamics (GFM). Since all its seven parameters have a clear and measurable meaning, they should have the right order of magnitude. A calibration with respect to the follow-the-leader data shows that this is indeed the case. We found the following optimal parameter values: $v_\alpha^0=16.98$ m/s, $\tau_\alpha=2.45$ s, $d_\alpha=1.38$ m, $T_\alpha=0.74$ s, $\tau'_\alpha=0.77$ s, $R_\alpha=5.59$ m, and $R'_\alpha=98.78$ m. Now the acceleration time τ_α is more than twice as large as in the optimal velocity model, the braking time τ'_α is smaller than τ_α, as demanded, and the reaction time T_α is also realistic. Note that the range R_α of the acceleration interaction is much shorter than the range R'_α of the deceleration interaction. This is not only sensible, it is also the reason for the astonishingly good agreement with the empirical data. Table I compares the minimal values of the average relative quadratic deviation D that could be reached for the different discussed traffic models by evolutionary parameter optimization [10]. (The advantage of the applied Boltzmann strategy is that this particular gradient method escapes local minima by means of a fluctuation term with eventually decreasing variance.)

TABLE I. Minimal values of the average relative deviation D between empirical data and simulation results that were reached for the different traffic models by evolutionary parameter optimization.

Model	OVM	T3	GFM
D	0.0586	0.0354	0.0316

It turns out that the optimal velocity model is considerably improved by the T3 model. This is not surprising, since the goodness of fit should increase with the number of model parameters. Nevertheless, the generalized force model reaches the best agreement with the data, although it includes only two third of the number of parameters of the T3 model. The simulation results for the generalized force model are depicted in Fig. 2. Finally, the representation of the relative vehicle movement in the Δv_α-s_α plane shows the expected oscillatory character of the follow-the-leader behavior, which can cause the development of stop-and-go traffic (cf. Fig. 6).

In comparison with physical models, the generalized force model still seems to contain a lot of parameters. However, let us discuss this in more detail for the previously mentioned

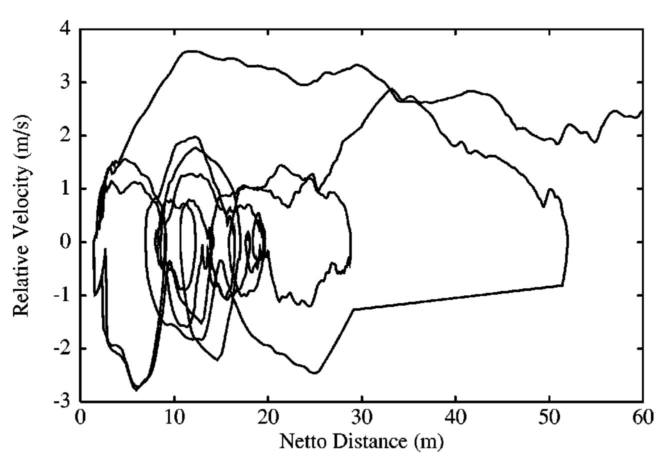

FIG. 6. The simulation of the follow-the-leader behavior according to the generalized force model shows the oscillatory nature of the relative motion of vehicles. The above result is in good agreement with the empirical findings depicted in Fig. 1.

molecular-dynamic models of granular media. If we want to describe interactions of smooth, inelastic, and spherical grains only, we need to know the particle size and the normal restitution coefficient related to translational energy dissipation. In cases of rough spheres, we require two additional parameters: a tangential restitution coefficient and a friction coefficient [16]. Moreover, if the grains are nonspherical, the situation becomes even more complicated. In conclusion, the small number of parameters occurring in physical models are often a result of simplifications and idealizations.

Finally, let us check the plausibility of the generalized force model. In order to do this, we rewrite the model in the form

$$\frac{dv_\alpha}{dt} = \frac{V_\alpha^*(s_\alpha, v_\alpha, \Delta v_\alpha) - v_\alpha(t)}{\tau_\alpha^*}, \tag{18}$$

with

$$\frac{1}{\tau_\alpha^*} = \frac{1}{\tau_\alpha} + \frac{\Theta(\Delta v_\alpha)}{\tau_\alpha''} \tag{19}$$

and

$$V_\alpha^* = \frac{\tau_\alpha'' V_\alpha + \tau_\alpha \Theta(\Delta v_\alpha) v_{\alpha-1}}{\tau_\alpha'' + \tau_\alpha \Theta(\Delta v_\alpha)}, \tag{20}$$

where

$$\tau_\alpha'' = \tau_\alpha' \exp\{[s_\alpha - s(v_\alpha)]/R_\alpha'\}. \tag{21}$$

For small velocity differences Δv_α or large distances s_α, we find

$$\frac{dv_\alpha(t)}{dt} \approx \frac{V_\alpha(s_\alpha, v_\alpha) - v_\alpha(t)}{\tau_\alpha}, \tag{22}$$

so that vehicles try to approach the optimal velocity V_α with a relaxation time τ_α. This corresponds to the optimal velocity model, but with a velocity-dependent function $V_\alpha(s_\alpha, v_\alpha)$. If a vehicle is faster than the leading vehicle (i.e., $\Delta v_\alpha > 0$), and its distance is sufficiently small, we have

$$\frac{dv_\alpha(t)}{dt} \approx \frac{v_{\alpha-1}(t) - v_\alpha(t)}{\tau_\alpha''}, \tag{23}$$

which coincides with a car-following model in which the following vehicle adapts to the velocity of the vehicle in front. According to the formula for τ_α'', the deceleration is stronger the closer the vehicles come to each other. Therefore, the limiting cases of the generalized force model behave very reasonably.

IV. SUMMARY AND DISCUSSION

We have calibrated several microscopic traffic models to city traffic, compared them with empirical follow-the-leader data, and investigated their properties. It turned out that one model showed accelerations and decelerations that were too large, but nevertheless caused accidents in certain situations. Another model was a regression model, so that the meaning of the model and its parameters was not clear. Therefore, we developed the generalized force model, which reached the best agreement with the empirical data, although it had only two parameters more than the optimal velocity model and four parameters less than the T3 model. Another advantage of the generalized force model is that all its parameters are easily interpretable and have the expected order of magnitude. Therefore, it can be immediately stated how the parameters will differ between fast cars, slow cars, and trucks (the latter being characterized by small v_α^0, but large τ_α and τ_α'). It can also be predicted what happens if the speed limit (i.e., v_α^0) is changed, if the weather conditions are bad (greater τ_α', but smaller v_α^0 and R_α'), if roads are used by vehicles with smaller length l_α, if the reaction time T_α is reduced (by means of technical measures like an autopilot), etc. For these reasons, the generalized force model is an ideal tool for carrying out detail studies of traffic flow, as well as for developing and testing traffic optimization measures.

The simulation of large vehicle numbers is completely analogous to molecular-dynamic simulation studies of many-particle systems, e.g., of granular flows. The parameters of the different driver-vehicle units are specified individually (α dependent), then. In this case, one would specify typical parameter values of fast cars, slow cars, and trucks, and their respective percentages. Alternatively, one could introduce a distribution of each parameter (e.g., a Gaussian one) around a typical value. Then the simulation is started with the initial and boundary conditions of interest. Of course, the model can be also extended to a multi-lane model with lane-changing and overtaking maneuvers [1,17]. This is a topic of current research.

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