Ultrashort pulsed Gaussian light beams

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We find a family of solutions of the paraxial wave equation that represents ultrashort pulsed light beams propagating in free space. These pulsed beams have an arbitrary temporal form and a nearly Gaussian cross section, while modeling for the pulses emitted by mode-locked lasers with stable two-mirror resonators. We also study the effects arising from their spatiotemporal coupled behavior, such as pulse time delay, distortion, and a frequency shift toward the beam periphery. Time-varying diffraction (with diffraction reduction at the first instants of arrival of the pulsed beam at a given distance) and the dependence of the spatial distribution of energy on the pulse form are also described. These effects become important for pulsed beams with a few optical oscillations within the pulse envelope. [S1063-651X(98)02007-8]

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I. INTRODUCTION

Due to rapid advances in the generation of femtosecond laser pulses [1], the study of their spatiotemporal behavior on propagation in free space [2–7] and optical systems [8–10] has become a subject of interest. It is nowadays well established [11,12] that the propagation of femtosecond laser pulses cannot be assimilated to that of longer (quasimonochromatic) pulses, but the spatial and temporal characteristics interact with each other during propagation. In free space, spatiotemporal couplings such as pulse time delay, pulse broadening, and frequency lessening toward the beam periphery have been reported in several papers for different initial conditions. In Refs. [5,7], the couplings arose in the propagation of an initial uncoupled (factorized) condition with Gaussian form in space and time; in Refs. [12–14], the spatiotemporal coupling came from the propagation through a lens, but not from the free propagation beyond.

Very recently [2], a model of an ultrashort pulsed Gaussian beam (PGB) was described, nearly Gaussian in time and space and propagating in free space, and incorporating some of the spatiotemporal couplings reported previously. These PGB's arise from the coherent superposition of axial modes in a confocal resonator, and provide a model for the radiation from mode-locked lasers with such resonators [2]. However, the authors of this reference overlooked the fact that this PGB model is not a well-behaved ''beam;'' the transversal amplitude distribution is described by a function which, after a central region of beamlike behavior, becomes boundless with the transversal coordinate r as $\exp(r^4)$, as is clear from Eqs. (12) and (13) of Ref. [2]. As a consequence, the energy in the beam is infinite [see Eq. (25) of Ref. [2]]. Apparently, this PGB could be used to model light beams with other pulse shapes [see Eq. (10) of Ref. [2]]. When doing so for some standard pulses, such as a super-Gaussian of order n, or for Lorentzian pulses, similar difficulties are found [an exponential growing $\exp(r^{2n})$ in the first case, and singular points in the transversal profile in the second]. These drawbacks render the use of the PGB of Ref. [2] problematic.

In this paper, we find a PGB as a family of solutions for the paraxial wave equation in free space (Sec. II), and describe a method, based on the use of the analytical signal complex representation of polychromatic light [15], by which the PGB's can represent pulsed beams (Sec. III). We do not restrict ourselves to the PGB with a Gaussian pulse shape, but draw general conclusions about the spatiotemporal couplings in a PGB with an arbitrary pulse shape (Sec. IV), pointing out the peculiar form the couplings adopt for different pulse shapes (Sec. V).

II. DERIVATION OF THE PGB FAMILY OF SOLUTIONS FOR THE PARAXIAL WAVE EQUATION

We start with the wave equation $\Delta E-(1/c^2)\partial^2 E/\partial t^2=0$ for a complex scalar field E, Re E giving the real field. Introducing the ansatz $E=\psi(x,y,z,t)\exp[i\omega_0(t-z/c)]$, where $\omega_0=2\pi/T_0$ is a carrier frequency and T_0 the period, we find

$$\Delta\psi-2i\frac{\omega_0}{c}\frac{\partial\psi}{\partial z}-2i\frac{\omega_0}{c^2}\frac{\partial\psi}{\partial t}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=0, \qquad (1)$$

and, with the new variables $t'=t-z/c$, $z=z$,

$$\Delta_{x,y}\psi-2ik_0\frac{\partial\psi}{\partial z}=\frac{2}{c}\frac{\partial^2\psi}{\partial z\,\partial t'}, \qquad (2)$$

where $\Delta_{x,y}=\partial^2/\partial x^2+\partial^2/\partial y^2$ and $k_0=\omega_0/c$. In Eq. (2) we have dropped, at the last step, $\partial^2\psi/\partial z^2$ to perform the paraxial approximation [16]. Equation (2) governs the propagation of pulsed paraxial light beams in free space [12]. Such a pulsed beam will have a characteristic variation time of the order of the magnitude of the pulse duration T for a smooth pulse shape. With the dimensionless variable $\tau'=t'/T$, Eq. (2) may be written as

$$\Delta_{x,y}\psi-2ik_0\frac{\partial}{\partial z}\left[\psi+\frac{1}{2\pi i}\left(\frac{T_0}{T}\right)\frac{\partial\psi}{\partial\tau'}\right]=0. \qquad (3)$$

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For pulses whose duration is much longer than the oscillation period, $T_0/T \ll 1$, the second term in the square brackets of Eq. (3) may be neglected, as is the case for optical pulses of several tens of femtoseconds or longer. Then the solutions of Eq. (2), where time derivatives do not appear explicitly, depend parametrically on time. In particular, factorized solutions of the form $\psi = \psi_p(t')\psi_c(x,y,z)$ exist. This means that the pulse experiences only a global complex amplitude change ψ_c from point to point in space, but its form is invariant, and conversely, that the transverse amplitude profile only changes by a global complex amplitude ψ_p with time, but its form is unaltered.

If, however, the pulse comprises only a few optical periods ($T_0/T \lesssim 1$), the crossed derivative in Eq. (2) must be kept, coupling the spatial and temporal behaviors of the pulsed beam in a complicated way. A family of solutions of Eq. (2), where these couplings become explicit, can be found as follows.

The pulsed spherical wave $E = (1/\rho)\psi_p(t-\rho/c)\exp[i\omega_0(t-\rho/c)]$, with $\rho^2 = x^2+y^2+z^2$ and $\psi_p(t)$ an arbitrary function, satisfies the wave equation. Its paraxial version is obtained by approaching $\rho \simeq z + r^2/2z$, with $r^2 = x^2+y^2$, both in the exponential and in the argument of ψ_p, while approaching $1/\rho \simeq 1/z$. After these approximations, and extracting a factor $\exp[i\omega_0(t-z/c)]$, we find $\psi = (1/z)\psi_p[t' - r^2/2cz]\exp[-ik_0 r^2/2z]$, which can be seen to satisfy the paraxial wave equation (2). New solutions of Eq. (2), the PGB's, are obtained from this paraxial pulsed spherical wave by using the well-known procedure [17] of shifting z by an imaginary constant, $z \to q(z) = z + iz_R$, with $z_R > 0$:

$$\psi(r,z,t') = \psi_p\left(t' - \frac{r^2}{2cq(z)}\right)\frac{iz_R}{q(z)}\exp\left[\frac{-ik_0 r^2}{2q(z)}\right], \quad (4)$$

where we have also introduced a factor iz_R for convenience. The complete field $E(r,z,t') = \psi(r,z,t')\exp(i\omega_0 t')$ can be written

$$E(r,z,t') = \frac{iz_R}{q(z)}\psi_p(t'_c)\exp(i\omega_0 t'_c), \quad (5)$$

where

$$t'_c = t' - \frac{r^2}{2cq(z)} \quad (6)$$

is the complex time t'_c of Ref. [2], which emerges here as a consequence of the more familiar complex spatial shift $z \to z + iz_R$. The PGB [Eq. (5)] appears as a cw Gaussian beam $[iz_R/q(z)]\exp(i\omega_0 t'_c)$ of frequency ω_0 modulated in space and time by the function $\psi_p(t'_c)$, which is, in principle, arbitrary.

The quantity $q(z)$ is the so-called complex beam parameter, and z_R is the Rayleigh range of cw Gaussian beams. For a cw Gaussian beam of frequency ω_0, the parameters are usually written in the form $z_R = k_0 a_0^2/2$, and

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{2i}{k_0 a^2(z)}, \quad (7)$$

where a_0, $R(z) = z[1 + (k_0 a_0^2/2z)^2]$ and $a^2(z) = a_0^2[1 + (2z/k_0 a_0^2)^2]$ are, respectively, the waist width, the radius of curvature of the wave fronts, and the Gaussian width at each cross section $z = \text{const}$ of the cw Gaussian beam [although they lose such meanings for the pulsed beam represented by Eq. (5)]. It is important to note that, according to Eqs. (7) and (6), the imaginary part of complex time is $\text{Im}\, t'_c = r^2/a^2(z)\omega_0 \geq 0$.

III. DETERMINATION OF SOLUTIONS OF PULSE AND BEAM FORM

We shall now determine $\psi_p(t')$, and then a particular PGB, by prescribing a real pulse form at $r = z = 0$, namely,

$$p(t') = A(t')\cos[\omega_0 t' + \Phi(t')], \quad (8)$$

where $A(t') \geq 0$ is the pulse envelope and $\Phi(t')$ its associated phase factor. If $p(t')$ is square integrable, it may be represented as a Fourier spectrum of the form

$$p(t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}[f(\omega-\omega_0) + f^*(-\omega-\omega_0)]\exp(i\omega t')d\omega, \quad (9)$$

where $f(\alpha)$ is one half the Fourier transform of the complex envelope $A(t')\exp[i\Phi(t')]$, i.e.,

$$f(\alpha) = \tfrac{1}{2}\int_{-\infty}^{\infty} A(t')\exp[i\Phi(t')]\exp(-i\alpha t')dt'. \quad (10)$$

The function $f(\omega-\omega_0)$ takes significant values within an interval (bandwidth) $\Omega \sim 5/T$ [assuming full widths at half maximum (FWHM's) Ω and T for $|f|$ and A] around the carrier frequency ω_0. For a pulse with a few oscillations, $T \gtrsim T_0$, the bandwidth satisfies $\Omega \lesssim 0.8\omega_0$. Then $f(\omega-\omega_0)$ will be generally small but not necessarily zero around $\omega = 0$.

The analytic signal [15] complex representation $P(t')$ of $p(t')$ [$\text{Re}\, P(t') = p(t')$] is

$$P(t') = \frac{1}{\pi}\int_0^{\infty}[f(\omega-\omega_0) + f^*(-\omega-\omega_0)]\exp(i\omega t')d\omega, \quad (11)$$

which does not contain spectral components of negative frequency. As a function of a complex variable t'_c, $P(t'_c)$ is [15] nonsingular in the upper half $\text{Im}\, t'_c \geq 0$ of the complex plane t'_c, and approaches zero for $\text{Im}\, t'_c \to +\infty$.

In order to have a PGB with the prescribed pulse form (8) at $r = z = 0$, we equate the PGB complex field $E(0,0,t') = \psi_p(t')\exp(i\omega_0 t')$ and the analytic signal $P(t')$,

$$\psi_p(t')\exp(i\omega_0 t') = \frac{1}{\pi}\int_0^{\infty}[f(\omega-\omega_0) + f^*(-\omega-\omega_0)] \times \exp(i\omega t')d\omega, \quad (12)$$

and, by the change $\omega' = \omega - \omega_0$, we find

$$\psi_p(t')=\frac{1}{\pi}\int_{-\omega_0}^{\infty}[f(\omega')+f^*(-\omega'-2\omega_0)]\exp(i\omega' t')d\omega', \tag{13}$$

which determines $\psi_p(t')$. The fact that $\psi_p(t')\exp(i\omega_0 t')$ is an analytic signal ensures that the PGB [Eq. (5)] $[iz_R/q(z)]\psi_p(t'_c)\exp(i\omega_0 t'_c)$ has no singularities, and tends to zero for large r.

The necessity of using the analytic signal complex representation originates in the peculiar form of the PGB solutions of the paraxial wave equation. The PGB's have been derived by the substitution $z\rightarrow z+iz_R$ in a paraxial pulsed spherical wave, which leads to replacing the real time t' with the complex time t'_c, with a positive imaginary part, in $\psi_p(t')\exp(i\omega t')$. This function must therefore have an adequate behavior in the upper half of the complex plane t'_c, which is achieved by the use of its analytic signal. Other solutions of the paraxial wave equation will not require, in general, the use of the analytic signal.

For many standard models of pulses [for instance, the Gaussian one $p(t')=\exp(-t'^2/T^2)\cos(\omega_0 t')$ used in Ref. [2]], $f(\omega-\omega_0)$ does not vanish in $\omega\leqslant 0$, but takes small values. In such cases it is customary to neglect the contribution of f^* in Eq. (12) [or Eq. (13)], and further extend the lower limit of the integral to $-\infty$. One then writes, on comparison of Eqs. (13) and (10), that $\psi_p(t')=A(t')\exp[i\Phi(t')]$, i.e., $\psi_p(t')$ is simply the complex envelope of $p(t')$. This approximation underlies Ref. [2]. However, because of the negative frequencies introduced in Eq. (12) in making this approximation, $\psi_p(t')\exp(i\omega_0 t')$ is no longer the analytic signal of $p(t')$, but another complex representation. The obtained PGB $[iz_R/q(z)]\psi_p(t'_c)\exp(i\omega_0 t'_c)$ is still a solution of Eq. (2) with the desired pulse form $p(t')$ at $r=z=0$, but it may have singularities at Im $t'_c>0$ (i.e., $r\neq 0$), or else tend to infinity for Im $t'_c\rightarrow+\infty$ ($r\rightarrow\infty$), which is just the problem with the PGB of Ref. [2]. The reason of this growth is that the amplitudes of the spectral components with negative frequency $\exp[i(-\omega)t']$ introduced in Eq. (12), however small they are, grow exponentially as $\exp(\omega\,\mathrm{Im}\,t'_c)$ in Im $t'_c>0$ ($r\neq 0$). In conclusion, we cannot perform the above approximation when dealing with a complex time.

We observe, however, that the PGB can be expressed, from Eqs. (5), (6), and (12), as the superposition

$$E(r,z,t')=\frac{iz_R}{q(z)}\frac{1}{\pi}\int_0^{\infty}[f(\omega-\omega_0)+f^*(-\omega-\omega_0)]\times\exp\left[\frac{-i\omega r^2}{2cq(z)}\right]\exp(i\omega t')d\omega \tag{14}$$

of cw Gaussian beams with different frequencies ω, amplitudes, and phases determined by $f(\omega-\omega_0)$, all with the same Rayleigh range z_R and waist position. The PGB then serves as a model for the pulsed beams from mode-locked lasers with stable resonators, in which z_R and the waist position are common parameters for all the locked Gaussian axial modes [18], as noted in Ref. [2]. Moreover, we point out that $f(\omega-\omega_0)$ is limited to positive values of ω by the laser line shape and mode gain, and, in the last analysis, by the resonator geometry (e.g., $\omega_{\min}=\pi c/L$ in a confocal resonator of length L). Therefore, we have that, without appealing to any approximation, $\psi_p(t')$ is the complex envelope of $p(t')$, i.e.,

$$\psi_p(t')=A(t')\exp[i\Phi(t')]. \tag{15}$$

As $f(\omega-\omega_0)\exp[-i\omega r^2/2cq(z)]$ in Eq. (14) is similarly limited to positive values of ω, $\psi(r,z,t')$ given by Eq. (4) can justly be said to be the temporal complex envelope of the PGB at any spatial position (r,z).

Nevertheless, we still may wish to use pulse models $p(t')$, with $f(\omega-\omega_0)$ different from zero for $\omega\leqslant 0$. Then $\psi_p(t')$ should be calculated from Eq. (13). The result is not the complex envelope $A(t')\exp[i\Phi(t')]$, but a certain function lacking a clear physical meaning. The same can be said for $\psi(r,z,t')$. It is more meaningful in this case to calculate and handle directly the complex field E from Eqs. (12) and (5). Later we shall see some examples of the different situations.

In the following discussion, we consider a PGB from a mode-locked laser. The formulas displayed below, however, are written so that they apply in the general case.

IV. TEMPORAL AND SPATIAL BEHAVIOR OF THE PGB

Let us now analyze the temporal behavior of the PGB at each point of space. According to Eq. (4), the pulse envelope $\psi(0,z,t')$ maintains its form along the beam axis ($r=0,z\neq 0$), acquiring only a global complex amplitude $iz_R/q(z)$ [i.e., a global amplitude $a_0/a(z)$ and phase $\tan^{-1}(z/z_R)$]. If we move toward the beam periphery ($r\neq 0,z\neq 0$), the pulse envelope $\psi(r,z,t')$ becomes shifted in time and changes its form, as Fig. 1(a) illustrates: at a point of space (r,z), the function $\psi_p(t'_c)$ is shifted in the complex plane by the quantity $r^2/2cq(z)$, which lies in the circle of diameter $\propto r^2$, as shown in Fig. 1(a). The cut along the real axis of the shifted function gives (after multiplication by the global complex amplitude $iz_R/q(z)\exp[-ik_0r^2/2q(z)]$) the pulse envelope at (r,z).

The real part $r^2/2cR(z)$ of the complex shift amounts to a pulse temporal shift (delay for $z>0$ and advancement for $z<0$). At the time $t=z/c$ of arrival of the pulse at the plane z, the temporal shift entails a spatial pulse shift $\Delta z=-r^2/2R(z)$ equal to the departure of the spherical wave front from the plane z. We thus have that the pulse fronts match the spherical wave fronts of the monochromatic Gaussian beam of the carrier frequency ω_0, as illustrated in Fig. 1(b).

The imaginary part $r^2/a^2(z)\omega_0$ of the complex shift results, in general, in a change in the shape of the envelope (broadening or narrowing in simple cases, and distorting in general), determined by the quotient $r^2/a^2(z)$. The points $r^2/a^2(z)=\mathrm{const}$ are the hyperbolic rays $r^2(z)=r_0^2[1+(2z/k_0a_0^2)^2]$ of the energy flow [16] of a monochromatic Gaussian beam of frequency ω_0; the pulse envelope is therefore invariant along these rays, as shown in Fig. 1(b).

These aspects and others are more clearly seen from the Fourier spectrum $\hat{E}(r,z,\omega)$ of $E(r,z,t')$ at each point in space. From Eqs. (14) and (7), it follows that

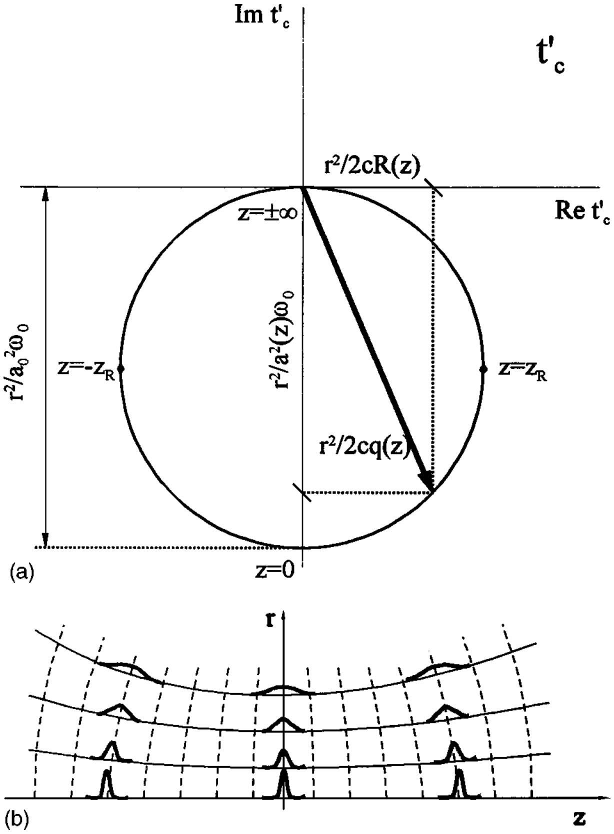

FIG. 1. (a) Displacement in the complex plane t_c' of $\psi_p(t_c')$ to obtain the pulse envelope as the cut along the real axis. (b) Schematic drawing of the PGB, showing the pulse time delay and broadening.

$$\hat{E}(r,z,\omega)=2\theta(\omega)[f(\omega-\omega_0)+f^*(-\omega-\omega_0)]\frac{iz_R}{q(z)}\times\exp\left[\frac{-i\omega r^2}{2cR(z)}\right]\times\exp\left[-\frac{\omega}{\omega_0}\frac{r^2}{a^2(z)}\right], \quad (16)$$

where $\theta(\ )$ is the Heaviside step function. The phase factor $\exp(-i\omega r^2/2cR)$ in Eq. (16) leads to the mentioned temporal shift. The amplitude $\exp(-\omega r^2/\omega_0 a^2)$, exponential in ω, deforms the on-axis spectrum, leading to a subsequent change in the form of the pulse envelope. Furthermore, this decaying exponential removes high frequency spectral components from the on-axis spectrum, which may cause a pulse frequency shift toward lower frequencies, i.e., an increase of the oscillation period. The frequency shift increases toward the beam periphery, remaining constant in each hyperbolic ray $r^2/a^2(z)=\text{const}$.

The spatial behavior of the PGB shows a great variety with $\psi_p(t')$, since the spatial coordinates enter its argument. In general, both the distribution of amplitude, $|E(r,z,t)|$ (only for long pulses can its square be identified with the average intensity over a few optical cycles), and of energy, $W(r,z)=\int[\text{Re } E(r,z,t')]^2dt'=\frac{1}{2}\int|E(r,z,t)|^2dt'$, are non-Gaussian at cross sections $z=\text{cte}$.

The energy distribution can be written, from Eq. (4), as

$$W(r,z)=\left[\frac{a_0}{a(z)}\right]^2\exp\left[-\frac{2r^2}{a^2(z)}\right]\times\int_{-\infty}^{\infty}dt''\left|\psi_p\left(t''+i\frac{r^2}{\omega_0 a^2(z)}\right)\right|^2=\left[\frac{a_0}{a(z)}\right]^2\exp\left[-\frac{2r^2}{a^2(z)}\right]g\left[\frac{r^2}{a^2(z)}\right], \quad (17)$$

where $t''=t'-r^2/2cR(z)$. The form of this expression indicates that the energy distribution of the PGB has an invariant transverse pattern during propagation, determined by the pulse form through $g(r^2/a^2)$, whose width and axial amplitude change at the same rate as the continuous Gaussian beam of the carrier frequency. The value of z of the narrowest $W(r,z)$ (the waist or focus of the PGB) is then $z=0$. From the spectrum [Eq. (16)], we have that $|\hat{E}(r,z,\omega)|^2<|\hat{E}(0,z,\omega)|^2$; integrating in ω and using the Parseval identity, we find $W(r,z)<W(0,z)$, i.e., the transversal pattern of energy distribution for any PGB is a strictly decreasing function of r.

Additional regularities for $W(r,z)$ and $|E(r,z,t')|$ can be drawn from the approximate expression for the PGB, valid for $r^2/a^2(z)\ll 2\pi T/T_0$,

$$E(r,z,t')\simeq\frac{iz_R}{q(z)}\exp\left[\frac{-ik_0r^2}{2q(z)}\right]\left[\psi_p\left(t'-\frac{r^2}{2cR(z)}\right)+\frac{\partial\psi_p}{\partial t'}\bigg|_{t'-(r^2/2cR)}\frac{ir^2}{\omega_0 a^2(z)}\right]\exp(i\omega_0 t'), \quad (18)$$

which follows by power expanding $\psi_p[t'-r^2/2cR(z)+ir^2/a^2(z)\omega_0]$ in Eq. (4) around $t'-r^2/2cR(z)$, and retaining up to the first order term. The amplitude distribution is then approximated by

$$|E(r,z,t')|\simeq\frac{a_0}{a(z)}\exp\left[-\frac{r^2}{a^2(z)}\right]\left|\psi_p\left(t'-\frac{r^2}{2cR(z)}\right)\right|\times\left[1-\frac{r^2}{a^2(z)\omega_0}\left(\frac{\partial\arg\psi_p}{\partial t'}\right)_{t'-r^2/(2cR)}\right], \quad (19)$$

which, in the case of an on-axis pulse with negligible phase modulation, reduces to

$$|E(r,z,t')|\simeq\frac{a_0}{a(z)}\exp\left[-\frac{r^2}{a^2(z)}\right]\left|\psi_p\left[t'-\frac{r^2}{2cR(z)}\right]\right|. \quad (20)$$

Squaring and integrating, we find that, in this case and within this approximation, $W(r,z)$ is Gaussian, of the same width ($1/e^2$ decay) as that of the cw Gaussian beam $a(z)$. The amplitude $|E(r,0,t')|$ at the waist ($z=0$) is also Gaussian [since $1/R(0)=0$], of width ($1/e$ decay) independent of time a_0. To visualize the behavior of $|E(r,z,t')|$ for $z>0$, as-

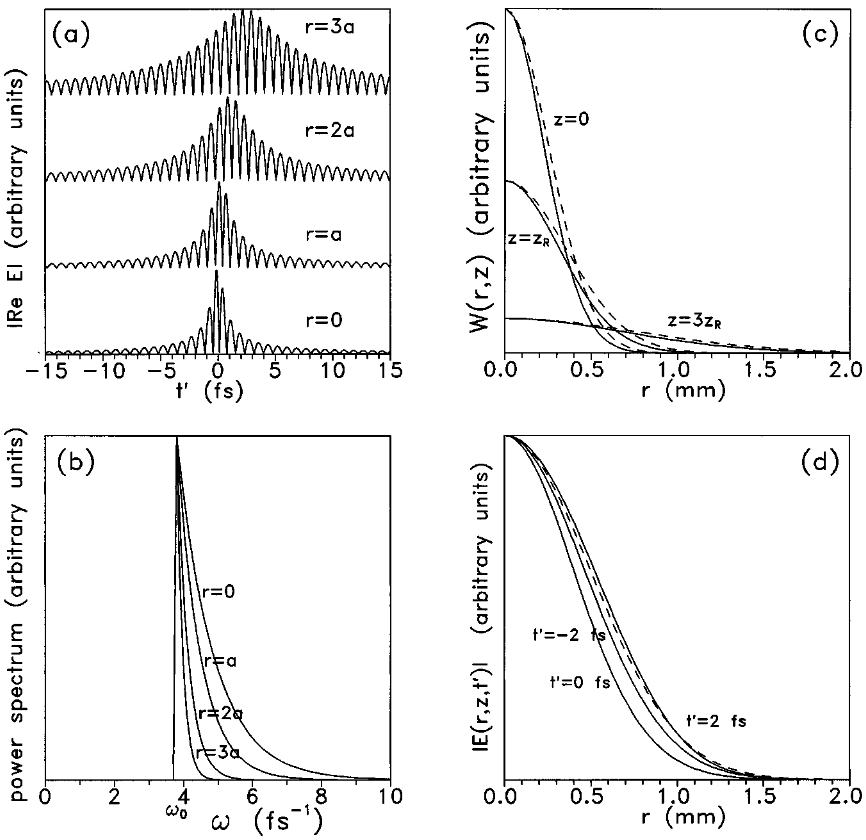

FIG. 2. (a) Pulse forms of the PGB of Eq. (22) for variable r. $a_0=0.5$ mm, $T_0=1.65$ fs ($\omega_0=3.8$ fs^{-1}), $T=T_0$, and $z=z_R=1.58\times10^3$ mm. (b) Power spectra of the pulses of (a). (c) Energy distribution of the PGB of Eq. (22) at several cross sections. The beam parameters are the same as in (a). The dashed lines are the energy distributions of the cw Gaussian beam of ω_0. (d) Amplitude distribution at $z=z_R$ of the PGB of Eq. (22) at the time instants $t'=-2$, 0, and 2 fs. The dashed line is the amplitude distribution of a cw Gaussian beam of ω_0.

sume, for simplicity, that $|\psi_p(t')|$ is a bell-shaped function around $t'=0$. Then $|\psi_p(t'-r^2/2cR)|$ is a decreasing function of r in the leading part $t'<0$ of the pulse, and, according to Eq. (20), $|E(t,z,t')|$ will be narrower than the Gaussian distribution of width $a(z)$ at these instants. Conversely, in the trailing edge $t'>0$ of the pulse, the amplitude distribution will be broader than the Gaussian distribution of width $a(z)$. In short, we might say that, at each $z>0$, diffraction spreading builds up with time.

V. EXAMPLES

Let the pulse be

$$p(t')=\frac{\Delta t}{\sqrt{t'^2+\Delta t^2}}\cos[\omega_0+\tan^{-1}(t'/\Delta t)], \tag{21}$$

$\Delta t>0$ (the FWHM duration is $T=2\sqrt{3}\Delta t$), of complex envelope $A(t')\exp[i\Phi(t')]=i\Delta t/(t'+i\Delta t)$. From Eq. (10), $f(\omega-\omega_0)=\pi\Delta t\exp[-\Delta t(\omega-\omega_0)]\theta(\omega-\omega_0)$ contains only positive frequencies, so that we can write $\psi_p(t')=i\Delta t/(t'+i\Delta t)$. The PGB complex envelope for this pulse form at $r=z=0$ is, from Eq. (4),

$$\psi(r,z,t')=\frac{i\Delta t}{[t'-r^2/2cR(z)]+i[\Delta t+r^2/a^2(z)\omega_0]}\times\frac{iz_R}{q(z)}\exp\left[\frac{-ik_0r^2}{2q(z)}\right], \tag{22}$$

which is nonsingular, tends to zero for $r\to\infty$, and explicitly shows the time delay $r^2/2cR(z)$ and a pulse broadening $r^2/a^2(z)\omega_0$ increasing with r. The pulse forms $|\mathrm{Re}\,E|$ for several r are shown in Fig. 2(a) for the parameters $a_0=0.5$ mm, $T_0=1.65$ fs ($\omega_0=3.8$ fs^{-1}), and $\Delta t=0.48$ ($T/T_0\simeq1$ optical cycle within the envelope), and at fixed $z=z_R=k_0a_0^2/2=1.58\times10^3$ mm. (It should be noted that the vertical scales are different for the different pulses in this figure. Actually, the pulse amplitude diminishes for increasing r.) In this example, the change in the pulse form consists of an envelope broadening, whereas the frequency shift is negligible. The broadening causes the pulse to approach a monochromatic wave of frequency ω_0 for large r. These facts are clearer from Fig. 2(b), which shows the power spectra $|\hat{E}(r,z,\omega)|^2$ of the pulses of Fig. 2(a). As r increases, frequencies larger than ω_0 are gradually filtered, narrowing the spectrum (broadening the pulse) but maintaining the exponential decaying shape and the maximum amplitude at ω_0. The energy distribution

$$W(r,z)=\frac{\pi}{2}\left(\frac{a_0}{a(z)}\right)^2\frac{\Delta t^2}{\Delta t+r^2/a^2(z)\omega_0}\exp\left[-\frac{2r^2}{a^2(z)}\right] \tag{23}$$

is depicted in Fig. 2(c) for several values of z. It is appreciably narrower than the energy distribution of the cw Gaussian beam, also shown this figure. This difference may be attributed to the phase modulation in $p(t')$. $W(r,z)$ only approaches the Gaussian form of width $a(z)$ for large pulse duration ($\Delta t\to\infty$). Figure 2(d) shows the transversal amplitude distribution $|E(r,z,t')|$ at $z=z_R$ at the instants of time $t'=-2$, 0, and 2 fs. The narrower amplitude distribution occurs at $t'=0$ instead of negative t'.

Now consider the pulse

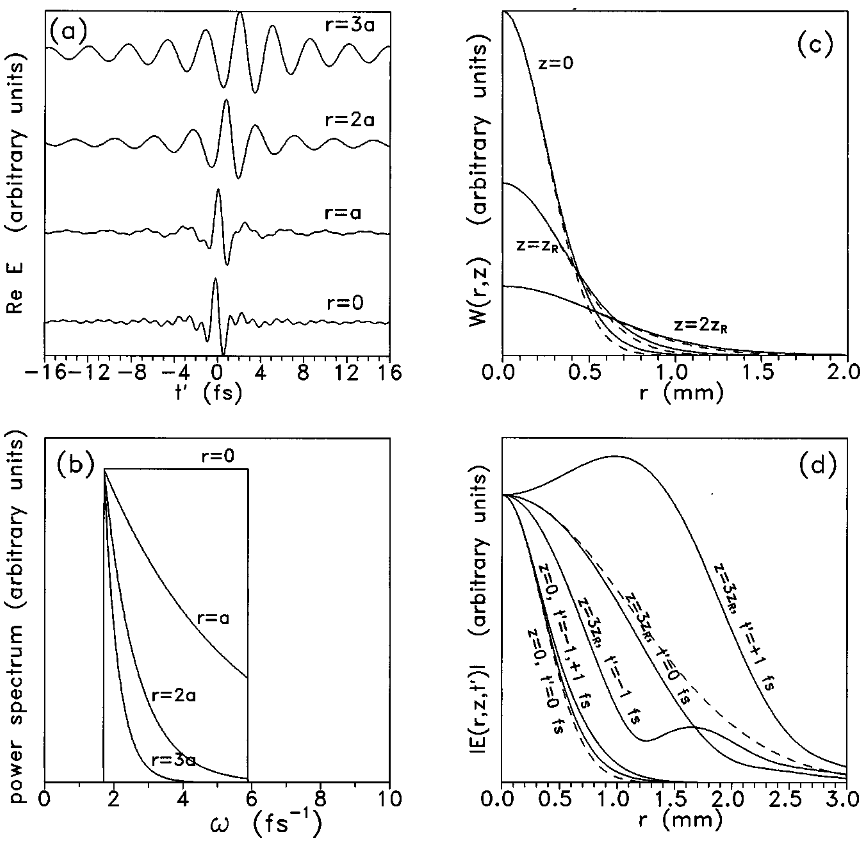

FIG. 3. (a) Pulse forms of the PGB of Eq. (25) for variable r. $a_0=0.5$ mm, $T_0=1.65$ fs ($\omega_0=3.8\ \mathrm{fs}^{-1}$), $\Delta t=3$ fs, and $z=z_R$. (b) Power spectra of the pulses of (a). (c) Energy distribution of the PGB of Eq. (25) at several cross sections. The beam parameters are the same as in (a). The dashed lines are the energy distributions of the cw Gaussian beam of ω_0. (d) Amplitude distribution at $z=0$ of the PGB of Eq. (25) at the time instants $t'=-1$, 0, and $+1$ fs, at $z=0$, and at $z=3z_R$. The dashed lines are the amplitude distributions of a cw Gaussian beam of ω_0.

$$p(t')=\mathrm{sinc}\left(\frac{2t'}{\Delta t}\right)\cos(\omega_0 t'), \tag{24}$$

where $\mathrm{sinc}(x)=\sin(\pi x)/\pi x$, and $\Delta t>0$ is the full width of the central maximum of $\mathrm{sinc}(2t'/\Delta t)$ (FWHM $T\simeq 0.52\Delta t$). We readily find that $f(\omega-\omega_0)=(\pi/\Delta\omega)\mathrm{rect}[(\omega-\omega_0)/\Delta\omega]$, with $\mathrm{rect}(x)=1$ if $|x|<\frac{1}{2}$, 0 otherwise, and $\Delta\omega=4\pi/\Delta t$. The pulse is then a uniform superposition of monochromatic waves within the interval $[\omega_0-\Delta\omega/2,\omega_0+\Delta\omega/2]$. If $\Delta\omega/2<\omega_0$, i.e., $\Delta t>T_0$ (the central maximum contains at least one optical cycle), $f(\omega-\omega_0)$ has only positive frequencies, and $\psi_p(t')$ is then the complex envelope of $p(t')$, i.e., $\psi_p(t')=\mathrm{sinc}(2t'/\Delta t)$ [in the case $\Delta\omega/2>\omega_0$, $\psi_p(t')$ should be calculated from Eq. (13)]. From Eq. (4), the PGB complex envelope is

$$\psi(r,z,t')=\mathrm{sinc}\left[\frac{2}{\Delta t}\left(t'-\frac{r^2}{2cq(z)}\right)\right]\frac{iz_R}{q(z)}\exp\left[\frac{-ik_0r^2}{2q(z)}\right]. \tag{25}$$

The pulse forms Re $E(r,z,t')$ of the PGB [Eq. (25)], showing the pulse time delay, broadening, and frequency shift for increasing r, can be seen in Fig. 3(a), for the parameters $a_0=0.5$ mm, $T_0=1.65$ fs ($\omega_0=3.8\ \mathrm{fs}^{-1}$), $\Delta t=3$ fs, and $z=z_R$. In this case, the filtering of high frequency components [see Fig. 3(b)] has the double effect of indefinitely broadening the pulse and shifting the frequency of the oscillations within the envelope up to the limit $\omega_0-\Delta\omega/2$. The pulse then approaches a monochromatic pulse of frequency $\omega_0-\Delta\omega/2$ as r increases. The energy distribution, calculated from Eq. (25), is

$$W(r,z)=\frac{\pi}{\Delta\omega}\left[\frac{a_0}{a(z)}\right]^2\frac{\sinh\left(\dfrac{\Delta\omega}{\omega_0}\dfrac{r^2}{a^2(z)}\right)}{\left(\dfrac{\Delta\omega}{\omega_0}\dfrac{r^2}{a^2(z)}\right)}\exp\left[-\frac{2r^2}{a^2(z)}\right]. \tag{26}$$

This differs only slightly from the energy distribution of the cw Gaussian beam [see Fig. 3(c)], as predicted above by the approximate formula (20). At $z=0$, the transversal amplitude distribution $|E(r,z,t')|$, shown in Fig. 3(d) at the instants $t=-1$, 0, and $+1$ fs, is almost independent of time, and has a width $\sim a_0$. At $z=3z_R$, however, diffraction broadening is sizably reduced in the leading part of the pulse, and increased in the trailing part [see Fig. 3(d)].

For comparison with Ref. [2], we finally consider the Gaussian pulse $p(t')=\exp(-t'^2/\Delta t^2)\cos(\omega_0 t)$ (the FWHM $T=2\sqrt{\ln 2}\Delta t\simeq 1.66\Delta t$), for which $f(\omega-\omega_0)=\frac{1}{2}\sqrt{\pi}\Delta t\,\exp[-\Delta t^2(\omega-\omega_0)^2/4]$ does not vanish at any ω. From Eq. (12), the analytic signal is

$$\psi_p(t')\exp(i\omega_0t')=\exp\left[-\frac{t'^2}{\Delta t^2}\right]\left\{\exp(i\omega_0t')-i\ \mathrm{Im}\left[\exp(i\omega_0t')\mathrm{erfc}\left(\frac{\Delta t\omega_0}{2}+i\frac{t'}{\Delta t}\right)\right]\right\}, \tag{27}$$

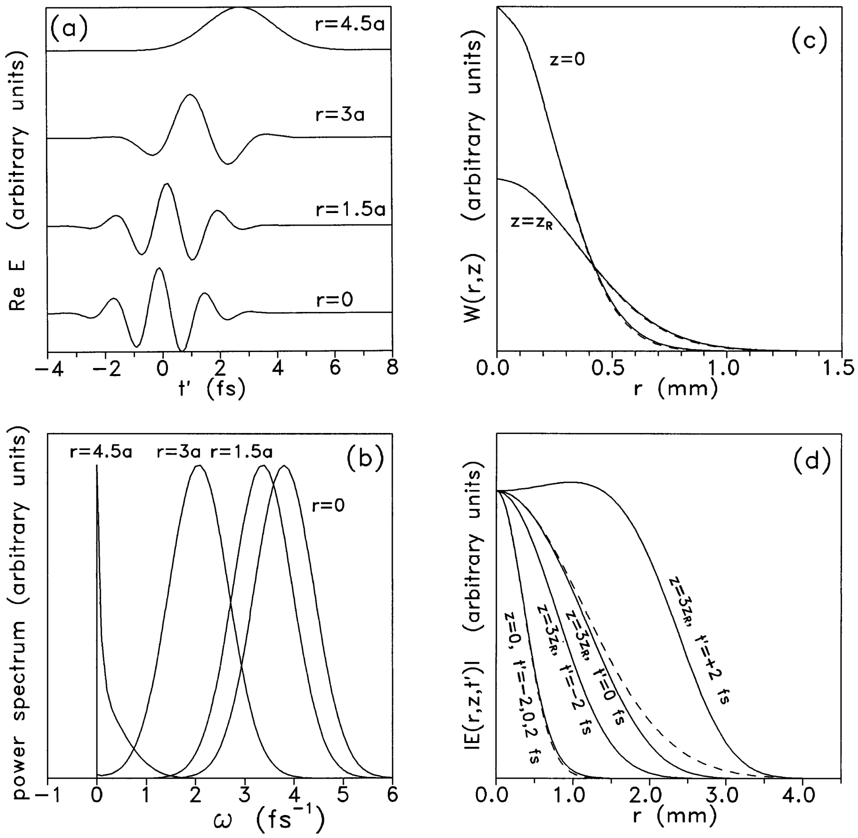

FIG. 4. (a) Pulse forms of the Gaussian PGB for variable r. $a_0=0.5$ mm, $T_0=1.65$ fs ($\omega_0=3.8$ fs^{-1}), $\Delta t=1.65$ fs, and $z=z_R/2$. (b) Power spectra of the pulses of (a). (c) Energy distribution of the Gaussian PGB at several cross sections. The beam parameters are the same as in (a). The dashed lines are the energy distributions of the cw Gaussian beam of ω_0. (d) Amplitude distribution at $z=0$ of the Gaussian PGB at the time instants $t'=-2$, 0, and $+2$ fs, at $z=0$, and at $z=3z_R$. The dashed lines are the amplitude distributions of a cw Gaussian beam of ω_0.

where erfc( ) is the complementary error function. It is clear from this expression that $\psi_p(t')$ is not the complex envelope $A(t')\exp[i\Phi(t')]=\exp(-t'^2/\Delta t^2)$. The Gaussian PGB is obtained, according to Eq. (5), by multiplying by $iz_R/q(z)$ and replacing t' by $t'_c=t'-r^2/2cq(z)$ in Eq. (27). The pulse forms Re E and corresponding spectra are shown in Figs. 4(a) and 4(b) for the parameters $a_0=0.5$ mm, $T_0=1.65$ fs, $\Delta t=1.65$ fs ($T/T_0=1.66$ cycles within the pulse), and $z=z_R/2$. In this case, the filtering of high frequencies with increasing r does not lead, at the first values of r, to a pulse deformation, nor even to broadening, but solely to a frequency shift. For larger values of r, the spectrum reaches the surroundings of $\omega=0$, where it distorts and narrows, giving rise to pulse broadening. In Figs. 4(c) and 4(d), we see the transversal energy and amplitude distributions for several values of z. As suggested by the approximate formula [Eq. (20)] for pulses without phase modulation, $W(r,z)$ is almost Gaussian, of width $\sim a(z)$; at $z=0$, $|E(r,z,t')|$ is almost time independent, of width $\sim a_0$, and at $z=3z_R$ diffraction spreading increases sizeably with time.

Figure 5 again shows the amplitude distribution of the Gaussian PGB of this paper, along with the Gaussian PGB of Ref. [2], both for the coordinates $z=0$ and $t'=0$, and the parameters $a_0=0.5$ mm, $T_0=3.55$ fs, and $\Delta t=2.14$ fs ($T/T_0=1$). The PGB of Ref. [2] is obtained from the one of this paper by eliminating the term with the function erfc in Eq. (27) [this amounts to neglecting f^* in Eq. (12) and extending the integral up to $-\infty$]. This approximation causes the beam amplitude to grow without bound in the transversal direction.

VI. CONCLUSION

In this paper we have provided a model for the spatiotemporal structure of pulsed light beams propagating in free space, particularly those emitted by mode-locked lasers. As well as the monochromatic Gaussian beam, the PGB's, having nearly Gaussian spatial forms and arbitrary prescribed temporal shapes at the beam axis, are solutions of the paraxial wave equation, and are found by the same method of shifting the axial coordinate z by an imaginary constant in the paraxial spherical wave. The PGB is a superposition of axial Gaussian modes of a stable laser resonator rather than a single mode, and its spatial and temporal characteristics are intimately entwined if the pulse duration is of the same order

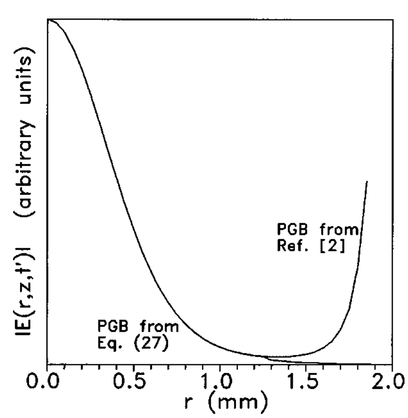

FIG. 5. Amplitude distribution at $z=0$ and $t'=0$ of the Gaussian PGB of this paper, and that of Ref. [2]. The parameters are $a_0=0.5$ mm, $T_0=3.55$ fs ($\omega_0=1.77$ fs^{-1}), and $\Delta t=2.14$ fs.

of magnitude as the optical period.

In the time domain, the spatiotemporal coupled structure of the PGB leads, regardless of the pulse form at the beam axis, to a pulse shift and a change in the pulse shape toward the beam periphery. Depending on the axial pulse form, such a change may manifest itself as an envelope distortion (broadening in simple cases), a frequency shift toward lower frequencies, or both. In the spatial domain, spatiotemporal coupling gives rise to time-dependent diffraction and a pulse-form-dependent spatial distribution of energy. The way the transversal amplitude distribution varies with time is quite involved in general, but, for pulses without phase modulation, the pattern is simpler. At the waist, the transverse amplitude profile is almost Gaussian, of the same width as that of the monochromatic Gaussian beam, and almost independent of time. At some distance from the waist in the direction of propagation, diffraction spreading is sizably diminished when the pulse arrives, but enlarged when the pulse surpasses that distance. The PGB has an invariant transverse pattern of energy distribution, determined by the pulse form, whose width and amplitude vary at the same rate as those of the monochromatic Gaussian beam of the carrier frequency. For pulses without phase modulation, the energy distribution is almost Gaussian, and of the same width as that of the monochromatic Gaussian beam at any propagation distance.

Most of these effects go unnoticed in Ref. [2] due to the fact that the authors only considered a PGB of Gaussian pulse form, and the amplitude distribution at only one instant of time $(t'=0)$. More importantly, their Gaussian PGB lacks a true beam behavior because of their use of the common complex representation $A(t')\exp[i\Phi(t')]\exp(i\omega t')$ of the real pulse $A(t')\cos[\omega_0 t'+\Phi(t')]$, which only approximates the analytical signal complex representation for real time. In the present paper, transversally decaying spatial profiles for PGB's with arbitrary pulse forms are achieved by the use of a true analytic signal complex representation.

The PGB of this paper should not be confused with the pulsed Gaussian beams studied in earlier papers [5,7], in which the pulsed beam is a propagated initial condition of transversal and temporal Gaussian form. By contrast, the PGB described here has a prescribed temporal form at the beam axis. Our PGB also differs from the pulsed beams of Refs. [13,14], where the initial condition itself contains spatiotemporal couplings by the action of a lens of dispersive material, but the process of propagation is described by the paraxial wave equation without the crossed derivative.

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