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Anomalous diffusion as a signature of a collapsing phase in two-dimensional self-gravitating systems

Mickaël Antoni*
Max-Planck-Institut für Physik Komplexer Systeme, Nothnitzer Strasse 38, D-01187 Dresden, Germany

Alessandro Torcini†
Dipartimento di Energetica "S. Stecco," Università di Firenze, via S. Marta, 3 , I-50139 Firenze, Italy

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A two-dimensional self-gravitating Hamiltonian model made by N fully coupled classical particles exhibits a transition from a collapsing phase (CP) at low energy to a homogeneous phase (HP) at high energy. From a dynamical point of view, the two phases are characterized by two distinct single-particle motions: namely, superdiffusive in the CP and ballistic in the HP. Anomalous diffusion is observed up to a time τ that increases linearly with N. Therefore, the finite particle number acts like a white noise source for the system, inhibiting anomalous transport at longer times. [S1063-651X(98)51405-5]

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In past years the thermodynamical properties of gravitational models have been studied in detail from a theoretical [1] and computational [2–4] point of view. In particular, it has been shown that at low energy the gravitational forces give rise to a collapsing phase (CP), identified by the presence of a single cluster of particles floating in a diluted homogeneous background. At high energy a homogeneous phase (HP) is recovered; the cluster disappears and the particles move almost freely. In the transition region the system is characterized (in the microcanonical ensemble) by a negative specific heat: the corresponding instability (termed "gravo-thermal catastrophe") is of extreme relevance for astrophysics (see Ref. [5] for more details). This apparent thermodynamical inconsistency has been solved by Hertel and Thirring in Ref. [1], where they demonstrated the nonequivalence of canonical and microcanonical ensemble in the vicinity of the transition region. These theoretical results have been successfully confirmed by numerical investigations of self-gravitating nonsingular systems with short-range interaction [2,3].

More recently, in one-dimensional (1D) lattices of fully and nearest-neighbor coupled symplectic maps with an attractive interaction, it has been observed that clustering phenomena are associated with anomalous diffusion (in particular, with subdiffusive motion), at least for short times [6]. Anomalous diffusion can be defined through the time dependence of the single-particle mean-square displacement (MSQD) $\langle r^2(t)\rangle$, which typically reads as

$$\langle r^2(t)\rangle \propto t^\alpha \tag{1}$$

where the average $\langle\ \rangle$ is performed over different time origins and over all the particles of the system. The transport is anomalous when $\alpha\neq 1$, superdiffusive for $1<\alpha<2$, and subdiffusive for $0<\alpha<1$ [7,8]. Anomalous transport has been revealed in dissipative and Hamiltonian models [7] as well as in experimental measurements [9]. However, most of the literature focuses on systems with few degrees of freedom (namely, one or two), and only a few studies have been devoted to extended models with $N\gg 1$ [6].

In this Rapid Communication, the thermodynamical and dynamical properties of a two-dimensional (2D) Hamiltonian system, consisting of N particles interacting via a long-range attractive potential, are analyzed. In particular, we observe a transition from CP to HP associated with a dynamical transition from anomalous to ballistic transport. Finite N effects induce a crossover from anomalous to normal diffusion at long times. In the limit $N\to\infty$, the transport mechanism remains anomalous at any time and reduces to that of a single particle in an "egg-crate" potential [10].

We consider a system of N identical fully coupled particles with unitary mass evolving in a two-dimensional periodic cell described by the Hamiltonian (for the 1D case, see Ref. [11]):

$$H=K+V=\sum_{i=1}^{N}\frac{p_{x,i}^2+p_{y,i}^2}{2}+\frac{1}{2N}\sum_{i,j}^{N}\,[3-\cos(x_i-x_j)-\cos(y_i-y_j)-\cos(x_i-x_j)\cos(y_i-y_j)], \tag{2}$$

where $(x_i,p_{x,i})$ and $(y_i,p_{y,i})$ are the two pairs of conjugate variables with $(x_i,y_i)\in[-\pi,\pi]\times[-\pi,\pi]$, where K and V are the kinetic and potential energy, respectively. The potential part corresponds to the first three terms of the Fourier expansion of a 2D attractive potential of the type $V(r)\propto\log|r|$. Such a type of interaction arises in self-gravitating 2D gases [1–3] as well as in the point vortices model for 2D turbulence [12]. Due to the long-range interaction among all the particles, this model can be described in terms of mean-field variables. In particular, the potential energy can be rewritten as $V=\frac{1}{2}\sum_{i=1}^{N}V_i$, with

*Electronic address: antoni@sebigbos.mpipks-dresden.mpg.de
†Affiliated with INFM, Firenze, Italy.

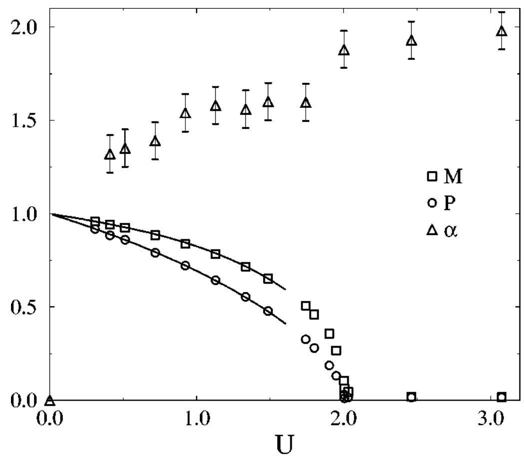

FIG. 1. Time averages of M and P as a function of U. The solid curves refer to the theoretical estimation (i.e., to canonical results) and the symbols refer to the MD findings (i.e., to microcanonical results). The exponents α, defined in Eq. (1), are also reported (triangles). The MD data have been obtained with $N=4000$ (apart from few points with $N=10\,000$) and averaged over a total integration time ranging from $t=1{,}2\times10^6$ to $t=4{,}2\times10^6$. The α values have been estimated in the time interval $150<t<10\,000$ for any reported U.

$$V_i=3-M_x\cos(x_i-\phi_x)-M_y\cos(y_i-\phi_y)$$
$$-\tfrac{1}{2}[M_{xy}^{+}\cos(x_i+y_i-\phi_{xy}^{+})+M_{xy}^{-}\cos(x_i-y_i-\phi_{xy}^{-})], \quad (3)$$

where $\mathbf{M}_z=(\langle\cos(z)\rangle_N,\langle\sin(z)\rangle_N)=M_z\exp[i\phi_z]$ represents four two-dimensional mean-field vectors with $z=x,y,x\pm y$, and $\langle\ \rangle_N$ denotes the average over N. However, the single-particle potentials V_i are nonautonomous, since the mean-field quantities M_z and ϕ_z are defined through the instantaneous values of the particles] coordinates. The motion of each particle is therefore determined self-consistently by an attractive and nonautonomous force field that is uniquely determined throughout the motion of all the particles. The self-gravitating nature of the model is due to this effective force acting among the particles. Since V is invariant under the transformations $x\leftrightarrow -x$, $y\leftrightarrow -y$, and $x\leftrightarrow y$, it turns out that in the ''mean-field limit'' (i.e., for $N\rightarrow\infty$ with $U=H/N$ constant), $M_x=M_y=M$ and $M_{xy}^{+}=M_{xy}^{-}=P$. Moreover, in this limit and assuming that $\phi_z=0$, the single-particle potential V_i turns out to be an egg-crate potential similar to that studied in Ref. [10]. This periodic potential is characterized in each elementary cell by a minimum ($V_m=3-2M-P$), four maxima ($V_M=3+2M-P$), and four saddle points ($V_s=3+P$).

For specific energy U smaller than a critical value $U_c\simeq2$, we observe that the particles are mainly in a clustered state. Above U_c a HP is recovered. Following Refs. [6,11], the degree of clustering of the particles can be characterized through the time averages $\langle M_z\rangle_t$ [13,14]. When at each time the particles have almost the same position, $\langle M_z\rangle_t$ is $\mathcal{O}(1)$, while, for a HP, their values vanish as $1/\sqrt{N}$ [11]. Figure 1 shows that, for $U\rightarrow0$, the average quantities M,P tend to one. This indicates that the particles are almost all trapped in a potential well of depth $\simeq(V_s-V_m)$, forming a compact cluster. For increasing energy U, the kinetic contribution becomes more relevant and the average number of particles trapped in the potential well drops. As a consequence, the value of $\langle M_z\rangle_t$ decreases together with (V_s-V_m). For $U\geqslant U_c$, the system is no longer clustered and the particles can move almost freely. Moreover, due to finite N effects $\langle M_z\rangle_t$ is not exactly zero, but $\mathcal{O}(1/\sqrt{N})$.

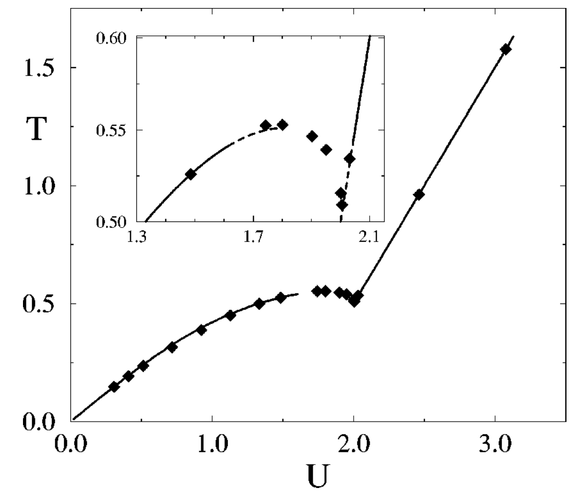

FIG. 2. Temperature T as a function of the energy U. The solid line corresponds to the analytical estimation (canonical ensemble), and the triangles correspond to the simulations results (microcanonical ensemble). In the inset, an enlargment of the transition region is reported; the solid (respectively dashed) curves refer to the principal (respectively relative) minimum of $F(T)$. The parameters for the MD simulations are the same as in Fig. 1.

In Fig. 2 the temperature $T=\langle K\rangle_t/N$ is reported as a function of U. Above U_c, T increases linearly with U, indicating that the system behaves like a free particle gas. In the CP, the tendency of the system to collapse is balanced by the increase of the kinetic energy [2]. This competition leads initially (for $0<U<1.8$) to a steady increase of T, followed (for $1.8<U<U_c$) by a rapid decay of T. This yields a negative specific heat as illustrated in the inset of Fig. 2. These results are in full agreement with theoretical predictions based on the analysis of a simple classical cell model [1], and with numerical findings [2,3], for short-ranged attractive potentials. The phenomenon of negative specific heat can be explained within a microcanonical approach with a heuristic argument [1]. Approaching the transition, a small increase of U leads to a significant reduction in the number of collapsed particles (as confirmed from the drop exhibited by M and P for $U>1.8$); as a consequence the value of V grows and, due to energy conservation, the system becomes cooler.

Our data also confirm another important prediction of Hertel and Thirring [1]: the nonequivalence of canonical and microcanonical ensemble near the transition region. In the inset of Fig. 2 the microcanonical findings are reported, obtained via standard molecular-dynamics (MD) simulations; the theoretical canonical results were derived in the mean-field limit [15,16]. These two sets of data coincide everywhere, except in the energy interval $1.6<U<2.0$. The discrepancy is due to the impossibility of the canonical ensemble to exhibit a negative specific heat, a limitation that does not hold for the microcanonical ensemble. Our theoretical estimation of the Helmholtz free energy $F=F(T)$ re-

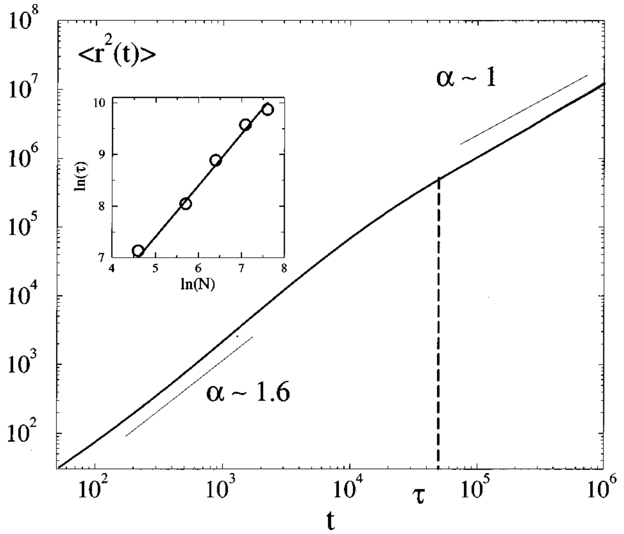

FIG. 3. Log-log plot of the mean-square displacement $\langle r^2(t)\rangle$ versus time. The crossover time τ is indicated by a dashed line. The data refer to $U=1.1$, $N=4000$, and to a total integration time $t=4.2\times 10^6$. In the inset, the logarithm of τ is reported as a function of ln(N) for $U=1.48$. The τ values (circles) have been estimated considering a threshold $\beta=1.1$ [17]. The solid line represents a best linear fit to the data, and its slope is 0.95 ± 0.08.

veals that usually F has a unique minimum. For $T<0.5$, the minimum F_C corresponds to nonzero values of M and P (i.e., to the CP), while for $T>0.55$, the minimum F_H is associated with $M=P=0$ (i.e., with the HP). In the region $0.5<T<0.55$, both minima F_C and F_H coexist as local minima of the free energy. However, for $T<T_c=0.54$, the CP is observed because $F_C<F_H$, while for $T>T_c$ the HP prevails, since $F_H<F_C$. At $T=T_c$ the two minima are equivalent and a jump in energy from $U(T_c^-)\approx 1.6$ to $U(T_c^+)\approx 2.0$ is observed. This picture suggests that this transition can be considered as a first-order transition [2].

Let us now investigate whether the observed thermodynamical transition has any effect on the dynamical behavior of the system. In order to characterize the single-particle dynamics, we consider the MSQD $\langle r^2(t)\rangle$. As shown in Fig. 3, in the CP the diffusion is anomalous for times shorter than a crossover time τ, while for longer times the Einstein law is recovered $\langle r^2(t)\rangle \propto 4Dt$ (where D is the diffusion coefficient). A similar behavior for the MSQD has already been observed for a system of N coupled simplectic maps in Ref. [6], but with $\alpha<1$. However, in the present case τ increases linearly with N [14], indicating that in the mean-field limit the asymptotic dynamical regime will be superdiffusive [17].

The observed dynamical behavior can be explained by noticing that in the mean-field limit each particle i will see essentially the same constant 2D egg-crate potential V_i. Moreover, it has been shown in Ref. [10] that a single particle moving in an egg-crate potential with an energy between V_s and V_M exhibits superdiffusion. This is due to the fact that the particle moves for long times almost freely along the channels of the potential and episodically is trapped for a while in the potential well. In phase space, the superdiffusive phenomenon can be explained by a trapping mechanism in a hierarchy of cantori around a cylindrical Kol'mogorov-Arnol'd-Moser surface [7]. Therefore, in our model (2) for $N\to\infty$, anomalous transport is due only to the fraction of particles that can move along the channels.

For finite N, the potential V_i seen by the particle i will fluctuate in time. Hence, particles having an energy close to V_S have the possibility of being trapped in the potential well as well as of escaping from it. As a consequence, for sufficiently long time scales each particle can experience free and localized motions. The fluctuations of the potential V_i are reflected in the structure of the phase space, introducing a white noise that destroys the self-similar structure of the island chains and of the cantori below a certain cutoff size. Since the self-similarity is no more complete, one can expect that on long-time scales normal diffusion will be recovered [18].

As is pointed out in Refs. [19,20], if white noise is added to a dynamical system exhibiting superdiffusive behavior, D (measured in the limit $t\to\infty$) is inversely proportional to the noise amplitude. Therefore, we expect that in our model the value of D will increase with N. That is indeed the case, and we observe a power-law dependence of the type $D\propto N^{\gamma}$. For example, considering systems with $100\leqslant N\leqslant 10\,000$ we have found, for $U=1.48$ and $U=2.00$, a γ value equal to 0.7 ± 0.1 and 1.0 ± 0.1, respectively. The N dependence of the diffusion coefficient can be explained by noticing that $D\propto\tau^{\alpha-1}$ [21]. This result coincides with that found theoretically in Ref. [19] and confirmed numerically by considering two very simple noisy maps as dynamical models. For subdiffusive motion, D is inversely proportional to τ (as found in [18]), while for superdiffusive motion ($\alpha>1$), a direct proportionality is expected [19]. As already reported, $\tau\propto N$; therefore, we will see that $\gamma=\alpha-1$. Assuming for α the corresponding asymptotic values [22], we can estimate as theoretical values $\gamma\simeq 0.64$ and $\simeq 0.9$ for $U=1.48$ and 2.00, respectively. In view of all the present limitations, these values can be considered consistent with the numerical findings.

As a final point, we examine the dependence of the asymptotic α values from the energy U of the system. A transition from anomalous diffusion to ballistic motion ($\alpha=2$) at $U\simeq U_c$ is evident from Fig. 1 where the α values, obtained for $N=4000$, are reported. In particular, for $0.4\leqslant U\leqslant 2.0$, we observe an increase of α from 1.3 ± 0.1 to 1.9 ± 0.1. This phenomenon is a consequence of the flattening of the single-particle potential (i.e., of the reduction of V_M-V_m) observed for growing U. The decrease in the average number of particles trapped in the cluster, and the consequent increase of those moving freely, naturally drives the diffusion mechanism toward a ballistic behavior. Moreover, for $U>U_c$, the potential V_i fluctuates with typical amplitude $\mathcal{O}(1/\sqrt{N})$ around a constant value and a ballistic motion is expected for all the particles. For small energies ($U\leqslant 0.3$) the MSQD seems to saturate to a constant value, indicating that all the particles are always clustered.

In conclusion, we have shown a thermodynamical transition associated with a dynamical transition from anomalous to ballistic transport. Moreover, the transport in our N body system can be interpreted in terms of a noisy single-particle motion in a 2D Hamiltonian egg-crate potential. The asymptotic dynamics of the model is strongly influenced by the order in which the two limits $N\to\infty$ and $t\to\infty$ are taken. Indeed, if the limit $N\to\infty$ is performed before the limit $t\to\infty$, the diffusion will be always anomalous. Otherwise, *normal diffusion is recovered for sufficiently long times*.

As a final remark, we expect that anomalous diffusion should be observable for atomic clusters [23], turbulent vortices [12] (for which it has already been observed [9]), and gravitational systems [5], all systems exhibit a clustered phase.

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