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Dynamical structure factors in models of turbulence

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We investigate the dynamical scaling behavior of the time-dependent structure functions, $S_q(r,\tau) = \langle [u(r,\tau)-u(0,0)]^q \rangle$, in the one-dimensional, stochastic Burgers equation as a function of the exponent β that characterizes the scale of noise correlations. We present and analyze the exact equations satisfied by $S_2(r,\tau)$ and a related correlation function to argue that (a) $\partial S_2(r,\tau)/\partial\tau$ exhibits a discontinuity at $\tau=0$ with an effective dynamical exponent given by $1+\beta/3$ and (b) the dynamical scaling exponent z is unity for intermediate times (a result equivalent to Taylor's hypothesis). Various numerical checks of these results are presented. Finally, the corresponding exact equations for the structure functions in the case of the Navier-Stokes equation are presented, and by analogy with the one-dimensional Burgers equation it is shown how Taylor's hypothesis can arise in homogeneous turbulence. [S1063-651X(98)50705-2]

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In this paper we address the problem of the dynamical behavior of structure functions in the steady state of a stochastically driven Burgers equation (1) in one dimension. Our aim is to shed light on Taylor's hypothesis in three-dimensional homogeneous turbulence [2,3]. The significance of Taylor's hypothesis is clear when there is an average flow, but even in the case of homogeneous isotropic turbulence it is taken to be true, the root mean square of the turbulent velocity fluctuations replacing the average flow velocity. One can view Taylor's hypothesis as saying that since characteristic time and distance are linearly related, the dynamic scaling exponent z, defined by $\tau \propto r^z$, is equal to 1. Here we present the exact equations satisfied by the low-order, time-dependent structure functions, and use analytical arguments to show how a dynamical exponent $z=1$ arises and provide numerical evidence for it. For the case of the Navier-Stokes equations, we also present the corresponding exact equations for the structure functions, draw a parallel to the analysis in the Burgers problem, and argue that a similar mechanism can lead to $z=1$ in three-dimensional homogeneous turbulence [4]. Our arguments, while they do not constitute a proof in that they require reasonable assumptions, provide a persuasive mathematical picture.

The stochastic Burgers equation in one dimension reads

$$\partial u/\partial t + u\,\partial u/\partial x = \nu \nabla^2 u + \eta(x,t), \tag{1}$$

where $u(x,t)$ is the velocity field, ν is the viscosity, and $\eta(x,t)$ is a Gaussian noise with zero mean and correlations in k space determined by $\langle \hat{\eta}(k,t)\hat{\eta}(k',t')\rangle = 2\hat{D}(k)2\pi\delta(k+k')\delta(t-t')$, where the noise variance $\hat{D}(k)$ exhibits power-law behavior $\hat{D}(k)=D_0|k|^{\beta}$ [5–8]. Here we consider only negative values of β.

The objects of our study are the time-dependent (velocity) structure functions $S_q(r,\tau)$ defined in the homogeneous steady state by

$$S_q(r,\tau) = \langle [u_1-u_2]^q \rangle, \tag{2}$$

where we have introduced the notation that we will employ in our subsequent analysis:

$$u_1 \equiv u(x+r/2,\ t+\tau/2), \quad u_2 \equiv u(x-r/2,\ t-\tau/2).$$

The usual static ($\tau=0$) structure functions exhibit power-law behavior in the inertial range: $S_q(r) \sim |r|^{\zeta_q}$. We know the (noise-dependent) exponents ζ_q for small q from our previous studies [8]; we have reported results for β positive and negative in the static case, and, in particular, pointed out [8] the rich multifractal behavior that occurs for β between 0 and -1.

Our analysis relies on the following exact equation satisfied by $S_2(r,\tau)$ in the one-dimensional (1D) stochastic Burgers problem in the steady state:

$$\frac{\partial}{\partial\tau} S_2(r,\tau) = \frac{1}{2}\frac{\partial}{\partial r} T_3 + \langle u_1\eta_2\rangle - \langle u_2\eta_1\rangle, \tag{3}$$

where

$$T_3(r,\tau) = -\langle (u_1+u_2)(u_1-u_2)^2 \rangle, \tag{4}$$

which apart from additive constants is the same as $\langle u_1^2 u_2 + u_1 u_2^2 \rangle$. For Gaussian noise, one can use the result of Donsker, Varadhan, and Novikov [9] to show that $\langle u_1\eta_2\rangle$ in Eq. (3) is the space-time Fourier transform of $\hat{D}(k)\hat{G}(k,\omega)$, where $\hat{G}(k,\omega)$ is the response function defined by $\langle \delta\hat{u}(k,\omega)/\delta\hat{\eta}(k',\omega')\rangle = \hat{G}(k,\omega)2\pi\delta(k+k')2\pi\delta(\omega+\omega')$. Moreover, at $\tau=0$, we know that $S_2(r,0) \approx A_2|r|^{\zeta_2}$ with $\zeta_2 = -2\beta/3$ from our previous work.

We now discuss the inertial-range behavior of $S_2(r,\tau)$ that can be deduced from Eq. (3). First, let us consider the behavior of S_2 for r in the inertial range and $\tau\to 0$. At small τ there is a discontinuity that arises because $\langle u_1\eta_2\rangle$ contributes for $\tau>0$ and $-\langle u_2\eta_1\rangle$ for $\tau<0$. It is easy to show from the constraint of causality that $\hat{G}(k,\tau=0^+)=\frac{1}{2}$ and, hence, that

$$\left[\frac{\partial}{\partial\tau} S_2(r,\tau)\right]_{\tau\to 0^+} = \tfrac{1}{2}\sum_k \hat{D}(k)\cos kr$$

$$=-\frac{1}{12}\frac{\partial}{\partial r}S_3(r,\tau=0), \tag{5}$$

where the last equality follows from the von Karman–Howarth relation [4,7]. We know that $S_3(r)\propto r^{-\beta}$ for r in the inertial range and, therefore, from the hypothesis of dynamic scaling for S_2, using $\tau\propto r^z$, we obtain $z=1+(\beta/3)$. This relationship between z and the parameter β, which determines the scale over which the noise acts, is the same as the one that is derived from a renormalization group argument [10] for positive values of β. A simple Kolmogorovlike scaling argument yields this value of z: the characteristic time scale associated with the length scale l is given by $\tau(l)\approx l/u(l)\approx l^{1-h}$, where h, the scaling dimension of u, is $-\beta/3$ [11,12]. We have calculated numerically the discontinuity at different values of β; time derivatives are clearly difficult to compute numerically; nevertheless, reasonable agreement with the above relation is obtained.

In order to understand the behavior of $S_2(r,\tau)$ at longer times τ we must examine the term involving $T_3(r,\tau)$ on the right-hand side of Eq. (3); we have derived the equation satisfied by $T_3(r,\tau)$:

$$\frac{\partial T_3(r,\tau)}{\partial\tau}=-\frac{1}{12}\frac{\partial\langle(u_1+u_2)^4\rangle}{\partial r}-2\langle(u_1\hat{\epsilon}_2-u_2\hat{\epsilon}_1)\rangle + \tfrac{1}{2}\langle(\eta_1-\eta_2)(u_1+u_2)^2\rangle, \tag{6}$$

where we have used the notation $\hat{\epsilon}_1=\nu/2(\partial u/\partial x)^2(x+r/2,t+\tau/2)$, etc.

We will outline the crux of the argument and address the technicalities later. Consider the first term in Eq. (6) that arises from nonlinear effects and postpone the discussion of the other terms. It is straightforward to argue that the first term behaves as a linear combination of $\partial S_4/\partial r$ and $\partial S_2/\partial r$. This is seen by noting that

$$\partial\langle(u_1+u_2)^4\rangle/\partial r=-12\partial\langle(u_1-u_2)^2(u_1^2+u_2^2)\rangle/\partial r + 5\partial S_4/\partial r,$$

verified by taking into account that $\langle u_1^4\rangle=\langle u_2^4\rangle$ is a constant because of homogeneity. The leading behavior of the first term is determined in the $\nu\to 0$ limit by the behavior of the derivative of $(u_1-u_2)^2$ when $r/L<1$, and thus yields a contribution proportional to $\langle u^2\rangle\partial S_2/\partial r$. The result can be justified from an operator product expansion point of view expanding the operator $\langle(u_1+u_2)^4\rangle$ in terms of S_q; by symmetry, S_3 will not appear in the expansion, and the leading term is S_2.

In the static limit S_2 dominates S_4 in the inertial region, because $\zeta_2<\zeta_4$ [6]. This continues to be true if $\zeta_2/z_2<\zeta_4/z_4$, where we have denoted the dynamic scaling exponent for S_4 by z_4 and, in particular, for $z_4=z_2$, as we will argue later. Hence, we obtain (suppressing the other terms)

$$\frac{\partial T_3(r,\tau)}{\partial\tau}\propto\langle u^2\rangle\frac{\partial S_2}{\partial r}. \tag{7}$$

This, combined with the first term in Eq. (3) for S_2, i.e.,

$$\frac{\partial S_2(r,\tau)}{\partial\tau}\propto\frac{\partial T_3}{\partial r}, \tag{8}$$

yields

$$\frac{\partial^2 S_2(r,\tau)}{\partial\tau^2}\propto\langle u^2\rangle\frac{\partial^2 S_2}{\partial r^2}. \tag{9}$$

This equation, which encapsulates Taylor's hypothesis for the dynamics of the one-dimensional Burgers equation, clearly leads to $z=1$.

We expect this to occur over time scales large compared to δ_2^2/ν where δ_2 is the short-distance cutoff for $S_2(r,\tau=0)$ and small compared to the turnover time for the large-scale structures in the system. Over these time scales, the large-scale motions provide an effective nonzero (local) background velocity, of the order of $\sqrt{\langle u^2\rangle}$, that does not average to zero. The dynamical exponent of unity reflects this effective motion. Obviously over much larger time scales the average background velocity vanishes.

Next, we fill in the gaps in the analysis; we discuss the other terms in Eq. (6) that were ignored. Consider the second term on the right-hand side that arises from viscous effects and use arguments reminiscent of Obukhov's argument for the exponent μ in three dimensional (3D) turbulence. We replace $\hat{\epsilon}_2\propto\nu(\partial u_2/\partial x_2)^2$ by $-\nu u_2\partial^2 u_2/\partial x_2^2$; then we use the equation of motion to replace the viscous term by the nonlinear term to obtain $\partial u_2^3/\partial x_2$. Straightforward manipulations lead to the identification of the expression $\partial(u_1^3u_2+u_1u_2^3)/\partial r$ with the term $-2\langle(u_1\hat{\epsilon}_2-u_2\hat{\epsilon}_1)\rangle$. One now notes that such terms are exactly those contained in the first term that we have already analyzed. The noise term depends on a higher-order response function, and by power counting it cannot yield a term more singular than those we have retained; a similar argument applies to Eq. (3), and we have checked numerically (see later) that its contribution is not significant.

We point out in passing how the Kolmogorovlike scaling prediction of $z=1+\beta/3$ is present in our analysis. The term $\partial S_4/\partial r$ that we have shown is contained in the first term in Eq. (6) for T_3 and yields a contribution $\partial^2 S_4/\partial r^2$ to $\partial^2 S_2/\partial\tau^2$. This leads to a dynamical scaling exponent $1+(\zeta_2-\zeta_4)/2$, which in the scaling regime, where $\zeta_4=2\zeta_2$, yields $1-\zeta_2/2=1+\beta/3$. We have outlined how $z=1$ arises analytically; we have performed numerical simulations using a standard pseudospectral code [13] to provide support for the various implications of our analysis.

We display the behavior of $S_2(r=0,\tau)\propto|\tau|^{\zeta_2/z}$ with $\zeta_2=-2\beta/3$ as a function of τ for $\beta=-1$ in Fig. 1. The data are consistent with $z=1$, since the slope is close to $\frac{2}{3}$, the slope of $S_2(r,\tau=0)$. The difference between $z=1$ and the dynamical renormalization group (RG) value of $z=\frac{2}{3}$ is easily distinguished, since the latter would yield a slope of unity. The range over which $z=1$ is obtained is consistent with the time scales over which the small k modes do not vary very much and extends up to $L_0/\sqrt{\langle u^2\rangle}$ where L_0 is the scale up to which inertial range scaling holds in coordinate space. We have checked that at $\beta=-0.8$ similar results are obtained. By the time β reaches a value of -0.5 it is difficult to distinguish between the two exponents reliably. For $\beta<$

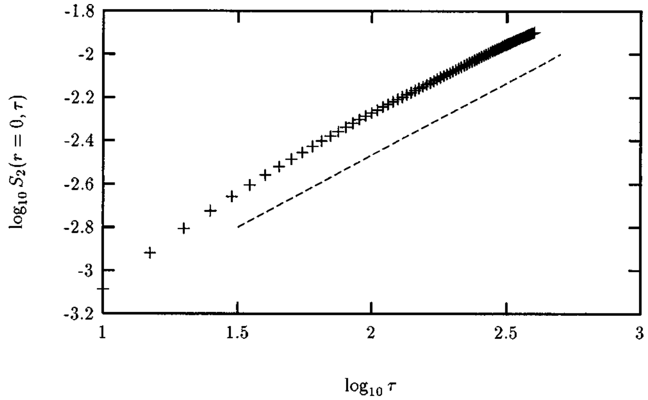

FIG. 1. (a). Log-log plot of the structure function $S_2(r=0,\tau)|$ vs τ for $\beta=-1$. A dashed line with the expected slope of$\frac{2}{3}$ is drawn for comparison. The numerical simulations were performed on a system of size $L=1024$ with 4096 k modes with $\nu=0.01$ and $D=5.0\times 10^{-9}$.

$-\frac{3}{2}$, where we expect the system to behave as if the noise were cutoff [8,14], we find that $z=1$, consistent with theoretical expectations based on $h_2=0$.

We now focus on the behavior of $\partial T_3/\partial r(r,\tau)$. We show that in the $r=0$ long-time (i.e., in the dynamic inertial range) regime, the contribution from $\partial T_3/\partial r(r=0,\tau)$ yields the leading singular behavior of $\partial S_2(r=0,\tau)/\partial\tau$. Since $z=1$ we expect $\partial T_3/\partial r(r=0,\tau)\approx 2[\partial S_2(r=0,\tau)/\partial\tau]\propto|\tau|^{\zeta_2-1}$. The data for $\partial T_3/\partial r(r=0,\tau)$ are displayed in Fig. 2 for $\beta=-0.5$. The dashed line has a slope of $-\frac{2}{3}$, the predicted value of (ζ_2-1), and the agreement is reasonable. We have made a detailed comparison of the coefficient of this term with that for $\partial S_2/\partial\tau$ and find that there is agreement to within 20%, thus showing that the terms we have retained in our analysis yield the dominant (leading) behavior.

We next consider the behavior of $S_4(r,\tau)$. An analysis similar in spirit to the one outlined for S_2 can be carried out by writing down an exact equation for S_4 and leads to the conclusion that the corresponding dynamical exponent $z_4=1$; the details will be provided elsewhere [15]. Based on this we expect $\widetilde{\zeta}_4$, the exponent that determines the behavior of S_4, $S_4(r=0,\tau)\propto|\tau|^{\widetilde{\zeta}_4}$, to be equal to ζ_4. The data for

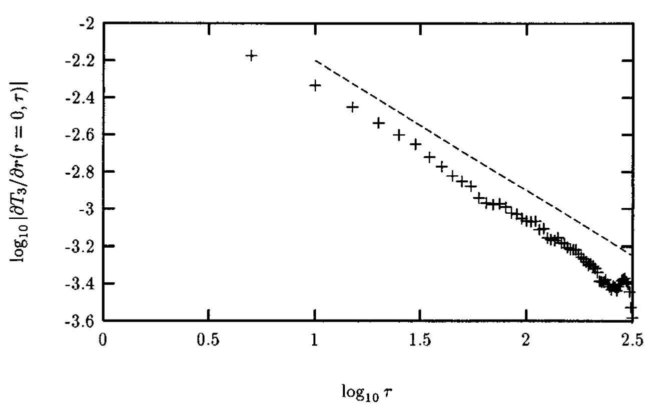

FIG. 2. $\text{Log}_{10}\,|\partial T_3(r=0,\tau)/\partial r|$ [see Eq. (4) for the definition of T_3] vs $\log_{10}\tau$. The dashed line has the theoretically expected slope of $\zeta_2-1=-\frac{2}{3}$. In the simulations we used $\beta=-0.5$, $\nu=0.06$, and $D=10^{-6}$ on a system of size 1024.

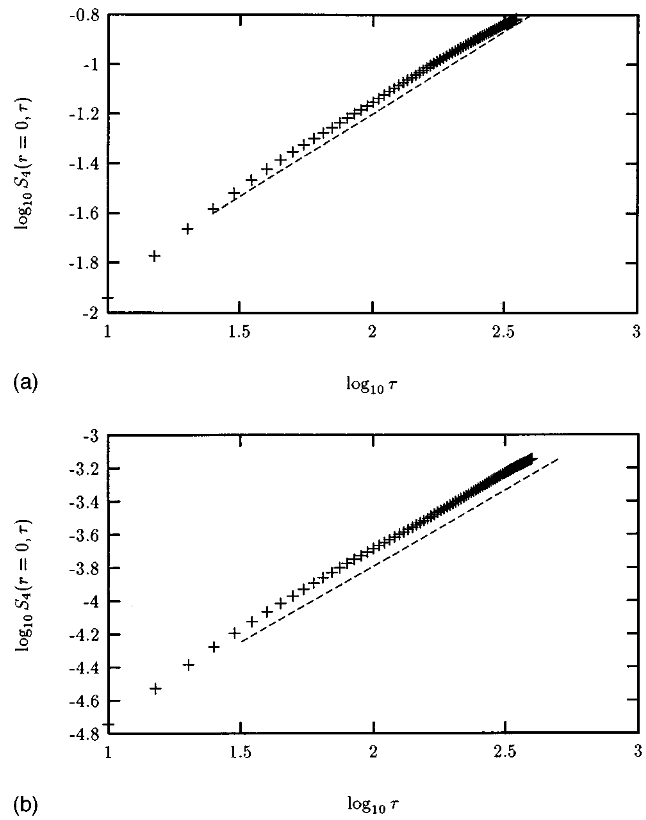

FIG. 3. Logarithm of the fourth-order structure function $\log_{10} S_4(r=0,\tau)$ as a function of $\log_{10}\tau$ for (a) $\beta=-0.5$ with a dashed line with a slope of $\zeta_4/z=\frac{2}{3}$ with $z=1$, and (b) for $\beta=-1$ where the dashed line has a slope of 0.92 close to the numerically observed value of ζ_4. The other parameters are as in Fig. 2. and Fig. 1, respectively.

$\beta=-0.5$ are displayed in Fig. 3(a) along with a dashed line with a slope of $\frac{2}{3}$, which is ζ_4/z for $\zeta_4=-4\beta/3$, the scaling value, and $z=1$. The scaling value for ζ_4 is used since as shown in Ref. [8] multifractality sets in for higher-order structure functions at this value of β. The data for $\beta=-1.0$ are displayed in Fig. 3(b) and yield a slope of 0.92 lower than the theoretically expected value of $\zeta_4=1.0$ for $\beta=-1$ [8]; however, the value of ζ_4 is numerically approximately 0.92. For $\beta=-1$ the values $z=1$ and $z=\frac{2}{3}$ are clearly distinguished since for the latter case $\widetilde{\zeta}_4$ would exceed 1.0, and be close to 1.35.

We conclude by outlining a similar analysis in the case of the Navier-Stokes equation. It is convenient to define the correlation function $b_{i,j}\equiv\langle u_i(\vec{x},t)u_j(\vec{x}\,',t')\rangle$, which is related to the longitudinal structure function S_2. It is easy to deduce the exact equation satisfied by $b_{i,j}$ [16]:

$$\frac{\partial}{\partial\tau}b_{i,j}=-\frac{1}{2}\frac{\partial}{\partial r_l}[b_{il,j}+b_{i,lj}]+\langle f_i u_j'-f_j' u_i\rangle, \qquad (10)$$

where f_i is the forcing term and $b_{il,j}\equiv\langle u_i u_l u_j'\rangle$; the fields that are unprimed and primed are evaluated at $(\vec{x},t)$ and $(\vec{x}\,',t')$, respectively. We will argue for the occurrence of an effective dynamical exponent $z=1$ over an appropriate range of time scales in analogy with our arguments for the Burgers equation. To do so we obtain an equation for the three-point functions. This is somewhat tedious and leads to

$$\frac{\partial}{\partial\tau}[b_{il,j}+b_{i,lj}]=-\frac{1}{2}\frac{\partial}{\partial r_k}[b_{ikl,j}+b_{j,ikl}]-2\nu\langle u_j'\partial_m u_i\partial_m u_l-u_j\partial_m' u_i'\partial_m' u_l'\rangle$$
$$+\left\langle u_j'u_l\frac{1}{\rho}\partial_i p-u_ju_l'\frac{1}{\rho}\partial_i' p'+u_j'u_i\frac{1}{\rho}\partial_l p-u_ju_i'\frac{1}{\rho'}\partial_l' p'\right\rangle+\langle u_j'u_lf_i-u_ju_lf_i'+u_j'u_if_l-u_ju_i'f_l'\rangle. \tag{11}$$

Now by differentiating Eq. (10) with respect to τ and substituting Eq. (11), we obtain a term $\partial^2 b_{ikl,j}/\partial r_k\partial r_l$ on the right-hand side. Plausible arguments based on the equations of motion can again be made to show that this term yields the dominant singularity. One of the singular contributions to $b_{ikl,j}$ comes from $\langle u_k u_l\rangle\langle u_i u_j'\rangle$, and projecting out the longitudinal components yields $\langle u^2\rangle\partial^2 S_2/\partial r^2$, thus obtaining an equation similar to Eq. (9). This provides a mechanism for how $z=1$ can arise in homogeneous turbulence again on time scales over which the large eddies provide a background flow.

In summary, we have presented and analyzed exact equations for time-dependent structure functions in the one-dimensional stochastic Burgers equation showing how $z=1$ arises and provided numerical support. We have outlined how the analysis can be extended to the Navier-Stokes equation, and leads in a way similar to the Burgers case, to a dynamical scaling exponent $z=1$, and thus to a justification of Taylor's hypothesis.

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