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Helicity transfer in turbulent models

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Helicity transfer in a shell model of turbulence is investigated. We show that a Reynolds-independent helicity flux is present in the model when the large scale forcing breaks inversion symmetry. The equivalent in shell models of the "2/15 law," obtained from helicity conservation in Navier-Stokes equations, is derived and tested. The odd part of the helicity flux statistics is found to be dominated by a few very intense events. In a particular model, we calculate analytically leading and subleading contributions to the scaling of triple velocity correlation. [S1063-651X(98)50603-4]

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One of the most intriguing problems in three-dimensional (3D) fully developed turbulence (FDT) is related to the appearance of anomalous scaling laws at high Reynolds numbers, i.e., in the limit when Navier-Stokes dynamics is dominated by the nonlinear interactions.

The celebrated 1941 Kolmogorov theory (K41) was able to capture the main phenomenological ideas by performing dimensional analysis based on the energy transfer mechanism. Kolmogorov postulated that the energy cascade should follow a self-similar and homogeneous process entirely dependent on the energy transfer rate, ϵ. This idea, plus the assumption of local isotropy and universality at small scales, led to a precise prediction of the statistical properties of the increments of turbulent velocity fields: $\delta v(l)\sim|v(x+l)-v(x)|\sim[l\epsilon(l)]^{1/3}$. The scaling of moments of $\delta v(l)$, the structure functions, can be determined in terms of the statistics of $\epsilon(l)$, i.e.,

$$S_p(l)\equiv\langle[\delta v(l)]^p\rangle=C_p\langle[\epsilon(l)]^{p/3}\rangle l^{p/3}, \tag{1}$$

where C_p are constants and the scale l is supposed to be in the inertial range, i.e., much smaller than the integral scale and much larger than the viscous dissipation cutoff. If $S_p(l)\sim l^{\zeta(p)}$ and $\langle\epsilon^p(l)\rangle\sim l^{\tau(p)}$, then

$$\zeta(p)=p/3+\tau(p/3). \tag{2}$$

In the K41 the $\epsilon(l)$ statistic is assumed to be l independent, or $\tau(p)=0$, implying $\zeta(p)=p/3,\forall p$. On the other hand, there are many experimental and numerical results [1,2] telling us that the K41 scenario for homogeneous and isotropic turbulence is quantitatively wrong. Strong intermittent bursts in the energy transfer have been observed and nontrivial $\tau(p)$ set of exponents are measured. Moreover, the problem of investigating scaling properties of observables with the same physical dimensions but with different tensorial structures has only recently been addressed [3,4].

Many different authors have focused their attention on the possible role played by helicity, the second global invariant of 3D Navier-Stokes equations [5–8], to determine leading or subleading scaling properties of correlation functions in the inertial range. Recently [4,9], an exact scaling equation for the third-order velocity correlations entering into the helicity flux definition has been derived using two hypotheses: (i) there exists a nonvanishing helicity flux, and (ii) the flux becomes Reynolds independent in the limit of FDT. This relation predicts an r^2 scaling for a particular third-order velocity correlation. The new relation has been called the "2/15 law" because of the coefficient appearing in front of the r^2 in analogy with the "4/5 law" derived by Kolmogorov for the third-order structure functions entering into the expression of energy flux. In the 4/5 law, the scaling of a different third-order velocity correlation is found to be linear in r.

This simple fact tells us that a different velocity correlation with the same physical dimension but with different tensorial structure may have very different scaling properties. Moreover, even if overwhelming evidence indicates that the main physics is driven by the energy transfer, there can be some subleading new intermittent statistics hidden in the helicity flux properties.

Homogeneous and isotropic turbulence always has, by definition, a vanishing mean helical flux. Nevertheless, both fluctuations about the zero mean in isotropic cases and/or net nonzero fluxes in cases where inversion symmetry is explicitly broken can be of some interest in the understanding of fully developed turbulence. In this paper, we analyze the helical transfer mechanism in dynamical models of turbulence [10,8,11], built so as to explicitly consider helicity conservation in the inviscid limit. We give strong numerical evidence that a Reynolds-independent helicity flux is present in cases where the forcing mechanism explicitly breaks inversion symmetry. We confirm that in all cases where two fluxes can coexist in the inertial range, velocity correlations with the same physical dimension but with different transformation properties under inversion symmetry can show strongly different scaling behavior.

In the following, we briefly summarize the main motivation behind the introduction of shell models for turbulence. We present the equivalent derivation of the 2/15 law for the helicity flux in our shell model language and we conclude with our numerical results about the Reynolds independence of helicity flux and about its statistical intermittent properties.

Shell models have been shown to be very useful for the understanding of many properties connected to turbulent flows [12–18]. The most popular shell model, the Gledzer-Ohkitani-Yamada (GOY) model ([12–18]), has been shown to predict scaling properties for $\zeta(p)$ (for a suitable choice of the free parameters) similar to that which is found experimentally. The GOY model can be seen as a severe truncation of the Navier-Stokes equations: it retains only one complex mode u_n as a representative of all Fourier modes in the shell of wave numbers k between $k_n = k_0 2^n$ and k_{n+1}.

It has been pointed out that GOY model conserves in the inviscid, unforced limit two quadratic quantities. The first quantity is the *energy*, while the second is the equivalent of *helicity* in 3D turbulence [17]. In two recent works [8,11] the GOY model has been generalized in terms of shell variable, u_n^+, u_n^-, transporting positive and negative helicity, respectively. It is easy to understand that only four independent classes of models can be derived that preserve the same helical structure of Navier-Stokes equations [7]. All these models have at least one inviscid invariant nonpositively defined that is similar to the 3D Navier-Stokes helicity. In the following, we will focus on the intermittent properties of a mixture of two such models. The mixture is a linear combination of the old GOY model (extended to include u^+, u^-) plus another model that has a different helical interaction and has already been extensively investigated [11,19]. We focus only on two of the four possible models because they are the only two classes of models which show a clear forward energy cascade (see [11] for more details). The time evolution for positive-helicity shells reads [11]

$$\dot{u}_n^+ = ik_n(A_n[u,u] + xB_n[u,u])^* - \nu k_n^2 u_n^+ + \delta_{n,n_0} f^+, \quad (3)$$

with the equivalent equations, but with all helicity signs reversed, for $\dot{u}^-$. In Eq. (3), x defines the relative weights of the two models in the mixture, ν is the molecular viscosity, f^+, f^- are the large scale forcing, and $A[u,u]$ and $B[u,u]$ refer to the nonlinear terms of the two models. Namely,

$$A_n[u,u] \equiv u_{n+2}^- u_{n+1}^+ + b_3 u_{n+1}^- u_{n-1}^+ + c_3 u_{n-1}^- u_{n-2}^-, \quad (4)$$

$$B_n[u,u] \equiv u_{n+2}^+ u_{n+1}^- + b_1 u_{n+1}^- u_{n-1}^- + c_1 u_{n-1}^- u_{n-2}^+. \quad (5)$$

It is easy to verify that for the choice $b_3 = -5/12$, $c_3 = -1/24$, $b_1 = -1/4$, $c_1 = -1/8$ there are only two global inviscid invariants [11]: the energy, $E = \sum_{i=1}^N (|u_i^+|^2 + |u_i^-|^2)$, and helicity, $H = \sum_{i=1}^N k_i (|u_i^+|^2 - |u_i^-|^2)$.

The equations for the fluxes throughout shell number n are

$$\frac{d}{dt} \sum_{i=1,n} E_i = k_n \langle (uuu)_n^E \rangle - \nu k_n^2 \sum_{i=1,n} E_i + E_{in}, \quad (6)$$

$$\frac{d}{dt} \sum_{i=1,n} H_i = k_n^2 \langle (uuu)_n^H \rangle - \nu k_n^2 \sum_{i=1,n} H_i + H_{in}, \quad (7)$$

where E_i and H_i are the energy and helicity of the nth shell, respectively: $E_i = \langle |u_i^+|^2 + |u_i^-|^2 \rangle$, $H_i = k_i \langle |u_i^+|^2 - |u_i^-|^2 \rangle$. E_{in} and H_{in} are the energy and helicity input due to forcing effects, $E_{in} = \mathrm{Re}(\langle f^+ (u_1^+)^* + f^- (u_1^-)^* \rangle)$, $H_{in} = \mathrm{Re}(k_1 \langle f^+ (u_1^+)^* - f^- (u_1^-)^* \rangle)$. In Eqs. (6) and (7) we have introduced the triple correlation

$$\langle (uuu)_n^E \rangle = (\Delta_{n+1}^+ + \Delta_{n+1}^-) + (b_3 + 1/2)(\Delta_n^+ + \Delta_n^-) + x[(\Gamma_{n+1}^+ + \Gamma_{n+1}^-) + (b_1 + 1/2)(\Gamma_n^+ + \Gamma_n^-)], \quad (8)$$

$$\langle (uuu)_n^H \rangle = (\Delta_{n+1}^+ - \Delta_{n+1}^-) + (b_3 + 1/4)(\Delta_n^+ - \Delta_n^-) + x[(\Gamma_{n+1}^+ - \Gamma_{n+1}^-) + (b_1 + 1/4)(\Gamma_n^+ - \Gamma_n^-)], \quad (9)$$

and

$$\Delta_n^+ = \langle \mathrm{Im}(u_{n+1}^+ u_n^- u_{n-1}^+) \rangle, \quad \Gamma_n^+ = \langle \mathrm{Im}(u_{n+2}^- u_{n+1}^+ u_n^+) \rangle. \quad (10)$$

Assuming that there exists a stationary state, we have $(d/dt)\Pi_n^E = (d/dt)\Pi_n^H = 0$, where $\Pi_n^E = k_n \langle (uuu)_n^E \rangle$ and $\Pi_n^H = k_n^2 \langle (uuu)_n^H \rangle$. Moreover, in the inertial range we can neglect the viscous contribution in Eqs. (6) and (7), obtaining

$$\langle (uuu)_n^E \rangle = k_n^{-1} E_{in}, \quad \langle (uuu)_n^H \rangle = k_n^{-2} H_{in}. \quad (11)$$

Supposing that there exist the energy and helicity fluxes $E_{in} = \epsilon$, $H_{in} = h$ (the latter different from zero only if $f^+ \neq f^-$) and supposing that both are Reynolds independent, we have in the inertial range

$$\langle (uuu)_n^E \rangle \sim k_n^{-1}, \quad \langle (uuu)_n^H \rangle \sim k_n^{-2}. \quad (12)$$

Relation (11) is the equivalent of that found for helical Navier-Stokes turbulence in [4,9]).

Figure 1 reports results for the helicity flux in numerical simulations done with two different Reynolds numbers, Re $\sim 10^5$, Re$\sim 10^9$ for a choice of mixture parameter $x = 0.1$ and additional numerical input as follows: $N = 16$ and 26, $\nu = 10^{-5}$ and 2×10^{-9}. A nonzero helical flux was obtained using a forcing-term-breaking inversion symmetry: $f^+ = 5 \times 10^{-3}(1+i)$, $f^- = f^+/10$. Time marching was obtained by using a slaved Adams-Bashforth algorithm for a number of iterations equal to several thousands of the typical eddy turnover time. A clear inertial range with a nonvanishing helicity flux is detected. The extension of the range where helicity flux is roughly constant scales with the Reynolds number. Moreover, the flux intensity is roughly constant at changing Reynolds number, giving the first evidence that the model can simultaneously support both energy and helicity transfers and that both of them are Reynolds independent.

Let us remark that this is only possible due to the nonpositiveness of helicity; in 2D turbulence, for example, similar results, concerning enstrophy and energy cascades, is clearly *a priori* forbidden.

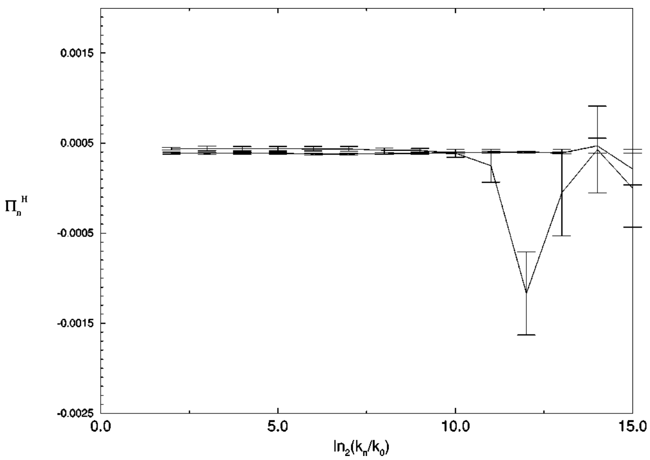

FIG. 1. Helicity-flux $[k_n^2\langle(uuu)_n^H\rangle]$ vs $lg(k_n/k_0)$ for $N=16$, $\nu=10^{-5}$ (dashed line) and $N=26$, $\nu=2\times10^{-9}$ (continuous line). Inertial ranges coincide with the regions where the fluxes are constants. Helicity is expressed in dimensionless units.

As for the statistics of helicity transfer, we measured the scaling exponents of the moments of energy and helicity fluxes:

$$\Sigma_E^{(p)}\equiv\langle[(uuu)_n^E]^{(p/3)}\rangle\sim k_n^{-\zeta(p)}, \tag{13}$$

$$\Sigma_H^{(p)}\equiv\langle[(uuu)_n^H]^{(p/3)}\rangle\sim k_n^{-\psi(p)}. \tag{14}$$

As one can see in Fig. 2, we have that the even parts of the two statistics coincide, i.e., $\zeta(2p)=\psi(2p)$. On the other hand, the scaling exponents of odd moments are different.

The difference in odd moments is the signature of strong cancellation effects in the statistics connecting fluctuations at different scales. The picture we have in mind is that the main effect driving turbulent fluctuations is due to the energy cascade process, with its intermittent fluctuations measured by the $\zeta(p)$ exponents. Superimposed on the energy transfer, there are ''topological'' fluctuations introduced by the asymmetric forcing and measured by the odd part of the helicity turbulent transfer.

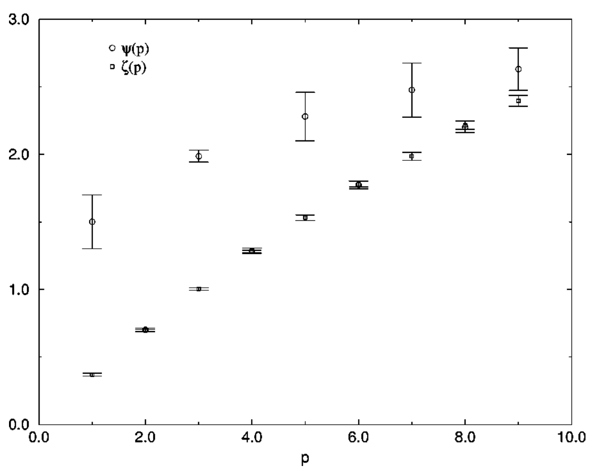

FIG. 2. Anomalous exponents for the helicity flux ψ_p (circles) and for the energy flux ζ_p (squares), obtained for $N=26$ and $\nu=2\times10^{-9}$.

Note that helicity flux fluctuations are much larger than the average helicity flux. This clearly distinguishes the helicity transfer mechanism from the energy transfer mechanism. In the former, the strong intermittent behavior shown by odd moments tells us that the odd part of the statistics is dominated by a few very singular structures. In the case of no mixture $(x=0)$, one can also exactly calculate subleading scaling for the triple correlation $\delta_n^+\equiv k_n(\Delta_n^+ + \Delta_n^-)$, $\delta_n^-\equiv k_n^2(\Delta_n^+ - \Delta_n^-)$. Indeed, from expression (8) one obtains, after some simple algebra,

$$\delta_n^+ = 2\epsilon\frac{1-y^{n+1}}{1-y}, \quad \delta_n^- = 4h\frac{1-z^{n+1}}{1-z}, \tag{15}$$

where $y=-2(b_3+1/2)$ and $z=-4(b_3+1/4)$. Since both y and z have a modulus of less then one, we recover the asymptotic scaling (12) and one can also control subleading correction to it:

$$\Delta_n^+ = k_n^{-1}\left(\epsilon\frac{1-y^{n+1}}{1-y}\right)+k_n^{-2}\left(2h\frac{1-z^{n+1}}{1-z}\right)$$

and

$$\Delta_n^- = k_n^{-1}\left(\epsilon\frac{1-y^{n+1}}{1-y}\right)-k_n^{-2}\left(2h\frac{1-z^{n+1}}{1-z}\right).$$

In conclusion, we have studied a helical shell model for turbulence with a forcing that explicitly breaks inversion symmetry at large scales. For a symmetric forcing the helicity flux is zero, while with the choice of forcing adopted in our numerical simulation the value of the helicity flux is an order of magnitude less than the value of the energy flux. We have verified that a Reynolds-independent helicity flux is established in the system, giving evidence of very different scaling for triple correlations entering into the energy flux and helicity flux definitions. Contrary to other shell models, such as the GOY model [17], helicity flux and energy flux in our model are not correlated; therefore, one can have cases where the importance of the two fluxes may be very different. The odd part of the helicity flux statistics is found to be strongly intermittent.

For a particular class of models we can also calculate explicitly subleading corrections to pure scaling behavior of typical triple correlation functions. The existence of subleading terms explicitly tell us that scaling laws in turbulent flows must be studied in correlation functions that have a pure projection, i.e., which fell energy or helicity flux only, on the relevant physical quantities. There is no reason why similar effects should not be present also in true Navier-Stokes equations. For example, spurious intermittent corrections can be detected in cases where isotropy is globally or locally violated (as in boundary layers).

We may summarize our findings as follows. General velocity correlation functions can serve to probe both the energy and helicity flux mechanisms, leading to the prediction that the typical behavior will be given by a superposition of power laws, the leading one connected to the energy flux, the subleading one connected to the helicity flux. Depending on the relative weights of the two fluxes, the subleading power laws may or may not have a detectable effect. For a subset of all possible correlation functions, the class of functions that depends only on the helicity flux, the leading term connected to the energy transfer is absent and therefore one may detect a new scaling regime connected only to the physics of the helicity transfer.

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