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Two complementary descriptions of intermittency

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We describe two complementary formalisms designed for the description of the probability density function (PDF) of the gradients of turbulent fields. The first approach, we call it adiabatic, describes the PDF at the values much less than dispersion. The second, instanton, approach gives the tails of the PDF at the values of the gradient much larger than dispersion. Together, both approaches give a satisfactory description of gradient PDFs, as illustrated here by an example of a passive scalar advected by a one-dimensional compressible random flow. [S1063-651X(98)50602-2]

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Probably the most striking feature of developed turbulence is its intermittent spatial and temporal behavior. The structures that arise in a random flow manifest themselves as high peaks at random places and at random times. The intervals between them are characterized by a low intensity and a large size. Rare high peaks are responsible for probability density function (PDF) tails, while the regions of low intensity contribute PDF near zero. That physical picture prompts an attempt to describe intermittency at the level of a single-point PDF by two complementary approaches. The first approach was recently introduced to describe rare strong fluctuations as optimal fluctuations realizing probability extrema [1–5]. Called an instanton approach, this formalism is based upon a path-integral representation of conditional probability, with optimal fluctuations being saddle points in the integral. A counterpart to the instanton approach is suggested here for the description of the gradient PDF at small values; the approach is just an adiabatic one when high-order spatial derivatives are consistently neglected.

The center anomaly and tails of the gradient PDF are two sides of the same coin called intermittency, which is the main target in modern turbulence studies. In this Rapid Communication, we demonstrate how both methods applied together give a consistent description of the gradient PDF. Note that the intermediate part of the PDF (which is beyond our approaches) where the matching of the asymptotics occurs is not that interesting because it is nonuniversal, i.e., depends on the particular form of the pumping correlation function. The central peak and the tail are robust; their form provides the main information on the probability of both the main body of the events and strong fluctuations.

Let us show how such a description can be developed by using, probably, the simplest (yet nontrivial) turbulent problem of a passive scalar $\theta(x,t)$ advected by one-dimensional random flow $v(x,t)$ that is smooth in space and white in time (this is a compressible version [6] of a well-known Kraichnan model [7]):

$$\partial_t\theta+v\nabla\theta=\kappa\Delta\theta+\phi. \tag{1}$$

Both the velocity v and the source function ϕ are supposed to be homogeneous, Gaussian, and δ correlated in time: $\langle\phi(x,t)\phi(x',t')\rangle=\chi(|x-x'|)\delta(t-t')$, where χ is some function that decays on a scale L, the value $\chi(0)=P$ is the flux of θ^2. The correlation function of the velocity may be defined by two parameters, typical velocity V and correlation length L_v:

$$\langle v(x,t)v(x',t')\rangle=[VL_v-VL_v^{-1}(x-x')^2]\delta(t-t'). \tag{2}$$

When studying a simultaneous statistic, the coordinate-independent part drops out. We assume $L_v\gg L$.

Let us first implement a simple adiabatic approach neglecting the diffusion term. Then, for the single-point PDF $\mathcal{P}(\omega,t)=\langle\delta[\theta_x(x,t)-\omega]\rangle$, one obtains a closed Fokker-Planck equation

$$2\frac{\partial\mathcal{P}}{\partial t}=(D\omega^2+T)\frac{\partial^2\mathcal{P}}{\partial\omega^2}+4D\omega\frac{\partial\mathcal{P}}{\partial\omega}+2D\mathcal{P}, \tag{3}$$

where we denote $T=\chi''(0)$ and $D=VL_v^{-1}$ the variances of ϕ_x and v_x, respectively. That equation has an equilibrium steady solution

$$\mathcal{P}(\omega)\propto(T+D\omega^2)^{-1} \tag{4}$$

that is expected to be applicable for $\omega^2\ll P/\kappa$. Since $T\simeq P/L^2$ and the Péclet number $\mathrm{Pe}^2=DL^2/4\kappa$ is assumed to be large, then Eq. (4) has a wide interval of validity. Note that T/D is a square gradient produced by the pumping during the typical stretching time D^{-1}. For $\mathrm{Pe}\gg1$, $T/D\ll P/\kappa$. Limiting solutions obtained at $\omega^2\gg T/D$ and at $\omega^2\ll T/D$ by a time-separation procedure [8] coincide with Eq. (4).

Let us now describe the tail of the probability density function $\mathcal{P}(\omega)$ at $\omega^2\gg P/\kappa$. It is clear from Eq. (2) that the correlation functions of the strain field $\sigma=v_x$ are x independent, that is, σ can be treated as a random function of time t only. To exploit that, it is convenient to pass into the comoving reference frame, that is, to the frame moving with the velocity of a Lagrangian particle of the fluid [1,3]. The Martin-Siggia-Rose action I for the nth order moment of the gradient θ_x is [8]

$$\mathcal{I}=\int dt\, dx\, p\partial_t\theta-\int Edt-\frac{in}{2}\ln(\theta_x)^2 \tag{5}$$

$$E=\int dxp(-\sigma x\partial_x\theta+\kappa\partial_x^2\theta)-\frac{i}{2}\int dx_1dx_2p_1\chi(x_{12})p_2-\frac{i}{4D}(\sigma-D)^2. \quad (6)$$

Assuming $n\gg1$, we shall calculate the moment in the saddle-point approximation $\langle\theta_x^n\rangle=\exp(\mathcal{I}_{extr})$; see [1–5] for the details. Here, $\mathcal{I}_{extr}$ is to be calculated on the flow configuration (optimal fluctuation or instanton) that has to satisfy extremum equations for the action:

$$\partial_t\theta+\sigma x\partial_x\theta-\kappa\partial_x^2\theta=-i\int dx'p(t,x')\chi(x-x'), \quad (7)$$

$$\partial_tp+\sigma\partial_x(xp)+\kappa\partial_x^2p=-y\,\delta(t)\,\delta'(x), \quad (8)$$

$$\sigma-D=2iD\int dxpx\partial_x\theta. \quad (9)$$

Here $y=-in/\theta_x(0,0)$. For calculations, it is more convenient to use this auxiliary parameter instead of $\theta_x(0,0)$. The boundary conditions are $\theta(x,-\infty)=0$ and $p(x,+0)=0$ [1]. The solution of Eqs. (7) and (8) can be sought in the following form:

$$\theta=f(\tau(t),x\sqrt{w(t)}),\quad p=\sqrt{w(t)}g(\tau(t),x\sqrt{w(t)}), \quad (10)$$

$$\partial_tw=-2\sigma w,\quad \partial_t\tau=-w,\quad w(0)=1,\quad \tau(0)=0. \quad (11)$$

The functions f and g satisfy the following equations:

$$\partial_\tau g-\kappa\partial_\xi^2g=y\,\delta(\tau)\,\delta'(\xi), \quad (12)$$

$$\partial_\tau f+\kappa\partial_\xi^2f=\frac{i}{w(\tau)}\int_{-\infty}^{\infty}d\xi'g(\tau,\xi')\chi\left(\frac{(\xi-\xi')}{\sqrt{w(\tau)}}\right). \quad (13)$$

The solution can be found in the Fourier representation

$$g(\tau,k)=ikye^{-\kappa k^2\tau}, \quad (14)$$

$$f(\tau,k)=-ky\int_{\tau_0}^{\tau}\frac{d\tau'}{\sqrt{w(\tau')}}\chi(k\sqrt{w(\tau')})e^{\kappa k^2(\tau-2\tau')}, \quad (15)$$

$$\sigma-D=-2Dy^2\int_{\tau_0}^{\tau}\frac{d\tau'}{\sqrt{w(\tau')}}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\times k^2(1-2\kappa k^2\tau)\chi(k\sqrt{w(\tau')})e^{-2\kappa k^2\tau'}. \quad (16)$$

Here τ_0 is the maximal value for τ, which is determined by the moment when the following integral diverges:

$$t(\tau)=-\int_0^{\tau}\frac{d\tau}{w(\tau)}. \quad (17)$$

In the following we will work in the dimensionless units. We put $D=1$, $P\equiv\chi(0)=1$, and $L=1$. Then $\kappa=1/(4\,\mathrm{Pe}^2)$, where Pe is the Péclet number. We believe $\mathrm{Pe}\gg1$. Calculating $\partial_x\theta(0,0)$ from Eq. (15) we obtain the following self-consistency condition for y:

$$\frac{n}{|y|^2}=\int_0^{\tau_0}\frac{d\tau}{w^2(\tau)}\phi\left(\frac{\tau}{\mathrm{Pe}^2w(\tau)}\right), \quad (18)$$

$$\phi(x)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}k^2\chi(k)\exp\left(-\frac{k^2x}{2}\right). \quad (19)$$

The function $\phi(x)$ has the following asymptotes: $\phi(x)\to1$ as $x\to0$ and $\phi(x)\sim x^{-3/2}$ as $x\to\infty$ if $\chi(k=0)\neq0$. Note that $\chi(k)\geq0$; hence $\phi(x)$ is a monotonic decreasing function. One may keep in mind some particular form of $\chi(x)$, say, $\exp(-x^2L^2/2)$. Then, $\phi(x)=(1+x)^{-3/2}$.

Now we derive a closed equation that describes the evolution of w, which is the square root of the solution inverse width. One can do that directly from Eq. (16), substituting there $\sigma=w'(\tau)/2$, which is the consequence of Eq. (11). One obtains an integral equation that is equivalent to some third-order ordinary differential equations. The order can then be reduced by 1 due to the conservation law (6). However, it is more instructive to derive the same equation on w in a different manner: since we are looking for the extremum of the action (5), we can substitute there all the fields as the functionals of w and then make a variation with respect to w. We have

$$i\mathcal{I}=\frac{n}{2}\ln\left[\frac{n}{e}\int_0^{\tau_0}\frac{d\tau}{w^2(\tau)}\phi\left(\frac{\tau}{\mathrm{Pe}^2w(\tau)}\right)\right]-\frac{1}{4}\int_0^{\tau_0}\frac{d\tau}{w(\tau)}\left(\frac{1}{2}w'-1\right)^2. \quad (20)$$

Varying with respect to w, we obtain

$$w''-\frac{w'^2}{2w}+\frac{2}{w}-\frac{8|y|^2}{w^2}\phi\left(\frac{\tau}{\mathrm{Pe}^2w}\right)-\frac{4|y|^2\tau}{\mathrm{Pe}^2w^3}\phi'\left(\frac{\tau}{\mathrm{Pe}^2w}\right)=0. \quad (21)$$

Equation (21) can be rewritten as a Hamiltonian system with the momentum $\zeta=w'/w$ and the Hamiltonian

$$H=\frac{\zeta^2w}{2}+\frac{4|y|^2}{w^2}\phi\left(\frac{\tau}{\mathrm{Pe}^2w}\right)-\frac{2}{w}. \quad (22)$$

Initial conditions for Eq. (21) are $w(0)=1$ and $w'(0)=-2(2n-1)$. The latter is readily derived from Eqs. (16) and (18). We should also satisfy a final condition. Indeed, Eqs. (17) and (20) require $w'\to2$ as $\tau\to\tau_0$; otherwise, the integral contribution to the action is infinite. Our aim is to find such a value of y that the solution of Eq. (21) satisfies the relation (18). In other words, we can divide the task into two parts: first, to find the solution of Eq. (21) with arbitrary y and the given boundary conditions; second, given w, to solve the algebraic equation (18), which determines y. Note that finite positive value of w' at $\tau\to\tau_0$ implies $\tau_0=\infty$.

Let us start the first part of our program, which is solving Eq. (21). Since Eq. (22) explicitly depends on τ, then H is not conserved and Eq. (21) cannot be solved explicitly for an arbitrary ϕ. In the limit $n\ll\mathrm{Pe}^2$, it is possible, nevertheless, to describe the solution with enough detail to recover its dependence on the parameters n and Pe.

The evolution of w can be divided into three parts. During the initial stage, when τ is close to zero, w is of order unity. Therefore, $\mathrm{Pe}^2 w \gg \tau$ and we can substitute ϕ in Eq. (22) by its asymptotic value 1. Thus, during that stage H is a constant, which we denote as H_0. One finds $H_0 = H(0) = 2(2n-1)^2 - 2 + 4|y|^2$. Since $n \gg 1$ and, as we shall see below, $y \ll 1$, we have $H_0 \approx 8n^2$. The equation for w can then be readily derived from Eq. (22),

$$w' = -\sqrt{4 + 2H_0 w - 8|y|^2/w}. \tag{23}$$

Now let us consider the final stage. Since $w' \to 2$ as $\tau \to \infty$, we can write $w \approx 2\tau$. It gives $\tau/(\mathrm{Pe}^2 w) \approx 1/(2\,\mathrm{Pe}^2) \ll 1$. Thus, as for the first stage, we can replace ϕ by 1, and therefore the energy is a constant. It follows from Eq. (22) that during that stage $Hw \ll 1$. Hence, w satisfies the following equation:

$$w' = \sqrt{4 - 8|y|^2/w}. \tag{24}$$

Since w' is negative during the first stage and positive during the third one, then it has to turn into zero at some reflection time τ_*. Around that time, there should exist an intermediate part of evolution, which matches the two above asymptotes. We will see below that this stage makes the main contribution to Eq. (18). During that stage one has $\tau/(\mathrm{Pe}^2 w) \gtrsim 1$, so that H is not conserved but decreases from $8n^2$ to 0. It will be important for us that H is a decreasing monotonic function of τ, which becomes obvious after one differentiates Eq. (22).

Let us make estimations of the parameters during that intermediate stage. From Eq. (23) it is easy to find that w diminishes from 1 to a substantially smaller value during the time $\tau_* \simeq 1/n$. To have $\tau/(\mathrm{Pe}^2 w)$ of order unity by the beginning of the second stage, there should be $w \sim 1/(n\,\mathrm{Pe}^2)$. Looking at Eq. (22), we observe that, since H is a decreasing function of τ, the left-hand side of Eq. (22) is less than $H_0 \approx 8n^2$. On the other hand, the term $2/w$ on the right-hand side can be estimated as $n\,\mathrm{Pe}^2$. If $n \ll \mathrm{Pe}^2$, we can disregard the left-hand side. We assume in addition that the duration of the second stage is much less than τ_*. Then, we can substitute τ by τ_* in the argument of ϕ and obtain the equation

$$w'^2 = 4 - \frac{8|y|^2}{w}\,\phi\!\left(\frac{\tau_*}{\mathrm{Pe}^2 w}\right). \tag{25}$$

We do not take the square root, since w' changes sign during that part of the evolution, so that both branches of Eq. (25) are pertinent. Actually, Eq. (25) is valid for all $\tau > \tau_*$, since at $w \gg \tau_*/\mathrm{Pe}^2$ it turns into Eq. (24). If our assumption about the duration of the second stage is correct, the transition region is described correctly too. On the other hand, if we add to Eq. (25) the term $H_0 w$, which is small in the transition region, the equation thus obtained will correctly describe the evolution of w at all times $\tau < \tau_*$.

Therefore, we have found that under the assumption of a short transition region it is possible to reduce Eq. (21) to the equations $w' = -\sqrt{\psi_1(w)}$ at $\tau < \tau_*$ and $w' = \sqrt{\psi_2(w)}$ at $\tau > \tau_*$, where

$$\psi_1 = 4 + H_0 w - \frac{8|y|^2}{w}\,\phi\!\left(\frac{\tau_*}{\mathrm{Pe}^2 w}\right),$$

$$\psi_2 = 4 - \frac{8|y|^2}{w}\,\phi\!\left(\frac{\tau_*}{\mathrm{Pe}^2 w}\right).$$

Now we can continue with the second task, that is, solving Eq. (18). We introduce also $w_* = w(\tau_*)$ at the moment of reflection. Since at $\tau = \tau_*$ the derivative of w is zero, then we find from Eq. (25)

$$|y|^2 = \frac{w_*}{2\phi\!\left(\dfrac{\tau_*}{\mathrm{Pe}^2 w_*}\right)}. \tag{26}$$

Now Eq. (18) can be rewritten in the following form:

$$\frac{2n}{w_*}\,\phi\!\left(\frac{\tau_*}{\mathrm{Pe}^2 w_*}\right) = \int_{w_*}^{1} \frac{dw}{w^2\sqrt{\psi_1(w)}}\,\phi\!\left(\frac{\tau_*}{\mathrm{Pe}^2 w}\right) + \int_{w_*}^{\infty} \frac{dw}{w^2\sqrt{\psi_2(w)}}\,\phi\!\left(\frac{\tau_*}{\mathrm{Pe}^2 w}\right). \tag{27}$$

Estimating contributions into the integrals from the first and third time intervals, one can find that alone, they are too small to satisfy Eq. (27). On the other hand, it is easy to see that, if the derivative of $\psi_{1,2}$ at the point w_* is not small enough, the second stage also does not contribute to the integrals in Eq. (27). The only way to have a solution is to make the derivatives small. Then $\psi_{1,2} \approx \psi''_{1,2}(w_*)(w - w_*)^2/2$ in a wide interval, and both integrals in Eq. (27) logarithmically diverge, with a cutoff on the nonzero value of the first derivative. Since we can make $\psi'(w_*)$ arbitrarily small by a small change of w_*, the solution exists. Equating the derivative of ψ to zero, one finds that w_* is close to $\alpha\tau_*/\mathrm{Pe}^2$, where α is some number of the order unity, which depends on the form of ϕ that is of the pumping χ. The deviation of the first derivative from zero δ is determined by Eq. (27):

$$n w_* = \sqrt{\frac{2}{\psi''(w_*)}}\,\ln\frac{1}{\delta}. \tag{28}$$

One finds $\ln\delta^{-1} \propto n$. The smallness of δ justifies our assumption on a short intermediate stage. Thus, we have shown that solution of Eq. (18) exists, and $|y^2| \sim 1/(n\,\mathrm{Pe}^2)$, which corresponds to $\theta_x(0,0) \sim n^{3/2}\mathrm{Pe}$. This makes the main contribution to $\langle\theta_x^n\rangle \propto n^{3n/2}\mathrm{Pe}^n$, since the integral term in the action $\mathcal{I}$ is $\sim n$ on our instanton solution. Such moments can be assembled into the following tail of the PDF of $\omega = \theta_x$:

$$\ln[\mathcal{P}(\omega)] \propto -(\kappa\omega^2/P)^{1/3}. \tag{29}$$

The correspondence between the tail (29) and the $n^{3n/2}$ result can be easily established by direct calculation of the integral $\int d\omega\,\omega^n \mathcal{P}(\omega)$ in the limit $n \gg 1$. Note that the tail does not depend on the strain amplitude D.

Let us describe the above instanton solution in more physical terms. The instanton corresponds to some optimal process making the main contribution in $\langle\theta_x^n\rangle$. It produces a large gradient, which compensates for the small probability of such a process. Looking at (10,15), we can distinguish the three stages of θ evolution. During the first one, starting at $t=-\infty$, the strain $\sigma\approx D$ stretches small-scale initial perturbation up to the width of order L and the force prepares some profile of θ, which has the amplitude of order $\sqrt{P/D}$. The second stage starts when w is close to w_*. Then $\sigma\ll D$; we can disregard both the advective and diffusive terms. Therefore, the width of θ does not change during that stage, while the amplitude grows due to the force. Of all realizations of the force, the constant one is preferred, since it gives the fastest growth. Then, θ increases as ϕt. The weight of such a process is $\exp(-\phi^2 t-t/2)$. The second term in the exponent is the probability of having small σ during the time t. Then we can find that $\phi\sim 1$ and $t\propto n$ (note that the second stage is long in terms of t, yet it was short in terms of τ). By the end of this stage, $\theta_x\propto n$. And finally, during the last stage we can disregard the force. The profile having the amplitude n and width L by the beginning of the stage is compressed by the large negative σ, which can be estimated as $\sigma\sim -Dn$. The duration of that stage (and the final width) is determined by diffusion: $\sigma x\partial_x\theta\sim\kappa\partial_x^2\theta$ at the end. Then, the width of $\theta(x)$ is $\sqrt{\kappa/Dn}$, while the amplitude is $n\sqrt{P/D}$; therefore the final answer for $\langle\theta_x^n\rangle$ is $\propto n^{3n/2}(P/\kappa)^{n/2}$, which corresponds to Eq. (29). To summarize, the optimal fluctuation that makes the main contribution in $\langle\theta_x^n\rangle$ starts from an infinitesimal fluctuation that is initially stretched; then it has a long stage of suppressed advection when the amplitude of $\theta(x,t)$ grows even when its spatial scale does not decrease, and then it contracts quickly.

Let us stress that the instanton describes a very special configuration of the fields. For any other solution that does not satisfy the correct self-consistency condition, there is no such long intermediate stage of growth. If the parameters were not fitted to guarantee the existence of this long stage, either large σ would bring a singularity in the solution or θ_x would not be large enough at $t=0$.

Note that the scalar itself has an exponential PDF tail; the fact that the gradient PDF is less steep was correctly attributed in [8] to the fluctuations of the diffusion scale. One may explain the 2/3 stretched exponent in the following simple abbreviated way, clarifying the physics of the phenomena: Due to diffusion, the local gradient can be thought of as the product of a scalar fluctuation and inverse diffusion scale. The former has an exponential PDF tail [8], while the latter is proportional to the local stretching rate, which is Gaussian. Therefore, $\langle\omega^n\rangle\sim\langle\theta^2\rangle^{n/2}n^n r_d^{-n}n^{n/2}$, which corresponds to Eq. (29). Also, instanton formalism provides an instructive insight into the relation between the scalar and its gradient on the optimal fluctuation: comparing the second and third terms in Eq. (7), one gets for the current dissipative scale, $r_d\propto\sqrt{\kappa/\theta D}$; substituting that into $\theta_x\simeq\theta/r_d$, we obtain $\theta_x\propto\theta^{3/2}$, so that exponential tail for θ has to correspond to the 2/3 stretched exponential for the gradient.

To conclude, the gradient's PDF is given by Eq. (4) for $\kappa\omega^2/P\ll 1$ and by Eq. (29) for $1\ll\kappa\omega^2/P\ll \mathrm{Pe}^3$, which agrees with the results found by a time-separation formalism [8]. Speaking about generalizations, it is likely that the one-dimensional instanton described here may be relevant for a multidimensional case, both compressible and incompressible, due to the universality of a locally flat ramp-and-cliff structure discussed in [9–11]. Note that the stretched exponential tail is what one expects for the steady gradient's distribution (which is possible only when diffusion is present) [12], contrary to unsteady log-normal distribution which takes place without diffusion [7].

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