Excitation of electromagnetic wake fields in a magnetized plasma

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We consider propagation of short electromagnetic pulses in a magnetized plasma. A self-consistent system of equations describing wake-field generation in the weakly nonlinear limit is derived. Due to the external magnetic field, the generated wake field becomes partially electromagnetic. The equations are applicable for arbitrary directions of propagation as compared to the external magnetic field. The conservation laws for the system are discussed in detail. The energy decrease rate and the frequency decrease rate of the short pulse are determined. [S1063-651X(98)00406-1]

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I. INTRODUCTION

The interaction of electromagnetic pulses with plasmas can give rise to a large number of phenomena [1–11]. For weakly nonlinear one-dimensional pulses, the quadratic nonlinear terms can generate second-harmonic perturbations as well as low-frequency perturbations. Typically, the second-harmonic terms in turn affect the evolution of the pulse, leading to a cubic nonlinearity of the nonlinear Schrödinger type. During certain circumstances the low-frequency response also leads to such a nonlinearity, but it can also lead to a nonlocal response. In particular, a short laser pulse with a duration of the order of the inverse plasma frequency or shorter generates a low-frequency wake field of plasma oscillations [3,4]. Similarly, microwave sources can generate energetic electron plasma wake fields in laboratory experiments [5]. The large electric fields of the plasma oscillations lead to the concept of plasma-based accelerators [3], which have shown promising results [6]. The plasma wake field can also interact with a second weaker pulse following the first, leading to frequency up-conversion of that pulse [7,8].

In the present paper we have investigated the interaction of a short one-dimensional weakly nonlinear electromagnetic pulse with plasmas during comparatively general conditions. The pulse is assumed to be short enough such that the low-frequency response occurs at the electronic time-scale and thus ion motion has been neglected. Furthermore, the frequency of the pulse is much larger than the plasma frequency. Using a WKB ansatz but keeping correction terms that are usually neglected, we have derived a set of self-consistent equations valid for propagation at an arbitrary angle to the external magnetic field, taking second-harmonic generation, relativistic effects, and low-frequency wake field generation into account. The system of equations exhibits conservation laws for the energy, the number of high-frequency quanta, and the Hamiltonian of the system, which are discussed in some detail. During the propagation through the unperturbed plasma, the pulse energy is transferred to the wake field, which is partially electromagnetic, due to the presence of the external magnetic field. Since the number of electromagnetic quanta is conserved, the frequency of the pulse must decrease. However, our equations allow for a general interaction between the wake field and the electromagnetic pulse. This means that wake-field generation by a number of pulses [9] or frequency up-conversion of a short pulse situated somewhere in the wake field where the derivative of the wake-field potential is negative [10] can be described. The organization of our paper is as follows. In Sec. II the derivation of the equation governing the pulse propagation is made, starting from the weakly relativistic cold plasma equations. In Sec. III a variational principle and the conservation laws for our system are presented. Section IV investigates wake-field generation by a single short pulse, which after some approximations leads to concrete expressions for the frequency decrease and energy-loss rate of the pulse. In Sec. V we point out some effects that occur due to the presence of the external magnetic field. Finally, the main results are summarized and discussed in Sec. VI.

II. DERIVATION OF MAIN EQUATIONS

We consider a cold homogeneous magnetized plasma with $\mathbf{B}_0=B(\hat{\mathbf{x}}\cos\theta+\hat{\mathbf{z}}\sin\theta)$. A short, one-dimensional high-frequency pulse is propagating in the z direction. We assume the ordering $\omega\gg\omega_p,\omega_c$ where ω is the frequency of the pulse, ω_p is the plasma frequency, and $\omega_c=qB/mc$ is the electron cyclotron frequency. The ratio ω_p/ω_c is arbitrary. Note that Ref. [11] has previously investigated wake-field generation in a magnetized plasma without using our assumption $\omega\gg\omega_c$. However, that work is limited to electrostatic wake fields. Generally, the low-frequency field turns out to be electromagnetic in our case, making our basic assumption complementary to that of Ref. [11]. First we note that for $\omega\gg\omega_c$ the influence of the magnetic field will be limited to the low-frequency perturbations generated by the pulse, as can be verified by examining the cold plasma conductivity tensor [12]. In the following we will use the scalar and vector potentials according to

$$\mathbf{E}=-\nabla\Phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\quad \mathbf{B}=\mathbf{B}_0+\nabla\times\mathbf{A},$$

applying the Coulomb gauge. To lowest linear order we assume the vector potential of the high-frequency pulse to be given by $\mathbf{A}=\mathbf{A}_H(z,t)\exp[i(kz-\omega t)]+\text{c.c.}$, where c.c. stands for complex conjugate. The polarization is assumed to be arbitrary, the amplitude is slowly varying in time and space as compared to the $\exp[i(kz-\omega t)]$ factor, and the dispersion relation $\omega^2=\omega_p^2+k^2c^2$ is fulfilled exactly. As can be seen from the calculations below, the scalar potential will have no contribution proportional to $\exp[i(kz-\omega t)]$, but only low-frequency and second-harmonic contributions. The vector potential, on the other hand, will have a low-frequency contribution, in addition to the lowest-order expression written above, but no second harmonics, to a good approximation in the parameter regime of consideration. Thus we will have

$$\mathbf{A}=\mathbf{A}_L(z,t)+\mathbf{A}_H(z,t)\exp[i(kz-\omega t)]+\text{c.c.},$$

$$\Phi=\Phi_L(z,t)+\Phi_{SH}(z,t)\exp[2i(kz-\omega t)]+\text{c.c.} \tag{1}$$

Naturally, all amplitudes are slowly varying and $\mathbf{A}_L$, Φ_L, and Φ_{SH} are second order in the amplitude $\mathbf{A}_H$. Indices L and SH will be used for denoting low-frequency and second-harmonic parts for all variables in what follows.

From Ampère's law we immediately have

$$\frac{\partial^2\Phi}{\partial z\partial t}-4\pi qnv_z=0 \tag{2}$$

and

$$\frac{\partial^2\mathbf{A}}{\partial t^2}-c^2\frac{\partial^2\mathbf{A}}{\partial z^2}-4\pi cqn\mathbf{v}_\perp=\mathbf{0}, \tag{3}$$

where n is the total density including low-frequency as well as second-harmonic contributions. The index $\perp$ refers to components perpendicular to the direction of propagation. Next we must derive an expression for the perpendicular velocity using the equation of motion. For an unmagnetized weakly relativistic cold plasma we have

$$\frac{\partial\mathbf{v}_\perp}{\partial t}+v_z\frac{\partial\mathbf{v}_\perp}{\partial z}=-\frac{q}{mc}\left[\frac{\partial\mathbf{A}}{\partial t}+v_z\frac{\partial\mathbf{A}}{\partial z}\right]\left(1-\frac{|\mathbf{v}_\perp|^2}{c^2}\right)^{1/2}. \tag{4}$$

We emphasize that Eq. (4) cannot be used to derive expressions for the evolution of the low-frequency quantities, but it is valid for the rapidly oscillating parts. The solution for the velocity is

$$\mathbf{v}_\perp=-\frac{q\mathbf{A}}{mc}+\frac{q^3|\mathbf{A}|^2\mathbf{A}}{2m^3c^5}, \tag{5}$$

which is correct to third order in the amplitude. Inserting Eq. (5) into Eq. (3) and keeping terms up to $\mathbf{A}_H^3$, we obtain

$$\frac{\partial^2\mathbf{A}}{\partial t^2}-c^2\frac{\partial^2\mathbf{A}}{\partial z^2}+\omega_p^2\mathbf{A}+\frac{4\pi q^2}{m}\times(n_L+n_{SH})\mathbf{A}-\frac{\omega_p^2q^2}{2m^2c^4}|\mathbf{A}|^2\mathbf{A}=\mathbf{0}. \tag{6}$$

Next we need an equation for the scalar potential. We first concentrate on the low-frequency evolution. The perpendicular component of Ampère's law becomes

$$\frac{\partial^2\mathbf{A}_L}{\partial t^2}-c^2\frac{\partial^2\mathbf{A}_L}{\partial z^2}-4\pi cqn_0\mathbf{v}_{L\perp}=\mathbf{0}. \tag{7}$$

This is the key equation that will lead to a fairly simple description of the excited low-frequency wave. Generally, we can think of a wave as a superposition of an electrostatic part described by Φ_L and an electromagnetic part described by $\mathbf{A}_L$. What simplifies the picture is that approximately the electromagnetic part is obeying the same equations as for vacuum and the electrostatic part obeys the same equations as for an unmagnetized plasma. Furthermore, these parts are only weakly coupled. This simplification occurs because $\mathbf{v}_{L\perp}$ is small; to be more precise, $|\mathbf{v}_{L\perp}|\ll\mathbf{v}_{Lz}|$ *even if the parallel and perpendicular electric fields are comparable in magnitude*. That $\mathbf{v}_{L\perp}$ is small can be seen from Eq. (7). The reason is that our system is nearly stationary in a frame moving with the group velocity v_g of the high-frequency wave. Since we have $v_g\approx c$ and $\partial/\partial t\approx v_g\partial/\partial z$, the operator $\partial^2/\partial t^2-c^2\partial^2/\partial z^2$ becomes small. The parallel and perpendicular equations of motion are

$$\frac{\partial v_{Lz}}{\partial t}=-\frac{q}{m}\frac{\partial\Phi_L}{\partial z}+\omega_c(\mathbf{v}_{L\perp}\times\hat{\mathbf{x}})_\parallel\sin\,\theta-\frac{q^2}{2m^2c^2}\frac{\partial(\mathbf{A}_H\cdot\mathbf{A}_H^*)}{\partial z} \tag{8}$$

and

$$\frac{\partial\mathbf{v}_{L\perp}}{\partial t}=-\frac{q}{mc}\frac{\partial\mathbf{A}_L}{\partial t}+\frac{q}{mc}(\mathbf{v}_L\times\mathbf{B}_0)_\perp, \tag{9}$$

where the ponderomotive force in Eq. (8) acts as a driver for the low-frequency perturbations. In Eq. (8) we see that the only effect of the magnetic field is through the term proportional to $\mathbf{v}_{L\perp}$. Actually, if we neglect this term altogether, we can use the continuity equation and Gauss's law to derive an equation for the low-frequency response that is exactly the same as for an unmagnetized plasma. For an unmagnetized plasma, the low-frequency response is completely electrostatic and accordingly $\mathbf{A}_L=\mathbf{0}$. However, this would be to push the approximations to far. From Eq. (9) we see that if $|\mathbf{v}_{L\perp}|$ is indeed small, we must have

$$\frac{\partial\mathbf{A}_L}{\partial t}\approx(\mathbf{v}_L\times\mathbf{B}_0)_\perp\approx v_z(\hat{\mathbf{z}}\times\mathbf{B})_\perp=\hat{\mathbf{y}}v_zB\,\sin\,\theta. \tag{10}$$

For parameters fulfilling $\omega_c\sin\theta\gg\omega_p$, such a balance means that the excited wake field is approximately transverse and electromagnetic, rather than electrostatic. This is due to the external magnetic field that clearly cannot be neglected. To resolve this paradox the magnetic correction terms from Eqs. (8) and (9) must be included when deriving the low-frequency equation. Acting with $\partial/\partial z$ on Eq. (8) and using the continuity equation followed by Gauss's law we obtain after integration

$$\frac{\partial^3\Phi_L}{\partial z\partial t^2}+\omega_p^2\frac{\partial\Phi_L}{\partial z}-4\pi q n_0\omega_c(\mathbf{v}_{L\perp}\times\hat{\mathbf{x}})_\parallel\sin\theta=-\frac{\omega_p^2 q}{2mc^2}\frac{\partial(\mathbf{A}_H\cdot\mathbf{A}_H^*)}{\partial z}. \quad (11)$$

The electrostatic part of the low-frequency perturbation couples weakly to the electromagnetic part via the last term on the left-hand side of Eq. (11). Our next aim is to express $\mathbf{v}_{L\perp}$ in terms of Φ. We start by eliminating v_z in Eq. (10), using Eq. (2), and obtaining after integration

$$\mathbf{A}_L=\frac{c\omega_c}{\omega_p^2}\sin\theta\frac{\partial\Phi_L}{\partial z}\hat{\mathbf{y}}. \quad (12)$$

The next step is to substitute Eq. (12) in Eq. (7). We then obtain

$$\mathbf{v}_{L\perp}=\frac{\omega_c\sin\theta}{4\pi q n_0\omega_p^2}\left[\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial z^2}\right]\frac{\partial\Phi_L}{\partial z}\hat{\mathbf{y}}. \quad (13)$$

Finally, from Eqs. (11) and (13) we obtain

$$\frac{\partial^2\Phi_L}{\partial t^2}+\omega_p^2\Phi_L+\frac{\omega_c^2\sin^2\theta}{\omega_p^2}\left[\frac{\partial^2\Phi_L}{\partial t^2}-c^2\frac{\partial^2\Phi_L}{\partial z^2}\right]=-\frac{\omega_p^2 q(\mathbf{A}_H\cdot\mathbf{A}_H^*)}{2mc^2}. \quad (14)$$

Next we focus our attention on the second-harmonic term n_{SH} in Eq. (6). We remind ourselves that the external magnetic field is not of importance at the fast time scale in our parameter regime. Using the longitudinal momentum equation, the continuity equation, and Gauss's law, we proceed similarly as when deriving Eq. (14). The result is

$$n_{SH}=\frac{\omega_p^2k^2\mathbf{A}_H\cdot\mathbf{A}_H}{8\pi\omega^2mc^2}\exp[2i(kz-\omega t)]+\text{c.c.} \quad (15)$$

Using Eq. (15) together with the ansatz (1) in Eq. (6) we find

$$-2i\omega\left(\frac{\partial\mathbf{A}_H}{\partial t}+v_g\frac{\partial\mathbf{A}_H}{\partial z}\right)+\frac{\partial^2\mathbf{A}_H}{\partial t^2}-\frac{\partial^2\mathbf{A}_H}{\partial z^2}+\frac{4\pi q^2n_L}{m}\mathbf{A}_H=\frac{\omega_p^2q^2}{2m^2c^4}\left[2(\mathbf{A}_H\cdot\mathbf{A}_H^*)\mathbf{A}_H+\left(1-\frac{k^2c^2}{\omega^2}\right)(\mathbf{A}_H\cdot\mathbf{A}_H)\mathbf{A}_H^*\right], \quad (16)$$

where $v_g=kc^2/\omega$ is the group velocity of the high-frequency electromagnetic wave. The last term proportional to $1-k^2c^2/\omega^2$ contains the nonlinearities due to second-harmonic effects and some part of the relativistic contribution. Since $1-k^2c^2/\omega^2\ll 1$, however, this term will be omitted from now on.

Next we introduce new coordinates (τ,ξ) moving with the group velocity, that is, $\tau=t$ and $\xi=z-v_gt$. We note that the evolution is slow in the new frame, such that we can safely use $\partial/\partial\tau\ll c\partial/\partial\xi$. For convenience we omit the subscript L from now on and write $\Phi(\xi,\tau)=\Phi_L(x,t)$. Furthermore, we note that for arbitrary polarization of the high-frequency pulse we can set $\mathbf{A}_H=A(\xi,\tau)\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a complex unit vector, i.e., $\hat{\mathbf{e}}\cdot\hat{\mathbf{e}}^*=1$. Equation (14) is then written as [13]

$$-2\frac{\omega}{k}\frac{\omega_{pc}^2}{\omega_p^2}\frac{\partial^2\Phi}{\partial\xi\partial\tau}+v_g^2\frac{\partial^2\Phi}{\partial\xi^2}+\omega_p^2\Phi+\frac{\omega_p^2q|A|^2}{mc^2}=0, \quad (17)$$

where $\omega_{pc}^2=\omega_p^2+\omega_c^2\sin\theta$. When deriving Eq. (17) a term proportional to $\omega_c^2(c^2-v_g^2)\partial^2\Phi/\partial\xi^2$ has been neglected. This term can be included without difficulty, but only causes a small frequency shift of the exited wake field and it is therefore omitted. Equation (17) is one of our main equations. Clearly, the low-frequency wake field is expressed only in terms of the scalar potential. However, in contrast to the unmagnetized case, the wake field may have a significant electromagnetic part, which can be found from Eq. (12). We emphasize that for $\omega_c\sin\theta\gg\omega_p$, the electromagnetic energy density of the wake field is much larger than the electrostatic energy density.

We are now ready to deduce the final form of Eq. (16). Since $\partial/\partial\tau\ll c\partial/\partial\xi$, we see that the first term in Eq. (17) is a small correction. As will be discussed later on, this term is of importance for the long-term evolution of the wake field, but as a first approximation we can use

$$v_g^2\frac{\partial^2\Phi}{\partial\xi^2}+\omega_p^2\Phi=-\frac{\omega_p^2q|A|^2}{mc^2}, \quad (18)$$

which together with Gauss's law gives

$$n_L=\frac{\omega_p^2}{4\pi q v_g^2}\left[\Phi+\frac{q|A|^2}{mc^2}\right]. \quad (19)$$

Substituting Eq. (19) in Eq. (16), changing to moving coordinates, and neglecting derivatives proportional to $\partial^2/\partial\tau^2$ we have [13]

$$-i\frac{\partial A}{\partial\tau}-\frac{1}{k}\frac{\partial^2A}{\partial\xi\partial\tau}-\frac{v_g'}{2}\frac{\partial^2A}{\partial\xi^2}+\frac{\omega_p^2q}{2\omega mc^2}\Phi A=0, \quad (20)$$

where $v_g'=dv_g/dk$ is the group dispersion of the high-frequency pulse. The total cubic nonlinearity becomes proportional to $\omega_p^2(c^2-v_g^2)/\omega^2c^2$ and is therefore omitted. The system of equations (17) and (20) governing the evolution of a low-frequency wake field described by Φ and a short electromagnetic pulse described by A is the main result of our paper. Note the presence of the second term in Eq. (20) proportional to $\partial^2A/\partial\xi\partial\tau$, which is a small correction term since we have $\partial/\partial\tau\ll c\partial/\partial\xi$. Although this term is small, it is important for the long-term evolution since it leads to a violation of the conservation law $d/d\tau\int|A|^2d\xi=0$, as will be discussed in detail in the next section.

III. VARIATIONAL PRINCIPLE AND CONSERVATION LAWS

The system of equations (17) and (20) can be derived from a variational principle. Introducing the Lagrangian density

$$\mathcal{L}=i\omega\left(A^*\frac{\partial A}{\partial\tau}-A\frac{\partial A^*}{\partial\tau}\right)-\frac{\omega}{k}\left(\frac{\partial A}{\partial\xi}\frac{\partial A^*}{\partial\tau}+\frac{\partial A^*}{\partial\xi}\frac{\partial A}{\partial\tau}\right)-\omega v_g'\left|\frac{\partial A}{\partial\xi}\right|^2-\frac{\omega}{k}\frac{\omega_{pc}^2}{\omega_p^2}\frac{\partial\Phi}{\partial\tau}\frac{\partial\Phi}{\partial\xi}+\frac{v_g^2}{2}\left(\frac{\partial\Phi}{\partial\xi}\right)^2-\frac{\omega_p^2\phi^2}{2}-\frac{\omega_p^2 q|A|^2\Phi}{mc^2}, \tag{21}$$

where the action functional is $\mathcal{A}(\Phi,A,A^*)=\int\mathcal{L}\,d\xi\,d\tau$ (also compare the Lagrangian densities given in Ref. [14]), we obtain Eqs. (17) and (20) varying Φ and A^* and minimizing the action as usual. Since time does not appear explicitly in the Lagrangian, the Hamiltonian will be conserved. Calculating the Hamiltonian density as

$$\mathcal{H}=\frac{\partial\mathcal{L}}{\partial(\partial A/\partial\tau)}\frac{\partial A}{\partial\tau}+\frac{\partial\mathcal{L}}{\partial(\partial A^*/\partial\tau)}\frac{\partial A^*}{\partial\tau}+\frac{\partial\mathcal{L}}{\partial(\partial\Phi/\partial\tau)}\frac{\partial\Phi}{\partial\tau}-\mathcal{L}, \tag{22}$$

we find from $(d/d\tau)\int\mathcal{H}\,d\xi=0$ that

$$\frac{d}{d\tau}\int_{-\infty}^{\infty}\left[\omega v_g'\left|\frac{\partial A}{\partial\xi}\right|^2-\frac{v_g^2}{2}\left(\frac{\partial\Phi}{\partial\xi}\right)^2+\frac{\omega_p^2\phi^2}{2}+\frac{\omega_p^2 q|A|^2\Phi}{mc^2}\right]d\xi=0. \tag{23}$$

Now it is time to consider two more exact conservation laws. By using Eqs. (17) and (20) and making a number of partial integrations we find

$$\frac{d}{d\tau}\int_{-\infty}^{\infty}\left\{k^2|A|^2+\frac{ik}{2}\left(A\frac{\partial A^*}{\partial\xi}-A^*\frac{\partial A}{\partial\xi}\right)\right\}d\xi=0 \tag{24}$$

and

$$\frac{d}{d\tau}[W_A+W_\Phi]=\frac{d}{d\tau}\int_{-\infty}^{\infty}\{\mathcal{W}_A+\mathcal{W}_\Phi\}d\xi=0, \tag{25}$$

where

$$\mathcal{W}_A=k^2|A|^2+ik\left(A\frac{\partial A^*}{\partial\xi}-A^*\frac{\partial A}{\partial\xi}\right)+\frac{\partial A^*}{\partial\xi}\frac{\partial A}{\partial\xi} \tag{26}$$

and

$$\mathcal{W}_\Phi=\frac{\omega_{pc}^2}{\omega_p^2}\left(\frac{\partial\Phi}{\partial\xi}\right)^2. \tag{27}$$

Since frequency and wave-number conversion are proportional, we have

$$\int_{-\infty}^{\infty}|E|^2d\xi\propto\int_{-\infty}^{\infty}|B|^2d\xi=\int_{-\infty}^{\infty}\mathcal{W}_A d\xi.$$

Consequently $\mathcal{W}_A$ in Eq. (25) corresponds to the pulse energy density, whereas the last term corresponds to the energy density of the wake field. The term $(\partial A/\partial\xi)(\partial A^*/\partial\xi)$ in $\mathcal{W}_A$ is a small correction that can be omitted within our approximation scheme, but it is needed to make the energy conservation law exact. The terms in Eqs. (17) and (20) that contain mixed derivatives are small corrections that are usually omitted. However, note that we cannot obtain a self-consistent conservation law such as Eq. (25) without including those terms. The significance of the corrections terms will be discussed in more detail in Sec. IV.

Generally, conservation laws such as Eqs. (23)–(25) are important for the understanding of the governing equations. Furthermore, they can also be useful for analytical approximation methods based on the variational principle [15]. The idea is to chose a trial function with a specified spatial variation involving a number of parameters that are functions of time. It is of course necessary to have physical insight when choosing what parameters to use. Minimizing the action then gives ordinary differential or algebraic equations determining the evolution of the parameters in the trial function. The question of the reliability of the approximate solutions then arises. An important check that does not involve numerical calculations is to investigate to what extent the conserved quantities are preserved by the variational solution. This scheme has been used successfully for the nonlinear Schrödinger equation [15] and Zakharov's equations [16]. However, a similar investigation for our set of equations is beyond the scope of our paper.

IV. ENERGY AND FREQUENCY DECAY RATE OF THE HIGH FREQUENCY PULSE

Equations (17) and (20) allow for a general interaction between the low-frequency mode and high-frequency pulses, provided the evolution is slow in a frame moving with v_g. For example, they can describe wake-field generation by a number of pulses [9] or frequency up-conversion of a short pulse situated somewhere in the wake field where $\partial\Phi/\partial\xi<0$ [10]. However, from now on we will concentrate on a single pulse propagating through an unperturbed plasma. The pulse is assumed to enter the plasma at $\tau=0$ when $\Delta\omega=0$ and the initial pulse energy is denoted W_{A0} $[\approx k^2\int_{-\infty}^{\infty}|A(\tau=0)|^2d\xi]$. We start by considering the frequency conversion rate. Generally the frequency shift is given by

$$\Delta\omega\approx c\Delta k=c\frac{\int_{-\infty}^{\infty}\kappa|A_\kappa|^2d\kappa}{\int_{-\infty}^{\infty}|A_\kappa|^2d\kappa}, \tag{28}$$

where $A_\kappa=\int A\,\exp(i\kappa\xi)d\xi$. Since we have used a WKB ansatz, we are limited to small frequency shifts $\Delta\omega\ll\omega$. As we will see below, this also implies small changes in pulse energy $\Delta W_A\ll W_{A0}$. Changing the order of integration, we can write an equation for the rate of change of the frequency shift as

$$\frac{d\Delta\omega}{d\tau}\approx c\frac{d}{d\tau}\frac{\int_{-\infty}^{\infty}\kappa|A_\kappa|^2 d\kappa}{\int_{-\infty}^{\infty}|A_\kappa|^2 d\kappa}$$

$$\approx c\frac{i\frac{d}{d\tau}\int_{-\infty}^{\infty}\left(A\frac{\partial A^*}{\partial\xi}-A^*\frac{\partial A}{\partial\xi}\right)d\xi}{2\int_{-\infty}^{\infty}|A|^2 d\xi}\approx-\frac{\omega}{W_{A0}}\frac{dW_\Phi}{d\tau}. \tag{29}$$

Before deducing the final expression, a number of approximations have been made using $\Delta\omega/\omega\ll 1$, $\Delta W_A/W_{A0}\ll 1$, and $kL\ll 1$, where L is the characteristic scale length for variations in A. The last equality follows from Eqs. (24) and (25) if we note that the term proportional to $(\partial A/\partial\xi)(\partial A^*/\partial\xi)$ in Eq. (25) is a small correction term. Integrating Eq. (29) keeping in mind that the variation in pulse energy is small, we obtain

$$\frac{\Delta\omega}{\omega}=-\frac{\Delta W_\Phi}{W_{A0}}=\frac{\Delta W_A}{W_{A0}}, \tag{30}$$

where the last step follows from the conservation law (25). Since the energy of the pulse is proportional to $N\hbar(\omega+\Delta\omega)$, where N is the number of high-frequency quanta, we see from Eq. (30) that N is conserved. Actually, from Ref. [10] we see that within our approximation scheme N is proportional to $\int_{-\infty}^{\infty}\{k^2|A|^2+ik[A(\partial A^*/\partial\xi)-A^*(\partial A/\partial\xi)]/2\}d\xi$ and thus Eq. (24) is a direct consequence of the conservation of high-frequency quanta.

Next we rewrite the conservation law (25) by dividing the integration limits according to

$$\frac{d}{d\tau}\int_{-\infty}^{\xi_1}\mathcal{W}_\Phi d\xi+\frac{d}{d\tau}\int_{\xi_1}^{\xi_2}\{\mathcal{W}_A+\mathcal{W}_\Phi\}d\xi=0, \tag{31}$$

where ξ_2 and ξ_1 are stationary points (in the moving frame) located slightly in front of the pulse and slightly after the pulse, respectively. The integral from $-\infty$ to ξ_1 can be performed by letting $d/d\tau$ operate inside the integral and using Eq. (17). Next we note that the low-frequency contribution to the last integral in Eq. (31) is much smaller than the pulse contribution and accordingly the conservation law can be approximated by

$$\frac{d}{d\tau}\int_{\xi_1}^{\xi_2}\mathcal{W}_A d\xi=-\frac{d}{d\tau}\int_{-\infty}^{\xi_1}\mathcal{W}_\Phi d\xi=-\frac{k}{2\omega}\left[v_g^2\left(\frac{\partial\Phi}{\partial\xi}\right)^2+\omega_p^2\Phi^2\right]_{\xi=\xi_1}, \tag{32}$$

where $\mathcal{W}_A$ is the energy density of the pulse. Thus the loss of pulse energy only depends on the wake-field magnitude after the pulse passage. Equation (32) can also be obtained by neglecting the first term of Eq. (17), using Eq. (18), which is a first approximation to Eq. (17), from the start. Although the approximation used to obtain Eq. (18) clearly can be justified, that result is somewhat peculiar. When making the derivations, we saw that the excited wake field can have a significant electromagnetic component [cf. Eq. (12)]. Actually for $\omega_c\sin\theta\gg\omega_p$, the wake field is approximately transverse and electromagnetic. Still the low-frequency equation can be approximated by Eq. (18), which is the same equation as for an unmagnetized plasma where the wake field is completely electrostatic. This paradox will be resolved in Sec. V, where the importance of the external magnetic field also will be discussed. At this stage we only note that the energy and frequency decay rates of the high-frequency pulse are unaffected by the external magnetic field to a good approximation and therefore Eq. (18) will be used in the rest of this section.

In order to simplify our problem, we consider the evolution during a comparatively short time, such that the high-frequency pulse can be taken as constant. During the condition that the plasma is unperturbed before the arrival of the pulse, Eq. (18) can be integrated [4] to give

$$\Phi(\xi)=\frac{\omega_p q}{mc^2 v_g}\int_{\xi}^{\xi_2}|A(\xi')|^2\,\sin[k_p(\xi-\xi')]d\xi'. \tag{33}$$

The resulting potentials for some specific forms of the pulse are given in Ref. [4]. From Eq. (33) we find that for a short pulse with pulse length fulfilling $k_pL\ll 1$ we can generally write $\Phi=\Phi_0\sin(k_p\xi+\delta)$ after the passage of the pulse, where δ is a phase factor (see Ref. [4] for an expression) and the amplitude Φ_0 is given by

$$\Phi_0=\frac{\omega_p q}{mc^2 v_g}\int_{\xi_1}^{\xi_2}|A(\xi')|^2 d\xi'. \tag{34}$$

From Eq. (34) combined with Eq. (32) the corresponding frequency and energy decay rates of the high-frequency pulse are

$$\frac{d\Delta\omega/d\tau}{\omega}=\frac{dW_A/d\tau}{W_A}=-\frac{q^2\omega_p^4}{2m^2kc^4\omega v_g^2}\int_{\xi_1}^{\xi_2}|A(\xi')|^2 d\xi'. \tag{35}$$

The fact that the energy and frequency decay rates coincide is again a consequence of the conservation of high-frequency quanta.

V. EFFECTS DUE TO THE EXTERNAL MAGNETIC FIELD

The derivation of the energy and frequency decay rates made in the preceding section was made by neglecting the effect of the first term in Eq. (17). Since that term is the only one containing B_0, the effect of the external magnetic field was completely omitted in that calculation. This approximation was possible since the high-frequency pulse couples directly only to Φ (and not to A_L) and furthermore the electrostatic and electromagnetic parts of the electric field are only weakly coupled. Still the external magnetic field has a number of consequences for the wake-field properties. First, it is obvious that the electric field is not longitudinal but has a transverse component that can be found from Eq. (12). Clearly, the relative strength of the transverse electrical field component to the longitudinal one fulfills

$|\partial A_L/\partial t|/|\partial\Phi/\partial z| = \omega_c \sin\theta/\omega_p$. Since the longitudinal field strength is approximately the same as for an unmagnetized plasma this means that the energy density of the wake field is amplified by a factor ω_{pc}^2/ω_p^2 [compare Eq. (27)] due to the magnetic field. At first one may think that the energy-loss rate of the high-frequency pulse must be increased by this, but that would be contrary to our results in Sec. IV. To resolve this paradox and improve our understanding of the wake-field properties we must use Eq. (17) instead of the approximation given in Eq. (18). We note that after the pulse passage, Eq. (17) can be solved by a WKB approximation. Substituting $\Phi = \tilde{\Phi}(\xi,\tau)\exp(ik_p\xi) + \text{c.c.}$ into Eq. (17), we obtain

$$\frac{\partial\tilde{\Phi}}{\partial\tau} + \Delta v_g \frac{\partial\tilde{\Phi}}{\partial\xi} = 0 \tag{36}$$

if the low-frequency wave number fulfills $k_p = \omega_p/v_g$. The group velocity of the wake field in the moving frame Δv_g is given by $\Delta v_g = v_{gwf} - v_g = -v_g\omega_p^2/\omega_{pc}^2$, where $v_{gwf} = v_g \sin^2\theta\omega_c^2/\omega_{pc}^2$ is the group velocity in the laboratory frame. We note that the finite group velocity (in the laboratory frame) is *solely due to the external magnetic field*. Furthermore, it is clear that the use of Eq. (18) would lead to an envelope that is stationary in the moving frame, instead of propagating with Δv_g. The energy contained in the wake field can roughly be written $W_\Phi = \langle \mathcal{W}_\Phi \rangle L_\Phi$, where angular brackets denote an average over the wake-field region and L_Φ is the length of the wake field. Furthermore, we can roughly write $L_\Phi = -\Delta v_g \tau$ (if the pulse enters the plasma at $\tau = 0$). From the expression for Δv_g we see that the external magnetic field leads to a *contraction of the wake field by a factor* ω_{pc}^2/ω_p^2, as compared to the unmagnetized case. This exactly compensates for the increased energy density and makes the total energy contained in the wake field independent of B_0. Consequently, the energy-loss rate of the high-frequency pulse is also unaffected by the external magnetic field, in agreement with the results in Sec. IV.

Due to propagation of the wake field, an interesting effect occurs if the external magnetic field is inhomogeneous. For definiteness we consider a magnetic field that is directed along $\hat{\mathbf{x}}$, has $B = B_I$ constant for $z < z_1$, decreases smoothly for $z_1 < z < z_2$, and finally become $B = B_F$ for $z > z_2$. If we follow a certain part of the wake field through the inhomogeneous region $z_1 < z < z_2$, two effects lead to the increase of the longitudinal electric field. First, the decrease of v_{gwf} means that energy piles up from behind and the total energy density of the wake field increases. Second, the ratio of longitudinal energy over transverse energy density for the wake field increases as $\omega_p^2/\omega_c^2(z)$. The combined effect results in a very simple expression for the amplification factor N of the longitudinal part of the electric wake field

$$N \equiv \frac{E_{FL}}{E_{IL}} = \frac{B_I}{B_F}, \tag{37}$$

where E_{IL} and E_{FL} are the longitudinal electric field amplitude before and after *the wake field has propagated* from the region $z < z_1$ to the region $z > z_2$. For this amplification mechanism to work properly, the cyclotron frequency cannot be too small as compared to the plasma frequency. However, we suggest that experiments in the microwave regime [5] should be able to benefit from this idea if an external magnetic field of reasonable strength is included in the experimental setup [17]. It would also be of interest to consider the case where the external magnetic field vanishes for $z > z_2$, that is, $B_F = 0$. Naturally, Eq. (37) does not apply anymore and the saturation of the longitudinal electric field would be determined from mechanisms outside our model, such as thermal motion, nonlinear self-interaction of the wake field, or two-dimensional effects.

VI. SUMMARY AND DISCUSSION

We have considered propagation of a short electromagnetic pulse in a magnetized plasma. Two self-consistent equations describing the interaction between the pulse and a low-frequency wake field for arbitrary directions of propagations (as compared to the external magnetic field) have been derived. During the propagation through the unperturbed plasma, the pulse energy is transferred to the wake field and the frequency of the pulse is decreased. Due to our WKB ansatz, the description is limited to small changes in frequency. However, the system is general enough to allow also for frequency up-conversion of a second weak pulse, following the first strong pulse that has generated the wake field. Basically, the effect of the external magnetic field is the following. First, the generated wake field becomes partially transverse and electromagnetic, as described by Eq. (12). The energy density of the wake field in the magnetized case is larger by a factor ω_{pc}^2/ω_p^2 as compared to the unmagnetized case. This magnification is directly attributed to the additional electromagnetic part of the wake field. Furthermore, the group velocity of the wake field in the laboratory frame becomes appreciable if $\sin^2\theta\omega_c^2 \sim \omega_p^2$. Propagation of the wake-field envelope in the laboratory frame leads to a contraction of the wake field by a factor ω_{pc}^2/ω_p^2 as compared to the unmagnetized case (for a given time of interaction with the high-frequency pulse). The nonzero group velocity induced by the external magnetic field causes an interesting effect in an inhomogeneous plasma. If the magnetic field is decreasing in the direction of propagation, the longitudinal part of the electric wake field can be significantly amplified due to the decreasing group velocity. For $\sin^2\theta\omega_c^2 \sim \omega_p^2$, the change in wake-field properties due to the external magnetic field should be easy to detect in laboratory experiments.

Our derivation is based on a WKB ansatz. However, in Eq. (20) we have kept the correction term proportional to $\partial^2 A/\partial\xi\partial\tau$, which is usually omitted [4,10]. It is necessary to include this term if we want self-consistent conservation laws. To be more specific, if we neglect this term and calculate the energy loss of a high-frequency pulse propagating through an unperturbed plasma, the result becomes a factor 2 too large. In addition to energy conservation, the pulse exhibits conservation laws for the Hamiltonian and the number of high-frequency quanta. Together with the variational principle, the conservation laws can be useful for analytical approximation methods [15]. In this paper we have restricted ourselves to the qualitative aspects of the pulse propagation, with a specific interest in the frequency decrease and energy-loss rate of the pulse. However, the pulse is also subject to a nonlinear modification of shape. The shape modification occur on a faster time scale than the frequency decrease and energy-loss rate of the pulse [4]. To investigate such effects detailed solutions of the governing equations must be performed. Such studies have been done for equations of a similar type to ours [18], but a thorough investigation of Eqs. (17) and (20) remains a project for further research.

Finally, we note that Eq. (20) can be rewritten slightly. The first two terms can be combined according to $i\partial A/\partial\tau + (1/k)\partial^2 A/\partial\xi\partial\tau = i[1-(i/k)(\partial/\partial\xi)]\partial A/\partial\tau$, which means that Eq. (20) can be written as

$$-i\frac{\partial A}{\partial\tau} - \left(1-\frac{i}{k}\frac{\partial}{\partial\xi}\right)^{-1}\left[\frac{v_g'}{2}\frac{\partial^2 A}{\partial\xi^2} - \frac{\omega_p^2 q}{2\omega mc^2}\Phi A\right] = 0. \tag{38}$$

Since the second term in the inverse operator is a small correction as compared to the first, that operator can be Taylor expanded to first order, which gives

$$-i\frac{\partial A}{\partial\tau} - \frac{v_g'}{2}\frac{\partial^2 A}{\partial\xi^2} + \frac{\omega_p^2 q}{2\omega mc^2}\Phi A - i\frac{v_g'}{2k}\frac{\partial^3 A}{\partial\xi^3} + i\frac{\omega_p^2 q}{2\omega mkc^2}\frac{\partial}{\partial\xi}[\Phi A] = 0, \tag{39}$$

where the last two terms are small corrections. The dispersive correction in Eq. (39) is typically not of much importance. However, the nonlinear correction is necessary if we should obtain self-consistent conservation laws. We emphasize that all results in Secs. III and IV can be derived from Eq. (39) rather than Eq. (20), with the difference that some of the conservation laws become approximate rather than exact. Probably Eq. (39) is easier to implement than Eq. (20) in a numerical code.

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