Periodically driven linear system with multiplicative colored noise

A. V. Barzykin and K. Seki
Department of Physical Chemistry, National Institute of Materials and Chemical Research, Tsukuba, Ibaraki 305, Japan

F. Shibata
Department of Physics, Faculty of Science, Ochanomizu University, Bunkyo-ku, Tokyo 112, Japan

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A periodically driven linear system subject to multiplicative correlated noise is considered. It has been argued recently by several authors that such a simple system exhibits stochastic resonance. By introducing a general type of composite stochastic process, bridging two previously considered limiting cases of dichotomous and Gaussian noise, it is proved that, indeed, the amplitude of the average of the driven linear process at long times shows a pronounced maximum both as a function of the noise strength and as a function of the autocorrelation time. However, this kind of stochastic resonant behavior can be experimentally observable only in a special case where the initial phase of the external forcing is somehow fixed. Additional averaging over the uniform distribution of the initial random phase, inherent in most physical systems, leads to that the periodic output vanishes identically at long times. Moreover, the system response is typically defined in terms of the power spectrum rather than the amplitude of the average. The output signal given by the spectral density corresponding to the frequency of the external forcing is calculated via the long-time phase-averaged correlation function. It appears that the output signal simply diverges upon approaching the second moment instability point with increasing noise strength. No stochastic resonance is observed for any parameter settings. Interestingly, the resonancelike behavior of the system response as a function of the autocorrelation time is retained. [S1063-651X(98)12906-9]

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I. INTRODUCTION

Stochastic forces can play a crucial role in influencing the deterministic kinetics. A representative example is stochastic resonance (SR) observed in metastable systems driven by a combination of periodic and random forcing [1–7]. SR manifests itself in a significant enhancement of the system response for a certain value of the noise strength. So far, the majority of the theoretical studies in this area have focused on nonlinear systems with additive white noise. It was concluded that nonlinearity is an essential ingredient of SR since in a linear system the input additive noise leads to only a trivial decrease in the output signal-to-noise ratio (SNR) while, in contrast, a dramatic improvement of the SNR can be observed for a periodically modulated nonlinear potential.

Recently, behavior similar to what is commonly ascribed to SR has been found in a linear system subject to multiplicative colored noise [8–10]. A pronounced maximum of the SNR as a function of the noise intensity was observed for not too high frequencies of the external periodic force as soon as noise correlation was introduced. Analytical solutions were obtained for two different limits of the noise, namely, for dichotomous noise [8,9] and for Gaussian noise [10], and it was suggested that noise multiplicativity and time correlation are the necessary conditions for the SR to occur in a linear system. Interestingly, the dependence of the SNR on the noise autocorrelation time also showed a nonmonotonic behavior.

Multiplicative fluctuations emerge naturally in a variety of systems with ensuing applications in different areas ranging from physics to biology [11,12]. In fact, in a realistic model one must always deal with various sources for fluctuations acting upon collective variables. Multiplicative processes have some common features with additive processes and they also have a number of striking differences. Importantly, the most probable values in a multiplicative process depend explicitly on the strength of the fluctuations, while in an additive process the dependence is very weak. For a multiplicative process, the stability of the associated deterministic problem does not guarantee the stability in the presence of fluctuations. On the contrary, an additive stochastic process is stable whenever the deterministic problem has a globally stable steady state far away from the instability point. Here lies a key to the explanation of the resonancelike phenomena observed in a linear periodically driven system subject to multiplicative colored noise.

Consider an overdamped linear system described by the stochastic differential equation

$$\dot{x}(t) = -[a_0 + \xi(t)]x + A\,\sin(\Omega t + \varphi_0), \tag{1.1}$$

where $\xi(t)$ represents the noise with a vanishing mean and a certain time correlation. A, Ω, and φ_0 denote the amplitude, the frequency, and the initial phase of the external modulation, respectively. Even this simple problem has a number of important applications, such as fluctuating barrier crossing in chemistry [13]. By introducing the potential $U(x) = \frac{1}{2}[a_0 + \xi(t)]x^2 - A\sin(\Omega t + \varphi_0)x$, Eq. (1.1) can be rewritten as $\dot{x}(t) = -\partial U/\partial x$. Thus the system evolution is governed by the interplay between the fluctuating potential curvature due to multiplicative noise and the periodic shift of the minimum due to the sinusoidal signal. The evolution of the average $\langle x(t)\rangle$ can be described in terms of a sequence of noise-generated effective potentials [10]. Only two potentials are involved in the case of dichotomous noise, while an infinite number of terms appears for Gaussian-type noise. Qualitatively, as the noise strength increases, the stability of the lowest most efficient term decreases, leading to the enhancement of the output signal. At the same time, the weight of this potential decreases with increasing noise strength. These two competing factors generate nonmonotonic behavior of the SNR. It should be emphasized that signal enhancement in this model is due solely to a decreasing system stability, typical of a multiplicative process. In the case of white noise, only one effective potential is involved. Thus, as the noise strength increases toward the marginal point of stability, a monotonic increase of the output signal is observed.

So far we have tacitly assumed, together with Fuliński [8] and Berdichevsky and Gitterman [9], that the output signal is given by the amplitude of the average $\langle x(t)\rangle$ at long times. However, the system response is often defined in terms of the experimentally observable power spectrum [2,6]. For a stationary stochastic process, the Wiener-Khintchine theorem holds and the spectral density is obtained as the Fourier transform of the autocorrelation function, which depends on the time difference only and thus, indeed, can be represented by the average. However, stochastic processes with periodic modulation are essentially nonstationary stochastic processes with the correlation function depending explicitly on two time arguments. The long-time amplitude of the average, therefore, generally does not define the power spectrum. Moreover, in many physical situations, the initial phase φ_0 of the modulation is unknown. It should thus be considered as a random variable and the results have to be averaged over the phase distribution [6].

The purpose of this paper is twofold. First of all, we will present a solution to Eq. (1.1) for a general type of noise that bridges the Markovian two-state jump process and the Gaussian process and allows a straightforward further generalization to include noise asymmetry. Our second goal is to clarify the above-mentioned uncertainties of the previous treatments. We will calculate the phase-averaged correlation function and thus show that the signal output defined as the spectral density corresponding to the periodic forcing frequency exhibits no SR but simply a monotonic increase with the noise strength due to a decreasing stability of the system. We will also show that the resonancelike behavior of the system response as a function of the autocorrelation time is retained.

II. PROPERTIES OF THE NOISE

The two-state jump process (often referred to as a dichotomous process) and the Gaussian process respresent two opposite extremes in a sense that the former is characterized by two values of realization, whereas the latter is characterized by an infinite number of realizations. These two processes have been widely used in stochastic modeling mainly due to their mathematical simplicity. The higher-order partial cumulants in the former and the cumulants in the latter vanish identically, yielding simple expansion formulas. Several years ago, a general model was formulated for a composite stochastic process that bridges the two-state jump and the Gaussian processes while retaining the simplicity of the two [14]. The composite process was introduced as a superposition of many two-state jump Markovian processes. In this section we summarize the properties of such a composite stochastic process, which will be required for our further discussion. Previously, the analysis was performed on the basis of the time-convolution expansion using projection operators [14,15]. This method is very general, allowing evaluation of any functional of the noise. Here we choose a different approach, originally due to Kubo [16], which operates directly with the evolution equation.

Consider a stochastic process composed of N independent two-state jump processes, namely,

$$\xi^{(N)}(t)=\sum_{n=1}^{N}\xi_n(t), \tag{2.1}$$

where each constituent process has the stationary properties

$$\langle \xi_n(t)\rangle=0, \tag{2.2}$$

$$\langle \xi_n(t)\xi_{n'}(t')\rangle=\delta_{nn'}\Delta_0^2\exp(-|t-t'|/\tau_c), \tag{2.3}$$

$$P_{\mathrm{st}}(\Delta_0)=P_{\mathrm{st}}(-\Delta_0)=\frac{1}{2}. \tag{2.4}$$

The evolution equation

$$\dot{y}(t)=-\xi^{(N)}(t)y, \tag{2.5}$$

with the initial condition $y(0)=1$ for a total of 2^N states, can be transformed into a direct product of N independent two-state evolution equations

$$\frac{d}{dt}\begin{bmatrix} y_+(t)\\ y_-(t)\end{bmatrix}=-\Delta_0\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix} y_+(t)\\ y_-(t)\end{bmatrix}-\frac{1}{2\tau_c}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\times\begin{bmatrix} y_+(t)\\ y_-(t)\end{bmatrix}, \tag{2.6}$$

where $y_\pm\equiv\int yP_\pm(y,t)dy$ and $P_\pm(y,t)$ is the probability that the value y in the state $\pm$ is realized at time t. The initial condition consistent with the stationary properties of the noise is

$$y_+(0)=y_-(0)=\frac{1}{2}. \tag{2.7}$$

Introducing

$$\phi(t)=y_+(t)+y_-(t),\qquad \psi(t)=y_+(t)-y_-(t), \tag{2.8}$$

we obtain for their Laplace transforms

$$\hat{\phi}(s)=\frac{(s+\tau_c^{-1})\phi(0)-\Delta_0\psi(0)}{s(s+\tau_c^{-1})-\Delta_0^2}, \tag{2.9}$$

$$\hat{\psi}(s)=\frac{s\psi(0)-\Delta_0\phi(0)}{s(s+\tau_c^{-1})-\Delta_0^2}. \tag{2.10}$$

The Laplace transform is defined by

$$\hat{f}(s)=\int_0^\infty f(t)e^{-st}dt. \tag{2.11}$$

Taking into account the initial conditions and inverting the Laplace transform in Eq. (2.9), we finally arrive at

$$\begin{aligned}\langle y(t)\rangle&=\left\langle \exp\left[-\int_0^t dt'\,\xi^{(N)}(t')\right]\right\rangle=[\phi(t)]^N\\&=\exp(-Nt/2\tau_c)\{\cosh(t/2b\tau_c)\\&\quad+b\sinh(t/2b\tau_c)\}^N\\&=\sum_{n=0}^N B_n(N,\mu)\exp(-\kappa_n t),\end{aligned} \tag{2.12}$$

where

$$b=(1+4\alpha^2/N)^{-1/2},\quad \alpha=\Delta\tau_c,\quad \Delta^2=N\Delta_0^2, \tag{2.13}$$

$$\mu=(1-b)/2,\quad \kappa_n=(n-\mu N)/b\tau, \tag{2.14}$$

and

$$B_n(N,\mu)=\binom{N}{n}\mu^n(1-\mu)^{N-n} \tag{2.15}$$

is the binomial weight factor with the mean μN. We also note the relationship

$$\dot{\phi}(t)=-\Delta_0\psi(t) \tag{2.16}$$

and thus

$$\psi(t)=-\frac{2\alpha b}{\sqrt{N}}\sinh(t/2b\tau_c)\exp(-t/2\tau_c), \tag{2.17}$$

which will be used later.

In the limit of $N\to\infty$, the binomial coefficients tend to the Poissonian weight factors

$$B_n(N,\mu)\to P_n(\alpha^2)\equiv(\alpha^{2n}/n!)\exp(-\alpha^2) \tag{2.18}$$

and Eq. (2.12) reduces to the well-known result for the Gaussian noise [16]

$$\langle y(t)\rangle=\exp[\chi(t)]=\sum_{n=0}^\infty P_n(\alpha^2)\exp(-\kappa_n t), \tag{2.19}$$

where

$$\chi(t)=\alpha^2(t/\tau_c-1+e^{-t/\tau_c}), \tag{2.20}$$

$$\kappa_n=(n-\alpha^2)/\tau_c. \tag{2.21}$$

The above results can be generalized to asymmetric noise [17] with the values of realization

$$\xi_\pm=\pm\Delta_0(1\pm\varepsilon) \tag{2.22}$$

and the stationary probabilities

$$P_{\rm st}(\xi_\pm)=(1\mp\varepsilon)/2. \tag{2.23}$$

It can be shown that both the noise amplitudes and the transition probabilities must be asymmetric for the average $\langle\xi_n(t)\rangle$ to be zero. The stationary correlation function is given by

$$\langle\xi_n(t)\xi_{n'}(t')\rangle=\delta_{nn'}\Delta_0^2(1-\varepsilon^2)\exp(-|t-t'|/\tau_c). \tag{2.24}$$

Again, the evolution equation can be transformed into a direct product of N independent two-state evolution equations, which now have the form

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}y_+(t)\\y_-(t)\end{bmatrix}&=-\Delta_0\begin{bmatrix}1+\varepsilon & 0\\0 & -1+\varepsilon\end{bmatrix}\begin{bmatrix}y_+(t)\\y_-(t)\end{bmatrix}\\&\quad-\frac{1}{2\tau_c}\begin{bmatrix}1+\varepsilon & -1+\varepsilon\\-1-\varepsilon & 1-\varepsilon\end{bmatrix}\begin{bmatrix}y_+(t)\\y_-(t)\end{bmatrix},\end{aligned} \tag{2.25}$$

with the initial condition

$$y_+(0)=\frac{1-\varepsilon}{2},\quad y_-(0)=\frac{1+\varepsilon}{2}. \tag{2.26}$$

All the final results for the asymmetric noise will be exactly the same as above after we make the substitutions

$$\tau_c\to\tilde{\tau}_c,\quad \Delta_0\to\tilde{\Delta}_0, \tag{2.27}$$

where

$$\tilde{\tau}_c=\frac{\tau_c}{1+2\varepsilon\Delta_0\tau_c},\quad \tilde{\Delta}_0^2=\Delta_0^2(1-\varepsilon^2). \tag{2.28}$$

We also have $\Delta^2=N\tilde{\Delta}_0^2$.

Another quantity of our interest is the correlation function

$$\begin{aligned}G(t,\tau)&=\langle g(t,\tau)\rangle=\langle y(t)t(t+\tau)\rangle=\left\langle \exp\left[-\int_0^t dt_1\xi^{(N)}(t_1)\right.\right.\\&\quad\left.\left.-\int_0^{t+\tau}dt_2\xi^{(N)}(t_2)\right]\right\rangle.\end{aligned} \tag{2.29}$$

By definition we have

$$\frac{\partial}{\partial\tau}g(t,\tau)=-\xi^{(N)}(t+\tau)g(t,\tau). \tag{2.30}$$

We can calculate the correlation function for each of the constituent two-state jump processes by considering two successive evolutions: the first over t described by Eq. (2.6) with the amplitude of $2\Delta_0$ instead of Δ_0 and the second over τ described again by Eq. (2.6) but with the initial conditions resulting from the first evolution, namely,

$$\hat{\hat{G}}(s,p)=\frac{(p+\tau_c^{-1})\hat{\phi}(s,2\Delta_0)-\Delta_0\hat{\psi}(s,2\Delta_0)}{p(p+\tau_c)-\Delta_0^2}, \tag{2.31}$$

where the functions $\hat{\phi}(s)$ and $\hat{\psi}(s)$ are defined by Eqs. (2.9) and (2.10), respectively. $\hat{\hat{G}}(s,p)$ denotes double Laplace transform over t and τ. Inverting the Laplace transforms, we finally obtain the expression for the correlation function of the composite process

$$G(t,\tau)=[\phi(\tau,\Delta_0)\phi(t,2\Delta_0)+\psi(\tau,\Delta_0)\psi(t,2\Delta_0)]^N, \tag{2.32}$$

where we introduced $\phi(\tau,\Delta_0)$ and $\psi(\tau,\Delta_0)$ to specify the argument Δ_0 in the expressions for $\phi(\tau)$ and $\psi(\tau)$. Generalization to asymmetric noise is straightforward, as described above for the average $\langle y(t)\rangle$.

In the limit of $N\to\infty$, $G(t,\tau)$ converges to the correlation function of the Gaussian noise

$$G(t,\tau)=\exp[2\chi(t)+2\chi(t+\tau)-\chi(\tau)], \tag{2.33}$$

where $\chi(t)$ is defined by Eq. (2.20). This result can be obtained directly by performing the cumulant expansion in Eq. (2.29) and using the Gaussian property of the noise.

Closing this section, we present the expansion for $G(t,\tau)$, which will be used later,

$$G(t,\tau)=\sum_{\{N\}} He^{-\kappa_1\tau}e^{-\kappa_2 t}, \tag{2.34}$$

where $\{N\}$ indicates that the summation is performed over all n_j $(j=1,\ldots,4)$ from 0 to N under the restriction of $\Sigma_{j=1}^4 n_j=N$,

$$H=N!\prod_{j=1}^{4}\frac{(h_j)^{n_j}}{n_j!}, \tag{2.35}$$

$$h_j=\frac{1}{4}[1+b_1(-1)^j][1+b_2(-1)^{j-[j/2]}] + \frac{2}{N}\alpha^2 b_1 b_2(-1)^{[j/2]}, \tag{2.36}$$

$$\kappa_i=\frac{N}{2\tau_c}+\frac{1}{2b_i\tau_c}\sum_{j=1}^{4} n_j(-1)^{j-[j/2](i-1)},$$

$$b_i=(1+4i^2\alpha^2/N)^{-1/2}, \tag{2.37}$$

where $i=1,2$ and $[x]$ denotes the largest integer $\leqslant x$. Note that H and κ_i are functions of all n_j. In the limit of $N\to\infty$, Eq. (2.34) reduces to

$$G(t,\tau)=\sum_{k,l,m=0}^{\infty} P_{klm}(\alpha^2)(-1)^m 2^{k+l}e^{-\kappa_{1ml}\tau}e^{-\kappa_{2kl}t}, \tag{2.38}$$

where

$$P_{klm}(\alpha^2)=P_k(\alpha^2)P_l(\alpha^2)P_m(\alpha^2), \tag{2.39}$$

$$\kappa_{ikl}=(k+l-i\alpha^2)/\tau_c. \tag{2.40}$$

III. STABILITY

Let us now consider the noisy relaxation described by Eq. (1.1) with $A=0$. It is well known that the stationary probability distribution for such a linear problem is not normalizable [11]. The moments, however, can be stable at certain parameter values. Different moments have different ranges of stability. The solution for the average value of $x(t)$ reads,

$$\langle x(t)\rangle=x_0 e^{-a_0 t}\langle y(t)\rangle, \tag{3.1}$$

where $x_0=x(0)$ and $\langle y(t)\rangle$ was calculated in the preceding section. With the help of the expansion in Eq. (2.12) we can immediately obtain the stability condition for the first moment

$$a_0>\mu N/b\tau_c, \tag{3.2}$$

which can also be rewritten as

$$\Delta^2<a_0(\tau_c^{-1}+a_0/N). \tag{3.3}$$

Clearly, the range of stability is wider for dichotomous noise than for Gaussian noise.

Noise asymmetry effectively changes the autocorrelation time. Therefore, the stability condition for asymmetric noise is given by

$$\Delta^2<a_0(\tilde{\tau}_c^{-1}+a_0/N), \tag{3.4}$$

where $\tilde{\tau}_c$ was defined by Eq. (2.28) in terms of Δ_0 and τ_c. Solving Eq. (3.4) for the noise strength $\Delta^2=N\Delta_0^2(1-\varepsilon^2)$, we obtain the inequality

$$\Delta/a_0<\frac{\varepsilon}{[N(1-\varepsilon^2)]^{1/2}}+\left[\frac{1}{N(1-\varepsilon^2)}+\frac{1}{a_0\tau_c}\right]^{1/2}. \tag{3.5}$$

The effect of the noise asymmetry vanishes in the limit of $N\to\infty$.

The stability conditions for the higher moments can be derived in a similar fashion. We have

$$\langle x^m(t)\rangle=x_0^m e^{-ma_0 t}[\phi(t,m\Delta_0)]^N \tag{3.6}$$

and consequently

$$\Delta^2<a_0\left(\frac{1}{m\tau_c}+\frac{a_0\tau_c}{N}\right). \tag{3.7}$$

For the asymmetric case we obtain

$$\Delta/a_0<\frac{\varepsilon}{[N(1-\varepsilon^2)]^{1/2}}+\left[\frac{1}{N(1-\varepsilon^2)}+\frac{1}{a_0 m\tilde{\tau}_c}\right]^{1/2}. \tag{3.8}$$

The stability range for the higher moments is considerably narrower than for the first moment, particularly for the Gaussian noise. This is a very important thing to notice. The moment stability conditions derived from Eq. (1.1) are the same for arbitrary $A\neq 0$. Thus the fact that different moments have different ranges of stability for a periodically driven linear system with multiplicative noise gives us the first indication that the definitions of the output signal in terms of the long-time amplitude of the average and in terms of the autocorrelation function (the power spectrum) are not equivalent.

IV. THE AVERAGE

Now that we have reviewed the properties of the multiplicative noise in a linear system and derived the relevant moment stability conditions, we can focus on the solution of Eq. (1.1). The average of the solution is given by

$$\langle x(t)\rangle = x_0 e^{-a_0 t}\phi^N(t) + A\int_0^t dt' \sin[\Omega(t-t')+\varphi_0]\phi^N(t'). \tag{4.1}$$

Using the expansion (2.12) of the kernel, we can rewrite Eq. (4.1) in the form

$$\langle x(t)\rangle = \sum_{n=0}^{N} B_n(N,\mu)\exp(-\beta_n t)\left[x_0 + \frac{A\Omega}{\Omega^2+\beta_n^2}\right] - C\cos(\Omega t+\varphi+\varphi_0), \tag{4.2}$$

where

$$\beta_n = a_0 + \kappa_n, \quad C = A(C_1^2+C_2^2)^{1/2}; \tag{4.3}$$

$$C_1 = \sum_{n=0}^{N} \frac{\beta_n B_n(N,\mu)}{\Omega^2+\beta_n^2}, \quad C_2 = \sum_{n=0}^{N} \frac{\Omega B_n(N,\mu)}{\Omega^2+\beta_n^2}; \tag{4.4}$$

$$\varphi = \arctan(C_1/C_2). \tag{4.5}$$

Our previously derived solution for the Gaussian noise [10] is obtained in the limit of $N\to\infty$. Relevant to further consideration is the long-time behavior

$$\langle x(t)\rangle_{\rm st} = -C\cos(\Omega t+\varphi+\varphi_0) \tag{4.6}$$

under the stability condition of Eq. (3.3).

The output SNR can be defined as

$$R = \frac{a_0 C}{\alpha^2 A}. \tag{4.7}$$

Here we have modified the previous definition of R by Fuliński [8], i.e., $R = C/A\Delta^2$, in order to make it a dimensionless parameter. For colored noise, a pronounced maximum of the SNR well separated from the point of instability can be observed by changing the noise strength, as shown in Fig. 1. This behavior is very similar to what is commonly ascribed to SR. In fact, it was defined as SR by several authors [8–10]. A qualitative explanation of this phenomenon is given in the Introduction. The underlying mechanism is the same for all N. Interestingly, for certain values of the input parameters the maximum can disappear when the number of constituent two-state jump processes is increased, as illustrated in Fig. 1(b). A nonmonotonic behavior is also found for the dependence of the SNR on the autocorrelation time, as shown in Fig. 2.

In Fig. 3 we analyze the range of Ω and τ_c for which the SR-like behavior occurs. The boundary condition for the existence of the maximum is defined by

$$\partial R/\partial(\Delta^2) = \partial^2 R/\partial(\Delta^2)^2 = 0. \tag{4.8}$$

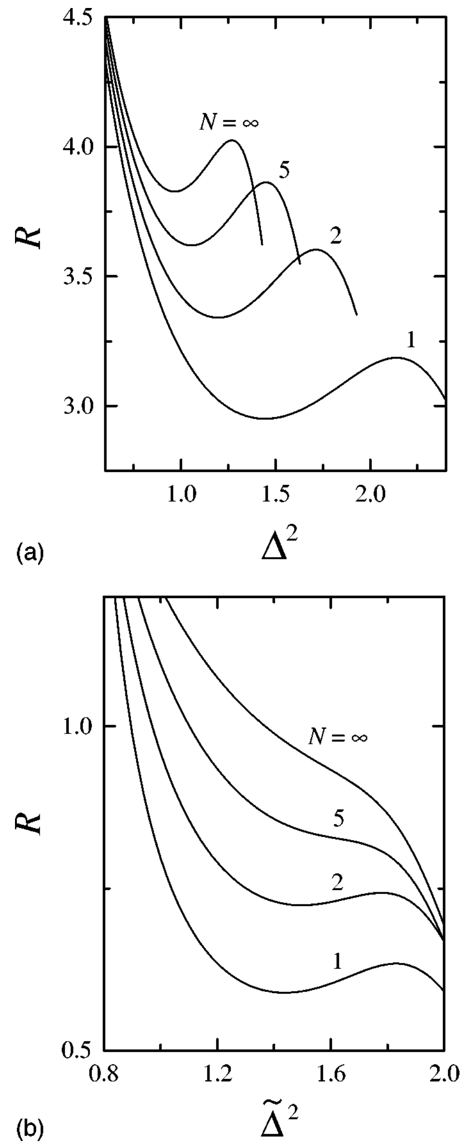

FIG. 1. Signal-to-noise ratio $R = a_0 C/\alpha^2 A$ as a function of (a) Δ^2 and (b) $\tilde{\Delta}^2$ for (a) $a_0=1$, $\tau_c=0.7$, and $\Omega=0.2$ and (b) $a_0=1$, $\tau_c=2$, and $\Omega=0.17$, for different values of $N=1,2,5$, and ∞. $\tilde{\Delta}^2 = \Delta^2/(1+a_0\tau_c/N)$ maps the stability region onto $[0,a_0/\tau_c]$ for all N.

The maximum appears for not too high frequencies Ω of the external periodic forcing immediately as soon as noise correlation is introduced. No maximum is observed for the white noise. As τ_c increases, the position of the maximum shifts farther from the instability point; its amplitude first increases and then decreases before the maximum finally disappears. In the case of finite N, there is a range of Ω where the maximum exists for all $\tau_c > 0$.

Let us consider the case of $N=1$ in more detail. We have

$$(C/A)^2 = \frac{\Omega^2+(a_0+1/\tau_c)^2}{(\Omega^2+\beta_0^2)(\Omega^2+\beta_1^2)}, \tag{4.9}$$

with

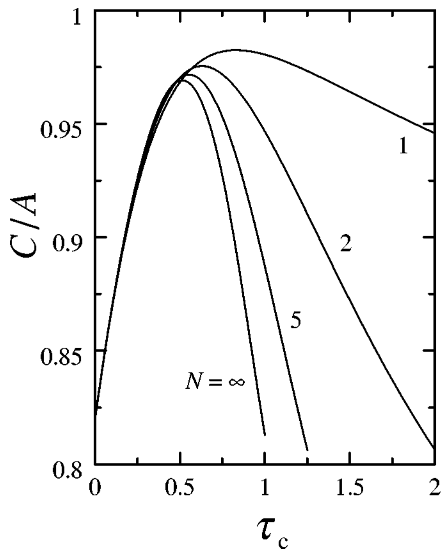

FIG. 2. Output signal amplitude C/A as a function of the autocorrelation time τ_c for $a_0=1$, $\Delta=1$, $\Omega=0.7$, and different values of $N=1,2,5$, and ∞.

$$\beta_{0,1}=a_0+\frac{1}{2\tau_c}\mp\frac{1}{2b\tau_c}. \tag{4.10}$$

This result has been obtained by Berdichevsky and Gitterman (BG) [9]. They considered the behavior of the normalized signal (i.e., C/A, *not* the SNR) as a function of the noise strength. The condition for the maximum is then $\Omega^2=\beta_0\beta_1$ or, as given by BG,

$$\Delta^2=a_0^2+a_0/\tau_c-\Omega^2. \tag{4.11}$$

This proves that upon increasing the external force frequency the maximum is shifted away from the instability point (i.e., $\Delta^2=a_0^2+a_0/\tau_c$) towards zero. The condition for the existence of the maximum is thus

$$\Omega<(a_0^2+a_0/\tau_c)^{1/2}. \tag{4.12}$$

The position of the maximum in the dependence of the SNR on the noise strength is defined by $\partial R/\partial\Delta^2=0$ or

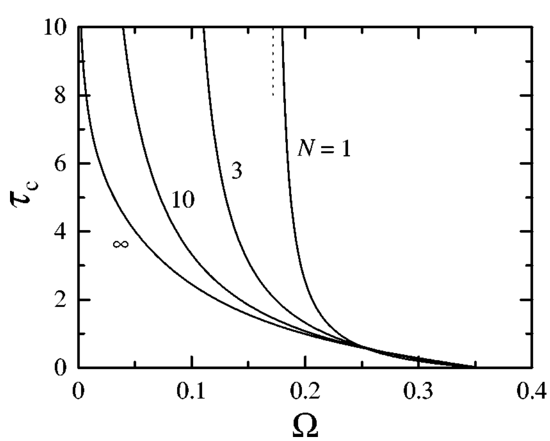

FIG. 3. Existence boundaries, above which the maximum in the R vs Δ^2 dependence disappears, for $a_0=1$ and different values of $N=1,3,10$, and ∞. The dotted line shows the boundary value of Ω in the limit of $\tau_c\to\infty$ for $N=1$.

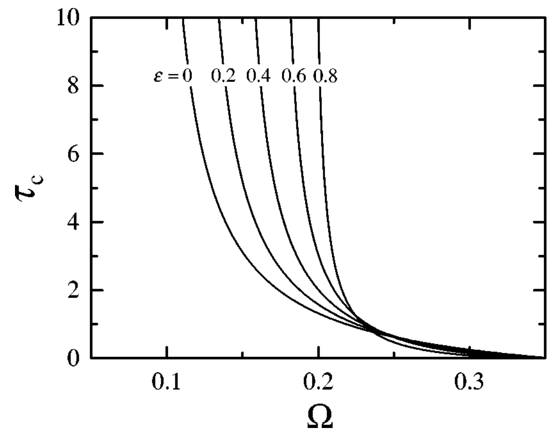

FIG. 4. Existence boundaries, above which the maximum in the R vs Δ^2 dependence disappears, for $a_0=1$, $N=3$, and asymmetric noise with $\varepsilon=0,0.2,0.4,0.6$, and 0.8.

$$\Delta^2(\beta_0\beta_1-\Omega^2)=(\Omega^2+\beta_0^2)(\Omega^2+\beta_1^2). \tag{4.13}$$

The existence boundary is defined by the complemetary condition on the second derivative, i.e., Eq. (4.8). We obtain for the position

$$\Delta^2=\frac{3}{4}(a_0^2+a_0/\tau_c-\Omega^2). \tag{4.14}$$

The relationship between Ω and τ_c is given by

$$(\Omega^4-34a_0^2\Omega^2+a_0^4)+\frac{2a_0}{\tau_c}(a_0^2-17\Omega^2)+\frac{1}{\tau_c^2}(a_0^2-8\Omega^2)=0. \tag{4.15}$$

Thus we obtain in the two important limits of small and large correlation times

$$\Omega<a_0/2\sqrt{2}\quad\text{for}\quad\tau_c\to0, \tag{4.16}$$

$$\Omega<a_0\sqrt{17-12\sqrt{2}}\approx a_0/\sqrt{34}\quad\text{for}\quad\tau_c\to\infty. \tag{4.17}$$

It is difficult to derive a general expression for the maximum existence boundary for arbitrary N. Here we shall discuss only the limit of $\tau_c\to0$. In this case, $\mu\approx\alpha^2/N\ll1$ and thus we can take only two first terms in the series (4.4). The corresponding binomial coefficients are $B_0(N,\mu)\approx1-\mu N$ and $B_1(N,\mu)\approx\mu N$. We have approximately

$$(C/A)^2\approx[\Omega^2+a_0^2-2a_0\Delta^2\tau_c+\Delta^4\tau_c^2]^{-1}. \tag{4.18}$$

Note that $\Delta^2\sim1/\tau_c$. Now we see that the maximum disappears at $\Delta^2\tau_c=\frac{3}{4}a_0$ and the range of existence is defined by

$$\Omega<a_0/2\sqrt{2}\quad\text{for}\quad\tau_c\to0 \tag{4.19}$$

for all N. The limit of $\tau_c\to\infty$ is more difficult to analyze analytically. Here we emphasize only that there is a finite range of Ω where the maximum exists even for $\tau_c\to\infty$ provided N is finite.

The existence boundary of the maximum is sensitive to the noise asymmetry, as shown in Fig. 4. The effect is more pronounced for small N since the autocorrelation time can be considerably altered in this case. On the other hand, as mentioned above, $\tilde{\tau}_c \to \tau_c$ in the limit of $N \to \infty$. It is possible to find such conditions for finite N where the noise asymmetry leads to the appearance of a maximum that does not appear in the corresponding symmetric system.

In this section we have shown that the output SNR, defined in terms of the amplitude of $\langle x(t) \rangle$ at long times, exhibits a maximum both as a function of the noise strength and as a function of the autocorrelation time. This SR-like behavior is observed for a general type of the noise that bridges the dichotomous and the Gaussian noise. However, it is essential in the above derivations that the initial phase φ_0 of the external forcing is fixed. In many physical situations, φ_0 is unknown. It should thus be considered as a random variable and the resulting expression for $\langle x(t) \rangle$ has to be averaged over the distribution of φ_0 [6]. It is natural to assume a uniform distribution of the initial phase. Averaging over the uniform distribution of φ_0 leads to the periodic term in the expression for $\langle x(t) \rangle$ as well as the output signal vanishing identically. Only if a nonuniform distribution of φ_0 is somehow prepared in a system can the SR-like behavior be observed.

An important example of a system where the phase of the external modulation is correlated with the internal stochastic dynamics is found in biology. Experimental data on active transport of ionic species in biomembranes under the influence of ac electric fields [18] has recently been interpreted as evidence of SR between the external field and the fluctuations of the membrane potential [19]. Ion channel currents under stimulation exhibit a strongly irregular character mostly of dichotomous type with the intensity of the noise depending on the amplitude of the applied field. If passive membrane permeability can be neglected, the ion channel fluctuations and, consequently, the ion traffic start as soon as the external field is switched on. Therefore, φ_0 can be taken as fixed (zero, without losing generality). However, the observed nonmonotonic behavior of the SNR as a function of the noise strength cannot be regarded as SR in conventional sense because the noise strength itself is totally governed by the external field in this case.

V. THE CORRELATION FUNCTION

The system response is often defined in terms of the power spectrum rather than the amplitude of the average [2,6]. For a stationary stochastic process, the Wiener-Khintchine theorem holds and the spectral density is obtained as the Fourier transform of the autocorrelation function, which depends on the time difference only and thus, in fact, can be represented by the average. However, stochastic processes with periodic modulation are essentially nonstationary with the correlation function depending explicitly on two time arguments. Therefore, the long-time amplitude of the average generally does not define the power spectrum. The generalized Wiener-Khintchine theorem can be formulated in terms of the phase-averaged autocorrelation function [6]. For a uniform distribution of the initial phase, averaging over the phase is equivalent to averaging over time. Thus the phase-averaged autocorrelation function depends only on the time difference. Whether the corresponding power spectrum shows a nonmonotonic behavior as a function of the noise strength is what we have to find out.

All the relevant information is contained in the long-time correlation function [2]

$$K(t,\tau)=\lim_{t\to\infty}\langle x(t)x(t+\tau)\rangle. \tag{5.1}$$

The stability condition for the second moment applies. It is convenient to set $\varphi_0=0$ for the time being. We can always restore the initial phase in the end via a transformation $\Omega t \to \Omega t+\varphi_0$, as we saw in Sec. IV. We have

$$K(t,\tau)=A^2\int_0^t dt_1\int_0^{t+\tau}dt_2\,\sin[\Omega(t-t_1)]\sin[\Omega(t+\tau-t_2)]\times\exp[-a_0(t_1+t_2)]G(t_<,|t_1-t_2|), \tag{5.2}$$

where $t_<=\min\{t_1,t_2\}$ and $G(t_<,|t_1-t_2|)$ is defined by Eq. (2.32). It can be readily shown that other terms in $\langle x(t)x(t+\tau)\rangle$ vanish at long times.

Using the expansion (2.34), Eq. (5.2) can be rewritten as

$$K(t,\tau)=A^2\sum_{\{N\}}H[f_1(t,\tau)+f_2(t,\tau)], \tag{5.3}$$

where the functions $f_i(t,\tau)$ are defined to ensure time ordering,

$$f_1(t,\tau)=\int_0^t dt_1\,\sin[\Omega(t-t_1)]\exp[-\beta_1 t_1]\times\int_0^{t_1}dt_2\,\sin[\Omega(t+\tau-t_2)]\exp[-(\beta_2-\beta_1)t_2], \tag{5.4}$$

$$f_2(t,\tau)=\int_0^t dt_1\,\sin[\Omega(t-t_1)]\exp[-(\beta_2-\beta_1)t_1]\times\int_{t_1}^{t+\tau}dt_2\,\sin[\Omega(t+\tau-t_2)]\exp[-\beta_1 t_2], \tag{5.5}$$

where

$$\beta_i=ia_0+\kappa_i,\quad i=1,2. \tag{5.6}$$

Note that since $n_j\leqslant N$, all β_i are positive when the stability condition is fulfilled.

The integrations are most conveniently performed using Laplace transformation over t. We obtain for $\hat{f}_1(s,\tau)$

$$\hat{f}_1(s,\tau)=\Omega\,\mathrm{Im}\,e^{\imath\Omega\tau}[s(s-2\imath\Omega)(s+\beta_1-\imath\Omega)(s+\beta_2)]^{-1}. \tag{5.7}$$

Only the residues at $s=0$ and $s=2\imath\Omega$ contribute to the long-time solution

$$f_1(t,\tau)=\frac{\beta_1\cos(\Omega\tau)-\Omega\sin(\Omega\tau)}{2\beta_2(\beta_1^2+\Omega^2)}-\frac{\beta_1\beta_2-2\Omega^2}{2(\beta_1^2+\Omega^2)(\beta_2^2+4\Omega^2)}\cos[\Omega(2t+\tau)]-\frac{\Omega(2\beta_1+\beta_2)}{2(\beta_1^2+\Omega^2)(\beta_2^2+4\Omega^2)}\sin[\Omega(2t+\tau)]. \tag{5.8}$$

Similarly, for $\hat{f}_2(s,\tau)$ we obtain

$$\hat{f}_2(s,\tau)=\Omega\,\mathrm{Im}e^{\imath\Omega\tau}[s(s-2\imath\Omega)(\beta_1+\imath\Omega)(s+\beta_2)]^{-1}-\Omega e^{-\beta_1\tau}\mathrm{Im}[(\beta_1+\imath\Omega)\{(s+\beta_1)^2+\Omega^2\}\times(s+\beta_2)]^{-1}. \tag{5.9}$$

The second term vanishes at long times. In the first term, only the residues at $s=0$ and $s=2\imath\Omega$ contribute to the long-time solution. The final expression for $f_2(t,\tau)$ is almost the same as that for $f_1(t,\tau)$, except that the minus sign is changed to plus in the numerator of the first t-independent term. We obtain finally

$$K(t,\tau)=C_0\cos(\Omega\tau)-C_1\cos(2\Omega t+\Omega\tau+\varphi_1), \tag{5.10}$$

where

$$C_0=A^2\sum_{\{N\}}\frac{H\beta_1}{\beta_2(\beta_1^2+\Omega^2)}, \tag{5.11}$$

$$C_1=(C_{11}^2+C_{12}^2)^{1/2}, \tag{5.12}$$

$$C_{11}=A^2\sum_{\{N\}}\frac{H\Omega(2\beta_1+\beta_2)}{(\beta_1^2+\Omega^2)(\beta_2^2+4\Omega^2)}, \tag{5.13}$$

$$C_{12}=A^2\sum_{\{N\}}\frac{H(\beta_1\beta_2-2\Omega^2)}{(\beta_1^2+\Omega^2)(\beta_2^2+4\Omega^2)}, \tag{5.14}$$

$$\varphi_1=-\arctan(C_{11}/C_{12}). \tag{5.15}$$

The expression for the long-time correlation function in the limit of $N\to\infty$ can be derived in a similar way using the expansion (2.38). The general form of Eq. (5.10) still applies, but the coefficients should be modified. For instance, we obtain for C_0

$$C_0=A^2\sum_{k,l,m=0}^{\infty}P_{klm}(\alpha^2)\frac{\beta_{1ml}(-1)^m2^{k+l}}{\beta_{2kl}(\beta_{1ml}^2+\Omega^2)}, \tag{5.16}$$

where $\beta_{ikl}=ia_0+\kappa_{ikl}$.

The initial phase of the periodic external forcing contributes additively to the phase of the second term in Eq. (5.10), i.e., $2\Omega t\to2\Omega t+2\varphi_0$. Thus, averaging over the uniform distribution of φ_0 leaves only the first term in the expression for the long-time correlation function. This term corresponds to a kink in the power spectrum at the forcing frequency. In contrast to the case of additive noise [2], there is no background band for multiplicative noise. The amplitude of the output signal C_0 simply diverges upon approaching the instability point with increasing noise strength because of β_2 in denominator; so does the SNR, defined as $R=C_0(a_0/\alpha A)^2$, as shown in Fig. 5. No SR is observed for any parameter settings. However, when the autocorrelation time of the noise is varied instead of the intensity, the output signal goes through a maximum, as illustrated in Fig. 6. The resonancelike behavior induced by correlated noise has recently been predicted for a number of systems, even in the absence of the external periodic force [20]. Under the condition of

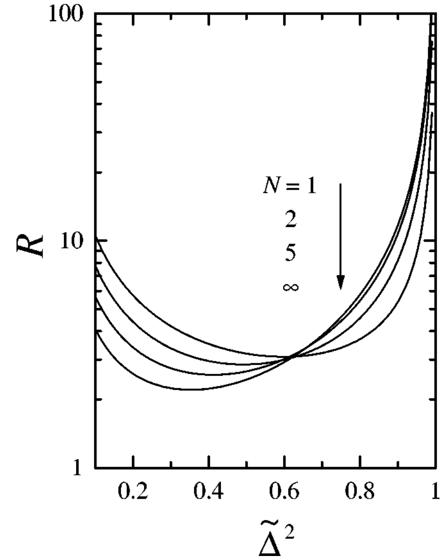

FIG. 5. Output signal-to-noise ratio, defined in terms of the amplitude of the long-time phase-averaged correlation function $R=C_0(a_0/\alpha A)^2$, as a function of the input noise strength $\tilde{\Delta}^2$ for $a_0=1$, $\tau_c=1$, $\Omega=0.15$, and different values of $N=1,2,5$, and ∞. $\tilde{\Delta}^2=\Delta^2/(1/2+a_0\tau_c/N)$ maps the stability region onto $[0,a_0/\tau_c]$ for all N.

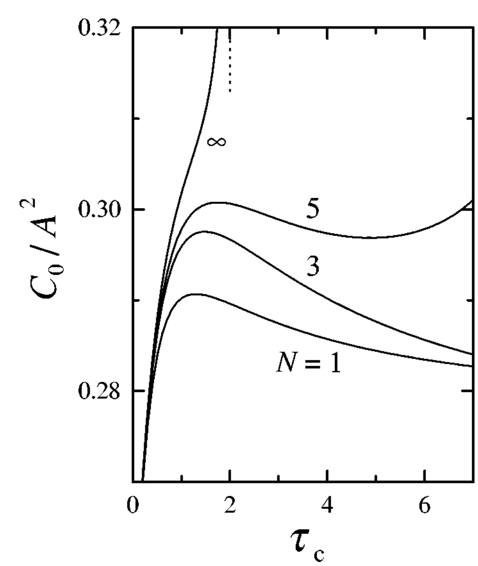

FIG. 6. Output signal amplitude C_0/A^2 as a function of the autocorrelation time τ_c for $a_0=1$, $\Delta=0.5$, $\Omega=1$, and different values of $N=1,3,5$, and ∞. The dotted line shows the instability point $\tau_c^*=a_0/2\Delta^2$ for the Gaussian noise.

$$\Delta^2 \leqslant a_0^2/N, \tag{5.17}$$

the system is stable for all τ_c and the maximum is most clearly observed. Otherwise, there is a marginal point at $\tau_c^* = \frac{1}{2} a_0(\Delta^2 - a_0^2/N)^{-1}$. The signal diverges upon approaching this point. As a result, the maximum can disappear.

However difficult it might seem to practically realize the variation of τ_c, it is not impossible. Zwanzig's model for passage through a fluctuating bottleneck is an example [21]. In describing ligand binding by a simple first-order rate equation, Zwanzig assumed that the rate constant depends on the radius of the bottleneck, which in turn fluctuates because of thermal noise and its time dependence is given by a Langevin equation. The characteristic relaxation time of the radius of the bottleneck is expected to be proportional to the solvent viscosity and can thus be controlled. The relaxation time in Zwanzig's model corresponds to the noise autocorrelation time τ_c in this paper.

VI. CONCLUDING REMARKS

It has been argued recently that stochastic resonance can occur in a linear system driven by periodic external forcing provided the input noise is multiplicative and correlated. Starting with the Langevin equation, it was found that the amplitude of the average at long times, defined as the output signal, shows a pronounced maximum both as a function of the noise strength and as a function of the correlation time for not too high frequencies of the external forcing. In this paper we have proved that this resonant behavior is quite general and occurs for any type of the noise ranging from the two-state jump process to the Gaussian process. Previously, only these two limiting cases had been considered. Here we have described a generalized composite stochastic process, defined as a suporposition of N independent two-state jump processes, which bridges the two limits.

Although the results obtained seem to be very exciting, we have to admit that the observed resonance is realizable experimentally only if a nonuniform distribution of the initial phase φ_0 of the external periodic modulation is somehow prepared in a system. In many physical situations, however, the external periodic forcing has a random, uniformly distributed initial phase. Averaging over the uniform distribution of φ_0 leads to the output signal, defined as the amplitude of $\langle x(t) \rangle$ at long times, vanishing identically. Moreover, the system response is often defined in the literature in terms of the experimentally measurable power spectrum rather than the amplitude of the average. The output signal is then given by the spectral density corresponding to the frequency of the external forcing. We have calculated the long-time correlation function. According to the generalized Wiener-Khintchine theorem, the Fourier transform of the phase-averaged correlation function gives the power spectrum. It appears that the output signal simply diverges with increasing noise strength upon approaching the instability point, which is in fact different from the instability point corresponding to the first moment. No SR, in a conventional sense, is observed in a linear system for any parameter settings. However, a very interesting phenomenon is still retained, namely, the resonancelike behavior as a function of the noise autocorrelation time.

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