Coefficient of normal restitution of viscous particles and cooling rate of granular gases

Thomas Schwager* and Thorsten Pöschel†
Institut für Physik, Humboldt-Universität zu Berlin, Invalidenstraße 110, D-10115 Berlin, Germany

(Received 19 May 1997)

We investigate the cooling rate of a gas of inelastically interacting particles. When we assume velocity-dependent coefficients of restitution the material cools down slower than with constant restitution. This behavior might have a large influence to clustering and structure formation processes. [S1063-651X(97)00112-8]

PACS number(s): 83.70.Fn, 62.40.+i, 81.40.Lm, 05.40.+j

The behavior of granular gases has been of large scientific interest in recent time. Goldhirsch and Zanetti [1] and McNamara and Young [2] have shown that a homogeneous granular gas is unstable. After some time one observes dense regions (clusters) and voids. To evaluate the loss of mechanical energy due to collisions one introduces the coefficient of (normal) restitution

$$g' = \epsilon g, \tag{1}$$

where $g = |\vec{g}|$ and $g' = |\vec{g}'|$, describing the loss of relative normal velocity g' of a pair of colliding particles after the collision with respect to the impact velocity g.

It can be shown that even for three particles for a certain region of the coefficient of restitution there exist initial conditions that lead to a behavior called ''inelastic collapse.'' This means that the particles accomplish an infinite number of collisions in finite time [2]. The conditions under which one can observe inelastic collapse have been studied in one-dimensional systems [3] as well as in higher dimensions [4]. Recently it was shown numerically that the probability for a collapse rises significantly when the particles have rotational degree of freedom [5]. In this case the collapse is possible for much larger coefficients of restitution than for nonrotating particles. Other interesting related results concern bouncing ball experiments on vibrating tables where complicated dynamical behavior is observed (e.g., [6]). Recently, complicated and under certain circumstances irregular motion of a bouncing cantilever of an atomic force microscope when excited by a transducer was investigated [7].

In the investigations [1–7] the approximation of the constant coefficient of restitution was assumed. Solving viscoelastic equations for spheres, it was shown that the coefficient of normal restitution ϵ is not a constant but a function of the impact velocity $\epsilon(g)$ itself [8,9]. For the ''compression'' $\xi = R_1 + R_2 - |\vec{r}_1 - \vec{r}_2|$ of particles with radii R_1 and R_2 at positions $\vec{r}_1$ and $\vec{r}_2$ one finds

$$\ddot{\xi} + \rho \left(\xi^{3/2} + \frac{3}{2} A \sqrt{\xi} \dot{\xi} \right) = 0, \tag{2}$$

$$\rho = \frac{2Y\sqrt{R^{\rm eff}}}{3m^{\rm eff}(1-\nu^2)}. \tag{3}$$

Y is the Young modulus, ν the Poisson ratio, and

$$m_{\rm eff} = \frac{m_1 m_2}{m_1 + m_2}, \tag{4a}$$

$$R_{\rm eff} = \frac{R_1 R_2}{R_1 + R_2} \tag{4b}$$

are the effective radius and mass of the grains, respectively. A is a material constant depending on the Young modulus, the viscous constants and the Poisson ratio of the material. Equation (2) was derived under the precondition that the colliding spheres have impact velocity much less than the speed of sound in the particle material. For details see [8]. The initial conditions for solving Eq. (2) are

$$\xi(0) = 0, \tag{5a}$$

$$\dot{\xi}(0) = g. \tag{5b}$$

The coefficient of restitution ϵ of at time $t=0$ colliding spherical grains can be found from this equation relating the relative normal velocities $g = \dot{\xi}(0)$ at time of impact and at time t_c, when the particles separate after the collision, i.e., t_c is the collision time:

$$\epsilon = -\dot{\xi}(t_c)/\dot{\xi}(0). \tag{6}$$

The (numerical) integration of Eq. (6) yields the coefficient of restitution as a function of the impact velocity (see Fig. 1 in [8]), which is in good agreement with experimental data [10]. A constant coefficient of restitution, however, does *not* agree with experimental experience [11]. Other theoretical work on this topic can be found, e.g., in [12,13].

Consider a gas of granular particles at a given initial granular temperature T_0. Then the question arises how the temperature decreases with time due to inelastic collisions. This problem has been investigated earlier [14,15] for the case of constant coefficient of restitution and the result is (see also [16])

$$T(t) = T_0(1 + t/\tau)^{-2}. \tag{7}$$

The time scale τ is a material constant. The temperature decay (7) is the origin of the cluster instabilities that have been investigated recently [1,2].

The aim of the present paper is to derive an explicit analytic expression for the coefficient of normal restitution $\epsilon(g)$

*Electronic address: schwager@itp02.physik.hu-berlin.de
†Electronic address: thorsten@itp02.physik.hu-berlin.de

as a function of the impact velocity g. A direct consequence of this result will be a refined expression for the temperature decay of a granular gas.

The duration of collision t_c^0 for the undamped problem $(A=0)$ is given by [17]

$$t_c^0=\frac{\Theta_c^0}{\rho^{2/5}g^{1/5}}. \tag{8}$$

We want to point out here that Θ_c^0 is a constant pure number, *not* depending on any material properties. Hence, t_c^0 depends only on the material constant ρ and on the initial velocity g. We use Eq. (8) to define a rescaled dimensionless time Θ:

$$\Theta=\rho^{2/5}g^{1/5}t. \tag{9}$$

Using the abbreviations

$$v=\rho^2 g, \tag{10a}$$

$$\alpha=\tfrac{3}{2}A \tag{10b}$$

and a new set of variables

$$\Theta=\rho^{2/5}g^{1/5}t=v^{1/5}t, \tag{11a}$$

$$x(\Theta)=\rho^2\xi(t), \tag{11b}$$

we rewrite Eq. (2) in the form

$$\ddot{x}+\alpha v^{-1/5}\dot{x}\sqrt{x}+v^{-2/5}x^{3/2}=0 \tag{12}$$

with $\dot{x}=(d/d\Theta)x$. We see that

$$\frac{dx}{dt}(0)=\frac{1}{\rho^2}\frac{d\xi}{dt}(0)=\frac{g}{\rho^2}=v=v^{1/5}\frac{dx}{d\Theta}(0). \tag{13}$$

Hence the initial conditions in our new variables x and Θ read

$$x(0)=0, \tag{14a}$$

$$\frac{dx}{d\Theta}(0)=\dot{x}(0)=v^{4/5}. \tag{14b}$$

Both equations of motion, (2) and (12), become special at $x=0$ or $\xi=0$, respectively, i.e., all derivatives of third order and higher diverge. This will be shown for the case of x:

$$\frac{d}{d\Theta}\ddot{x}=-\frac{d}{d\Theta}(\alpha v^{-1/5}\dot{x}\sqrt{x}+v^{-2/5}x^{3/2})$$
$$=\alpha v^{-1/5}\left(\ddot{x}\sqrt{x}+\frac{\dot{x}}{2\sqrt{x}}\right)-\frac{3}{2}v^{-2/5}\dot{x}\sqrt{x}. \tag{15}$$

Hence

$$\lim_{x\to 0}\frac{d^3}{d\Theta^3}x=\pm\infty, \tag{16}$$

and so are the higher derivatives. Because of this singularity we must not expand x in powers of Θ. Because of the initial conditions $x(\Theta)$ has the form

$$x(\Theta)=v^{4/5}\Theta(1+\eta(\Theta)), \tag{17}$$

$$\eta(0)=0, \tag{18}$$

which defines the function $\eta(\Theta)$. Using transformation (17) we find

$$\Theta\ddot{\eta}+2\dot{\eta}+\alpha v^{1/5}\Theta^{3/2}\dot{\eta}\sqrt{1+\eta}+(\alpha v^{1/5}\sqrt{\Theta}+\Theta^{3/2})(1+\eta)^{3/2}$$
$$=0. \tag{19}$$

In Eq. (19), terms $\Theta^{0.5}$ and $\Theta^{1.5}$ occur, therefore we expand η in powers of $\sqrt{\Theta}$:

$$\eta=\sum_{k=0}^{\infty}a_k\Theta^{k/2}. \tag{20}$$

The first coefficient a_0 vanishes because of the initial condition for x. When we require

$$\dot{\eta}=\frac{a_1}{2\sqrt{\Theta}}+a_2+\cdots \tag{21}$$

to be finite at $\Theta=0$ the second coefficient a_1 must vanish as well. With Taylor expansion of $\sqrt{1+\eta}$ and $(1+\eta)^{3/2}$ for small η we arrive at

$$\eta=-\frac{4}{15}\alpha v^{1/5}\Theta^{3/2}-\frac{4}{35}\Theta^{5/2}+\frac{3}{70}\alpha v^{1/5}\Theta^4+\frac{1}{15}\alpha^2 v^{2/5}\Theta^3$$
$$+\cdots \tag{22}$$

and therefore

$$x=v^{4/5}\Theta-\frac{4}{15}\alpha v\Theta^{5/2}-\frac{4}{35}v^{4/5}\Theta^{7/2}+\frac{1}{15}\alpha^2 v^{6/5}\Theta^4$$
$$+\frac{3}{70}\alpha v\Theta^5-\frac{38}{2475}\alpha^3 v^{7/5}\Theta^{11/2}+\frac{1}{175}v^{4/5}\Theta^6+\cdots . \tag{23}$$

Rearranging the full series (23) one finds

$$x=v^{4/5}\left(\Theta-\frac{4}{35}\Theta^{7/2}+\frac{1}{175}\Theta^6+\cdots\right)+\alpha v\left(-\frac{4}{15}\Theta^{5/2}\right.$$
$$\left.+\frac{3}{70}\Theta^5+\cdots\right)+\alpha^2 v^{6/5}\left(\frac{1}{15}\Theta^4+\cdots\right)+\cdots$$
$$=v^{4/5}x_0(\Theta)+\alpha v x_1(\Theta)+\alpha^2 v^{6/5}x_2(\Theta)+\cdots . \tag{24}$$

$v^{4/5}x_0$ is the solution of the undamped (elastic) collision (see dashed line in Fig. 1). The full line in Fig. 1 shows the damped motion according to Eq. (23). The direct numerical integration of Eq. (12) collapses with the full line.

For $x(\frac{1}{2}\Theta_c^0)$, where Θ_c^0 is the duration of the undamped collision, one finds using Eq. (24):

$$x\left(\frac{\Theta_c^0}{2}\right)=v^{4/5}x_0\left(\frac{\Theta_c^0}{2}\right)+\alpha v x_1\left(\frac{\Theta_c^0}{2}\right)+\alpha^2 v^{6/5}x_2\left(\frac{\Theta_c^0}{2}\right)+\cdots$$
$$=v^{4/5}B_0+\alpha v B_1+\alpha^2 v^{6/5}B_2+\cdots , \tag{25}$$

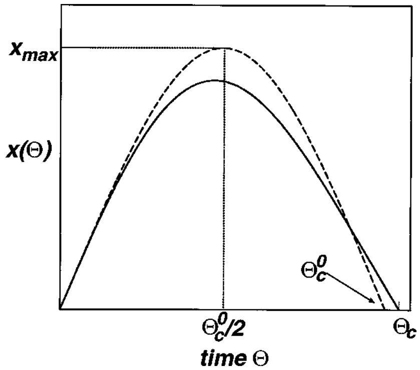

FIG. 1. The dynamics of the collision. The dashed line shows the (strictly symmetric) solution of the undamped collision. For the case of the damped motion (full line) the maximum penetration depth is achieved earlier whereas the duration of the collision is longer ($\Theta_c > \Theta_c^0$).

which we do not need now but will later on.

Note that the coefficients B_k are constants; i.e., they do not depend on v nor on material constants.

Equations (2) and (12), respectively, hold for the entire collision. The collision starts with v and ends with v'. For practical purposes we now define the term *inverse collision*. The inverse collision is a collision that starts at time Θ_c with relative velocity v' and ends at time 0 with relative velocity v, i.e., time runs in an inverse direction during the inverse collision. The equation of motion for x^{inv}, i.e., for a collision in inverse time, follows from Eq. (12). Since the inverse collision starts with v' we have to replace v by v'. Because of the time reversal we have to change the sign of time derivatives of odd orders, i.e., $\dot{x} \to -\dot{x}^{\text{inv}}$. The equation of motion for the inverse collision reads

$$\ddot{x}^{\text{inv}} - \alpha(v')^{-1/5}\dot{x}^{\text{inv}}\sqrt{x^{\text{inv}}} + (v')^{-2/5}(x^{\text{inv}})^{3/2} = 0. \tag{26}$$

A motion due to Eq. (26) in normal time would be an accelerated one. However, we shall mention here that Eqs. (12) and (26) describe strictly the same physical motion. The solution x^{inv} of the inverse problem can be derived from the solution of the direct problem replacing $\alpha \to -\alpha$ and $v \to v'$.

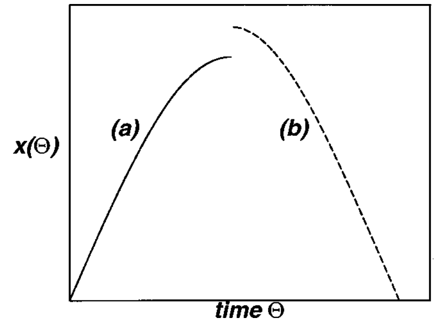

FIG. 2. Sketch of the calculation. In (a) $\Theta \in (0, \Theta_m)$ is calculated directly. In (b) we define the *inverse collision* where the particles start with velocity v' and velocity approaches zero at $\Theta = \Theta_m$. Both curves have to fit together smoothly.

$$x^{\text{inv}}(\Theta') = (v')^{4/5}x_0(\Theta') - \alpha v' x_1(\Theta') + \alpha^2 (v')^{6/5} x_2(\Theta') + \cdots . \tag{27}$$

Now we determine the collision time Θ_c and the final velocity. One direct method to calculate Θ_c would be to determine the solution of $x(\Theta)=0$ using Taylor expansion of x in the region close to Θ_c^0. It can be seen easily that this method fails since all derivatives of $(d^n/d\Theta^n)x$ with $n \geqslant 3$ diverge for $\Theta = \Theta_c^0$. Therefore Θ_c has to be calculated by an indirect method.

The problem will be subdivided into two parts (see Fig. 2): (a) the motion of the particles x from $\Theta = 0$ to time Θ_m when x approaches its maximum and where $\dot{x}$ changes its sign, and (b) from Θ_m to Θ_c.

In the case of undamped motion where $\alpha = 0$ we have $\Theta_m = \Theta_c^0/2$. In part (b) we do not consider the collision itself but the inverse problem in the interval $(\Theta = 0,\ \Theta'_m)$, with Θ'_m being the time where x^{inv} approaches its maximum. The continuity of both parts means $x(\Theta_m) = x^{\text{inv}}(\Theta'_m)$.

For finite damping $\alpha \neq 0$ we write $\Theta_m = \Theta_c^0/2 + \delta$ and $\Theta'_m = (\Theta_c^0)'/2 + \delta'$ and recall that $\Theta_c^0 = (\Theta_c^0)'$. To get an expression for δ we expand

$$\dot{x}\left(\frac{\Theta_c^0}{2} + \delta\right) = 0 = \dot{x}\left(\frac{\Theta_c^0}{2}\right) + \delta\ddot{x}\left(\frac{\Theta_c^0}{2}\right) + \frac{\delta^2}{2}\frac{d^3}{d\Theta^3}x\left(\frac{\Theta_c^0}{2}\right) + \cdots \tag{28}$$

$$\begin{aligned} &= v^{4/5}\left[\dot{x}_0\left(\frac{\Theta_c^0}{2}\right) + \delta\ddot{x}_0\left(\frac{\Theta_c^0}{2}\right) + \frac{\delta^2}{2}\frac{d^3}{d\Theta^3}x_0\left(\frac{\Theta_c^0}{2}\right) + \cdots\right] + v\,\alpha\left[\dot{x}_1\left(\frac{\Theta_c^0}{2}\right)\right.\\ &\quad \left. + \delta\ddot{x}_1\left(\frac{\Theta_c^0}{2}\right) + \frac{\delta^2}{2}\frac{d^3}{d\Theta^3}x_1\left(\frac{\Theta_c^0}{2}\right) + \cdots\right] + v^{6/5}\alpha^2\left[\dot{x}_2\left(\frac{\Theta_c^0}{2}\right) + \delta\ddot{x}_2\left(\frac{\Theta_c^0}{2}\right)\right.\\ &\quad \left. + \frac{\delta^2}{2}\frac{d^3}{d\Theta^3}x_2\left(\frac{\Theta_c^0}{2}\right) + \cdots\right] \end{aligned} \tag{29}$$

and using $\dot{x}_0(\Theta_c^0/2) = 0$ ($v^{4/5}x_0$ is the solution of the undamped problem)

$$\delta = -\alpha v^{1/5}\frac{\dot{x}_1(\Theta_c^0/2)}{\ddot{x}_0(\Theta_c^0/2)} + O(\alpha^2). \tag{30}$$

The expression (30) has to be inserted into the Taylor expansion of $x(\Theta_c^0/2+\delta)$:

$$x(\Theta_c^0/2+\delta)=v^{4/5}\left[x_0\left(\frac{\Theta_c^0}{2}\right)+\delta\dot{x}_0\left(\frac{\Theta_c^0}{2}\right)+\frac{\delta^2}{2}\ddot{x}_0\left(\frac{\Theta_c^0}{2}\right)+\cdots\right]+\alpha v\left[x_1\left(\frac{\Theta_c^0}{2}\right)+\delta\dot{x}_1\left(\frac{\Theta_c^0}{2}\right)+\frac{\delta^2}{2}\ddot{x}_1\left(\frac{\Theta_c^0}{2}\right)+\cdots\right] \tag{31}$$

$$=v^{4/5}x_0\left(\frac{\Theta_c^0}{2}\right)+\alpha v x_1\left(\frac{\Theta_c^0}{2}\right)-\frac{\alpha^2 v^{6/5}}{2}\frac{\dot{x}_1^2(\Theta_c^0/2)}{\ddot{x}_0(\Theta_c^0/2)}+\alpha^2 v^{6/5}x_2\left(\frac{\Theta_c^0}{2}\right)+O(\alpha^3). \tag{32}$$

Hence

$$x(\Theta_m)=v^{4/5}x_0(\Theta_c^0/2)+\alpha v x_1(\Theta_c^0/2)+\alpha^2 v^{6/5}\left[x_2\left(\frac{\Theta_c^0}{2}\right)-\frac{1}{2}\frac{\dot{x}_1^2(\Theta_c^0/2)}{\ddot{x}_0(\Theta_c^0/2)}\right]+\cdots . \tag{33}$$

Replacing again $v\to v'$ and $\alpha\to-\alpha$ yields

$$\delta'=\alpha(v')^{1/5}\frac{\dot{x}_1(\Theta_c^0/2)}{\ddot{x}_0(\Theta_c^0/2)}+O(\alpha^2), \tag{34}$$

$$x^{\mathrm{inv}}(\Theta_m')=(v')^{4/5}x_0(\Theta_c^0/2)-\alpha v' x_1(\Theta_c^0/2)+\alpha^2(v')^{6/5}\left[x_2\left(\frac{\Theta_c^0}{2}\right)-\frac{1}{2}\frac{\dot{x}_1^2(\Theta_c^0/2)}{\ddot{x}_0(\Theta_c^0/2)}\right]+\cdots . \tag{35}$$

As explained above both solutions (33) and (35) have to be equal. With

$$\beta=x_2\left(\frac{\Theta_c^0}{2}\right)-\frac{1}{2}\frac{\dot{x}_1^2(\Theta_c^0/2)}{\ddot{x}_0(\Theta_c^0/2)} \tag{36}$$

we write

$$v^{4/5}x_0\left(\frac{\Theta_0^c}{2}\right)+\alpha v x_1\left(\frac{\Theta_0^c}{2}\right)+\alpha^2 v^{6/5}\beta=(v')^{4/5}x_0\left(\frac{\Theta_0^c}{2}\right)-\alpha v' x_1\left(\frac{\Theta_0^c}{2}\right)+\alpha^2(v')^{6/5}\beta. \tag{37}$$

We expand v' in α,

$$v'=v+\alpha v_1+\alpha^2 v_2+\cdots , \tag{38}$$

and find

$$v^{4/5}x_0\left(\frac{\Theta_0^c}{2}\right)+\alpha v x_1\left(\frac{\Theta_0^c}{2}\right)+\alpha^2 v^{6/5}\beta$$
$$=v^{4/5}\left(1+\frac{\delta v}{v}\right)^{4/5}x_0\left(\frac{\Theta_0^c}{2}\right)-\alpha v\left(1+\frac{\delta v}{v}\right)x_1\left(\frac{\Theta_0^c}{2}\right)$$
$$+\alpha^2 v^{6/5}\left(1+\frac{\delta v}{v}\right)^{6/5}\beta, \tag{39}$$

with $\delta v=\alpha v_1+\alpha^2 v_2+\cdots$. Writing $(1+\delta v/v)^{n/5}$ in powers of α and comparing coefficients yields finally

$$v'=v\left(1+\frac{5}{2}\alpha v^{1/5}\frac{x_1(\Theta_c^0/2)}{x_0(\Theta_c^0/2)}+\frac{15}{4}\alpha^2 v^{2/5}\left(\frac{x_1(\Theta_c^0/2)}{x_0(\Theta_c^0/2)}\right)^2+\cdots\right)$$
$$=v(1-\alpha v^{1/5}C_1+\alpha^2 v^{2/5}C_2+\cdots), \tag{40}$$

with

$$C_1=\frac{5}{2}\frac{x_1(\Theta_c^0/2)}{x_0(\Theta_c^0/2)} \tag{41a}$$

$$C_2=\frac{15}{4}\left(\frac{x_1(\Theta_c^0/2)}{x_0(\Theta_c^0/2)}\right)^2. \tag{41b}$$

Since Θ_c^0 depends on neither any material properties nor on the impact velocity g or v, respectively, C_1 and C_2 are pure numerical constants. Evaluating C_1 and C_2 in Eq. (41) numerically yields $C_1=1.15344$ and $C_2=0.79826$.

For coefficients of normal restitution one gets

$$\epsilon=\frac{v'}{v}=1-\alpha v^{1/5}C_1+\alpha^2 v^{2/5}C_2+\cdots \tag{42a}$$

$$=1-C_1 A\rho^{2/5}g^{1/5}+C_2A^2\rho^{4/5}g^{2/5}+\cdots , \tag{42b}$$

with g being the impact velocity (Fig. 3). For the duration of the collision we find with Eqs. (30), (35), and (40)

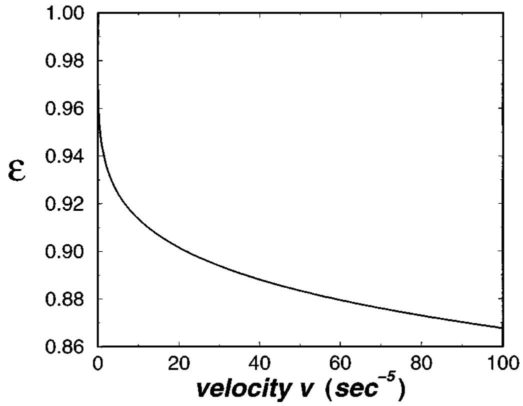

FIG. 3. The coefficient of restitution over impact velocity due to Eq. (42). As expected for small relative velocity the particles collide almost elastically. The result of numerical integration of Eq. (6) coincides with the curve. The two curves cannot be distinguished in the plot.

$$
\begin{aligned}
t_c &= \left(\frac{\Theta_c^0}{2}+\delta\right) v^{-1/5} + \left(\frac{\Theta_c^0}{2}+\delta'\right)(v')^{-1/5} \\
&= \Theta_c^0 v^{-1/5}\left(1-\frac{1}{4}\alpha v^{1/5}\frac{x_1(\Theta_c^0/2)}{x_0(\Theta_c^0/2)}\right)+O(\alpha^2) \\
&= \Theta_c^0 v^{-1/5}\left(1+\frac{1}{10}C_1\alpha v^{1/5}\right)+O(\alpha^2) \\
&= \Theta_c^0 \rho^{-2/5} g^{-1/5}\left(1+\frac{1}{10}C_1\alpha\rho^{2/5}g^{1/5}\right)+O(\alpha^2). \qquad (43)
\end{aligned}
$$

$$\Theta_c = v^{1/5}t_c = \Theta_c^0\left(1+\tfrac{1}{10}C_1\alpha v^{1/5}\right)+O(\alpha^2). \qquad (44)$$

To check the theoretical result [Eqs. (41)] we integrated numerically Eq. (12) and received the curves $\epsilon(v)$ and $\Theta_c(v)$. Then we fitted C_1 and C_2 to these data using Eqs. (42) and (44). For instance, for $\alpha = 0.05$ we found $C_1^{\mathrm{num}} = 1.15356$ and $C_2^{\mathrm{num}} = 0.80439$ from the curve $\epsilon(v)$ [see Eq. (42)]. The fit of C_1 to $\Theta_c(v)$ [see Eq. (44)] gives $C_1^{\mathrm{num}} = 1.15342$. For other values of α we found very similar numbers. Hence, the numerical results agree with theory.

When we use the velocity-dependent coefficient of restitution in the collision term of the Boltzmann equation

$$\dot{T} \sim \int\int dv_1 dv_2 (1-\epsilon^2)|v_1-v_2|^3 f(v_1) f(v_2) \qquad (45)$$

we get the cooling rate for dissipative gas:

$$T \sim T_0/(1+t/\tau')^{5/3}. \qquad (46)$$

Our final result, Eq. (42), shows that for viscoelastic colliding smooth bodies the coefficient of normal restitution is a decreasing function with rising impact velocity: $1-\epsilon \sim g^{1/5}$. A direct consequence is the cooling rate of a granular gas [Eq. (46)]: a granular gas consisting of viscoelastic particles cools down significantly *slower* than a gas of particles that collide with constant coefficient of restitution [see Eq. (7)]. Due to our understanding it is not self-evident whether the clustering observed in granular gases of the latter type, and the extreme case of this effect, the inelastic collapse, will change their overall behavior or whether they exist at all. These questions should be reconsidered in detail for velocity-dependent restitution.

The authors are grateful to N. Brilliantov, S. Esipov, H. Herrmann, F. Spahn, and W. Young for helpful discussions, and J.-M. Hertzsch for providing relevant literature.

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