ARTICLES

Application of a renormalization group algorithm to nonequilibrium cellular automata with one absorbing state

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We improve a recently proposed dynamically driven renormalization group algorithm for cellular automata systems with one absorbing state, introducing spatial correlations in the expression for the transition probabilities. We implement the renormalization group scheme considering three different approximations that take into account correlations in the stationary probability distribution. The improved scheme is applied to a probabilistic cellular automaton already introduced in the literature. [S1063-651X(98)05206-4]

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I. INTRODUCTION

In order to study cellular automata systems displaying a second-order irreversible phase transition to an absorbing state characterized by a scalar order parameter, a dynamical renormalization group (RG) algorithm [1] has been proposed recently. These cellular automata models are in the directed percolation (DP) universality class. The method is based on a dynamically driven renormalization group (DDRG) scheme (for a recent review about DDRG see [2]), which has been successfully applied to self-organized critical phenomena, as sandpile models [3] and forest fire models [4].

The basic idea introduced in [4] is to couple a real space RG scheme to a stationary condition that drives the RG equations through the parameter space. The stationary equations, involving the stationary distribution, have to be approximated since the form of the stationary probability distribution is not known *a priori*, as in the case of systems in equilibrium. de Oliveira and Satulovsky [1] showed, as proposed in [4], that results can be improved using more refined approximations for the stationary probability distribution. The expression for the transition probability used in [1] consists in a product of independent one-site transition probabilities at every step of the RG transformation.

In this work we exploit another aspect of the scheme in order to include additional correlations. In fact, correlations can be also introduced in the renormalization scheme if we allow the transition probability to depend upon more neighbors. In the case of nonequilibrium models, the space in which the RG flows is the space spanned by the transition probabilities. As we will see later, our approach broadens this space providing more degrees of freedom to the RG trajectories that flow towards the fixed points.

We apply the modified RG scheme to a probabilistic cellular automaton (PCA) with one absorbing state already introduced in the literature [5]. Using a block renormalization that properly treats the nature of the absorbing state, we figure the value of the critical exponent of the divergence of the spatial correlation length $\nu_\perp$ using three different approximations involving correlations among clusters of one, two, and four neighboring sites in the lattice. Our calculations for $\nu_\perp$, at small orders of approximation in the mean field scheme for the stationary distribution, give better values than the ones reported in Ref. [1].

The paper is organized as follows. We begin with a brief description of the model proposed in [5] and the general renormalization scheme. After this, we define the algorithm used in this work and present the values obtained for $\nu_\perp$. Finally, after mentioning the ideas involved in the simulation technique used to study the model, we present the values of the whole set of critical exponents for the PCA obtained by means of dynamical numerical simulations and stationary simulations. Our results are in well agreement with the values corresponding to $(1+1)$-dimensional DP and differ considerably from the ones reported in Ref. [5].

II. MODEL

The model studied in [5] is a one-dimensional cellular automaton in which each site can be either vacant $\sigma_i=0$ or occupied by a particle $\sigma_i=1$. At each time step, the state of a given site will depend only on its previous state and the previous state of its nearest neighbors. The transition probability $T(\sigma|\sigma')$ from state $\sigma'=(\sigma'_1,\sigma'_2,\ldots,\sigma'_L)$ to state $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_L)$ will be given by the product

$$T(\sigma|\sigma')=\prod_{i=1}^{L}\tau(\sigma_i|\sigma'_{i-1},\sigma'_i,\sigma'_{i+1}), \tag{1}$$

where L is the number of sites and $\tau(\sigma_i|\sigma'_{i-1},\sigma'_i,\sigma'_{i+1})$ is the one-site transition probability given by the following

rules

τ	000	001	100	101	010	011	110	111
0	1	$1-p$	$1-p$	1	0	1	1	1
1	0	p	p	0	1	0	0	0

(2)

This probabilistic cellular automaton models a transition from elementary rule 4 to elementary rule 22 (following Wolfram's nomenclature scheme) [6]. The system has two critical points, one of them at $p_1=0$ and the other at $p_2 \approx 0.75$. The critical point at $p_1=0$ will be shown to be trivial, in contrast to the result obtained in Ref. [5]. Consequently, we will apply the RG to study the nontrivial transition point. Rule $\tau(0|000)=1$ implies that the vacuum state is indeed an absorbing state.

III. RENORMALIZATION SCHEME

The RG scheme proposed in [4] is a real space RG scheme [7] in which one renormalizes the transition probability T. The RG flow takes place in the space of parameters defining T. A blocking procedure transforms cells of b sites into one site at the new scale. In order to account for the fact that the vacuum state is absorbing, a cell devoid of particles will always renormalize into an empty site. Cells with at least one particle have been chosen to renormalize into an occupied site. Other options have been tried, but they do not preserve the existence of the absorbing state.

Let $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_L)$ be the state of a system with L degrees of freedom and the vector $S=(S_1,S_2,\ldots,S_{L'})$ be the state of the renormalized system with $L'=L/b$ degrees of freedom, where b is the size of the renormalization block. The conditional probability of state S given state σ, $\mathcal{R}(S|\sigma)$, must satisfy

$$\mathcal{R}(S|\sigma)\geqslant 0, \quad \sum_S \mathcal{R}(S|\sigma)=1. \tag{3}$$

Given T and the probability of a state σ' at time t, $W(\sigma')$, one can write the joint probability of state σ' at time t and state σ at n time steps later $W_n(\sigma,\sigma')$ by simply applying T to $W(\sigma')$ n successive times

$$W_n(\sigma,\sigma')=T^n(\sigma|\sigma')W(\sigma'). \tag{4}$$

In addition, in the stationary regime, the probability distribution $W(\sigma)$ must satisfy

$$W(\sigma)=\sum_{\sigma'} T^n(\sigma|\sigma')W(\sigma') \tag{5}$$

for any value of n.

In the same way one can write these expressions at the coarse grained level. Denoting by $\tilde{T}(S,S')$ the probability of occurrence of state S' at a given time and state S one time step later, the RG transformation is obtained imposing [7]

$$\tilde{W}(S,S')=\sum_{\sigma}\sum_{\sigma'} \mathcal{R}(S|\sigma)\mathcal{R}(S'|\sigma')W_n(\sigma,\sigma'), \tag{6}$$

from which follows

$$\tilde{W}(S')=\sum_S \tilde{W}(S,S')=\sum_{\sigma'} \mathcal{R}(S'|\sigma')W(\sigma'). \tag{7}$$

Once knowing the transition probabilities at the coarse grained scale, one can easily build the rescaled transition matrix $\tilde{T}(S|S')$ as

$$\tilde{T}(S|S')=\frac{\tilde{W}(S,S')}{\tilde{W}(S')}. \tag{8}$$

Using now Eqs. (4), (6), and (7), we obtain the final expression for the renormalization equations [4]

$$\tilde{T}(S|S')=\frac{\sum_{\sigma}\sum_{\sigma'} \mathcal{R}(S|\sigma)\mathcal{R}(S'|\sigma')T^n(\sigma|\sigma')W(\sigma')}{\sum_{\sigma'} \mathcal{R}(S'|\sigma')W(\sigma')}. \tag{9}$$

Let us note that, while this equation expresses rescaled transition probabilities $\tilde{T}$ in terms of transition probabilities at a lower scale T, the stationary weight of each state present in Eq. (9), $W(\sigma')$, is yet unknown. Contrary to the case of closed systems in thermal equilibrium, we do not know *a priori* the expression for the stationary probability distribution. However, including Eq. (5), one can get a closed set of equations to solve at each renormalization step. The stationarity condition (5) is actually essential in driving the RG equations (9) through parameter space. Equation (9), together with a given approximation for the stationary probability, provides then a well defined RG transformation $T \rightarrow \tilde{T}$.

In practice, Eq. (5) can hardly be solved and one must resort to approximations. The values of critical exponents obtained using the present RG approach are expected to improve as these approximations improve. We have used three different levels of approximations, in which correlations among clusters of one, two, and four sites are considered respectively.

In this work we will be concerned with another way to improve the RG scheme. An important point is that, in order to solve Eq. (9), an assumption needs to be made on how the transition probability between states S' and S, $\tilde{T}(S|S')$, depends on local transition probabilities at the coarse grained level. This choice will determine the degree of proliferation that the RG will have since the form of the renormalized transition probability will be preserved along the RG trajectories.

In the former approach [1], the authors carried out the most general RG transformation considering one-site transition probabilities. By preserving the form of the renormalized transition probabilities, the RG trajectories are found to flow to the attractive fixed points in a five-dimensional space spanned by the dynamical parameters. In contrast to usual

RG methods in which new couplings arise at each step of the transformation, this RG procedure is not able to proliferate the dynamical parameters since the form of the transition probabilities is kept fixed at the coarse grained level. However, new dynamical parameters can be considered from the very beginning if we allow the transition probabilities to depend upon more neighbors, i.e., introducing more correlations in them. This feature of the method should be compared with other dynamical RG procedures [7] where the introduction of new couplings since the very beginning is an alternative way to carry out the RG transformation.

We propose, then, a form for the coarse grained transition probability consisting of a product of independent two-site transition probabilities instead of one-site transition probabilities. Denoting the position of each lattice site with i, the new transition probability is defined at even time steps as

$$\tilde{T}(S|S')=\prod_{k=1,\ i=2k}^{L/2}\tilde{\tau}(S_i,S_{i+1}|S'_{i-1},S'_i,S'_{i+1},S'_{i+2}) \tag{10}$$

and at odd time steps as

$$\tilde{T}(S|S')=\prod_{k=1,\ i=2k+1}^{L/2}\tilde{\tau}(S_i,S_{i+1}|S'_{i-1},S'_i,S'_{i+1},S'_{i+2}). \tag{11}$$

Here we have used the same symbol $\tilde{\tau}$ to indicate a different type of transition probability than the ones appearing in Eq. (1). In formulas (10) and (11) periodic boundary conditions are assumed.

One can retrieve at any time one-site transition probabilities knowing both two-site probabilities and the stationary distribution. It is straightforward to show that

$$\begin{aligned}\tilde{\tau}(S_i|S'_{i-1},S'_i,S'_{i+1})&=\sum_{S_{i+1},S'_{i+2}}\tilde{\tau}(S_i,S_{i+1}|S'_{i-1},S'_i,S'_{i+1},S'_{i+2})\\&\times\frac{W(S'_{i-1},S'_i,S'_{i+1},S'_{i+2})}{W(S'_{i-1},S'_i,S'_{i+1})}.\end{aligned} \tag{12}$$

Using expressions (10) and (11) in Eq. (9) one can implement the RG transformation. The algorithm we used is explained in more detail in the next section.

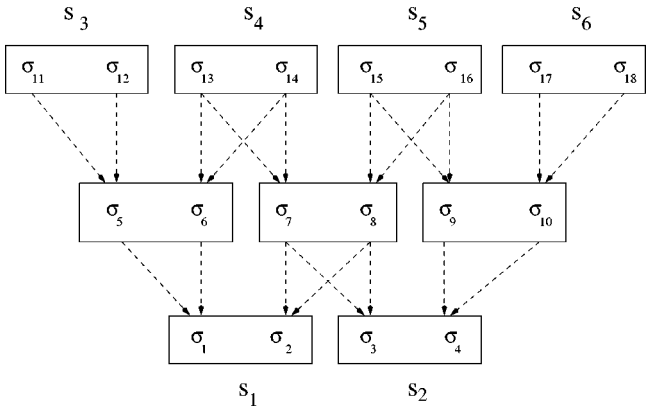

FIG. 1. Diagram showing the blocking scheme procedure. Numbers correspond to the indices used in Eqs. (18)–(20).

IV. RENORMALIZATION ALGORITHM

We have used a temporal coarse graining of two time steps ($n=2$). The blocking operator $\mathcal{R}$ was chosen in the same way as in [1], renormalizing cells of size $b=2$ into one site

$$\mathcal{R}(S|\sigma)=\prod_{k=1}^{L/2}R(S_k|\sigma_{2k-1},\sigma_{2k}), \tag{13}$$

with

$$R(S_k|\sigma_{2k-1},\sigma_{2k})\geqslant 0 \tag{14}$$

and

$$\sum_{S_k}R(S_k|\sigma_{2k-1},\sigma_{2k})=1. \tag{15}$$

In order to preserve the nature of the absorbing state, we have also required R to satisfy

$$R(0|0,0)=1 \tag{16}$$

and

$$R(0|\sigma_{2k-1},\sigma_{2k})=0 \tag{17}$$

whenever $\sigma_{2k-1}\neq 0$ or $\sigma_{2k}\neq 0$.

The diagram in Fig. 1 indicates how two-site transition probabilities are renormalized. Indices appearing in Eqs. (18)–(20) refer to this diagram.

Using Eqs. (9)–(11), we can write down the expression relating τ to $\tilde{\tau}$, which is given by

$$\begin{aligned}\tilde{\tau}(S_1,S_2,|S_3,S_4,S_5,S_6)=[N(S_3,S_4,S_5,S_6)]^{-1}&\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_{11},\ldots,\sigma_{18}}R(S_1|\sigma_1,\sigma_2)\\&\times R(S_2|\sigma_3,\sigma_4)R(S_3|\sigma_{11},\sigma_{12})R(S_4|\sigma_{13},\sigma_{14})R(S_5|\sigma_{15},\sigma_{16})R(S_6|\sigma_{17},\sigma_{18})\\&\times D(\sigma_1,\sigma_2,\sigma_3,\sigma_4|\sigma_{11},\ldots,\sigma_{18})W(\sigma_{11},\ldots,\sigma_{18}),\end{aligned} \tag{18}$$

where

$$D(\sigma_1\sigma_2,\sigma_3,\sigma_4|\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{14},\sigma_{15},\sigma_{16},\sigma_{17},\sigma_{18})$$

$$=\sum_{\sigma_5,\ldots,\sigma_{10}}\tilde{\tau}(\sigma_1,\sigma_2|\sigma_5,\sigma_6,\sigma_7,\sigma_8)\tilde{\tau}(\sigma_3,\sigma_4|\sigma_7,\sigma_8,\sigma_9,\sigma_{10})\tilde{\tau}(\sigma_5,\sigma_6|\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{14})$$

$$\times\tilde{\tau}(\sigma_7,\sigma_8|\sigma_{13},\sigma_{14},\sigma_{15},\sigma_{16})\tilde{\tau}(\sigma_9,\sigma_{10}|\sigma_{15},\sigma_{16},\sigma_{17},\sigma_{18}) \tag{19}$$

and

$$N(S_3,S_4,S_5,S_6)=\sum_{\sigma_{11},\ldots,\sigma_{18}}R(S_3|\sigma_{11},\sigma_{12})R(S_4|\sigma_{13},\sigma_{14})R(S_5|\sigma_{15},\sigma_{16})$$

$$\times R(S_6|\sigma_{17},\sigma_{18})W(\sigma_{11},\ldots,\sigma_{18}). \tag{20}$$

Since we do not know *a priori* the stationary weights $W(\sigma_{11},\ldots,\sigma_{18})$, we need an approximate method to estimate them. The simplest approximation, sometimes known as simple mean field approximation, consists in neglecting correlations among different sites, that is,

$$W(\sigma_{11},\ldots,\sigma_{18})=\prod_{i=11}^{18}W(\sigma_i), \tag{21}$$

where $W(\sigma_i)$ is the solution of

$$W(\sigma_1)=\sum_{\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6}\tilde{\tau}(\sigma_1\sigma_2|\sigma_3\sigma_4\sigma_5\sigma_6)$$

$$\times W(\sigma_2)W(\sigma_3)W(\sigma_4)W(\sigma_5)W(\sigma_6). \tag{22}$$

Correlations, however, are actually taken into account in the geometrical aspects of the blocking procedure, leading to nonclassical exponents. Better approximations can be also implemented (as a reference see [8]). We used one-, two-, and four-site approximations.

Equations (18)–(20) involve each transition probability. Being so many terms, we are prevented from an analytical determination of the fixed points of the transformation. So we performed our search numerically, using initial values for the transition probabilities that correspond to the original model [Eq. (2)].

As a technical remark, let us say that in each iteration of the RG and given a set of parameters $\{\tilde{\tau}\}$, we need to solve Eq. (22) (or its analog for two- and four-site approximations) before the next RG step. We have done this by iterating the equation until reaching convergence.

V. RG RESULTS

The behavior of the RG equations can be described as follows. For values of p that are small enough, the set of transition probabilities flows towards an attractive fixed point characterized by a lattice devoid of particles. Increasing p above a critical value p_{cr}, the flow is driven to another attractive fixed point, consisting of a lattice full of particles. The value of p_{cr}, for each level of approximation used, can be found in Eq. (23). Starting around the critical values, the representative point of the parameter set spends a long time near an unstable fixed point before leaving towards one of the two attractive fixed points. A projection of the RG flow along two specific transition probabilities is shown in Fig. 2.

We have found one relevant parameter. Since we are only dealing with stationary properties of the model, it is reasonable to assume that this parameter is associated with the divergence of the spatial correlation length and not the temporal correlation length [1]. In order to calculate the eigenvalue Λ associated with that parameter, we have to find the linear region of the RG transformation.

Let us take a trajectory passing close enough to the unstable point and construct a sequence of numbers consisting in the distance between two successive points along that trajectory. Now let us call r the ratio between two consecutive numbers in that sequence. In the portion of the trajectory corresponding to the linear region of the RG transformation (around the unstable fixed point), one expects to see two plateaus in the values of r.

The first plateau corresponds to a trivial parameter, while the value of r at the second plateau corresponds to the eigenvalue Λ of the RG transformation. Figure 3 shows an example of one of such curves for the simple mean field approximation. So, figuring the eigenvalue Λ associated with the relevant parameter, we get $\nu_\perp=\ln 2/\ln\Lambda$. The value measured numerically for the plot in Fig. 3 (simple mean field approximation) is $\nu_\perp=0.965\pm0.001$.

To the best of our knowledge, the best value of $\nu_\perp$ is

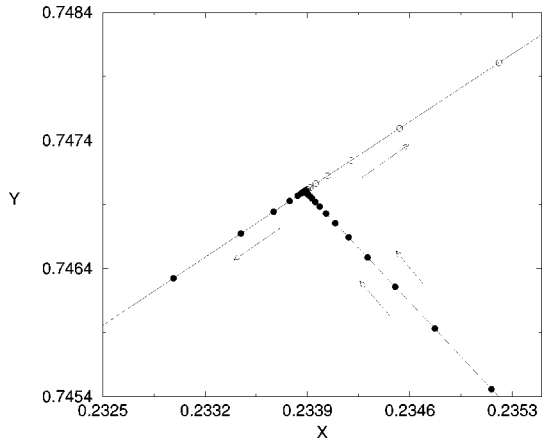

FIG. 2. Two-dimensional projection of the RG flow along $x=\tilde{\tau}(00|0010)$ and $y=\tilde{\tau}(00|1000)$ for the simple mean field approximation.

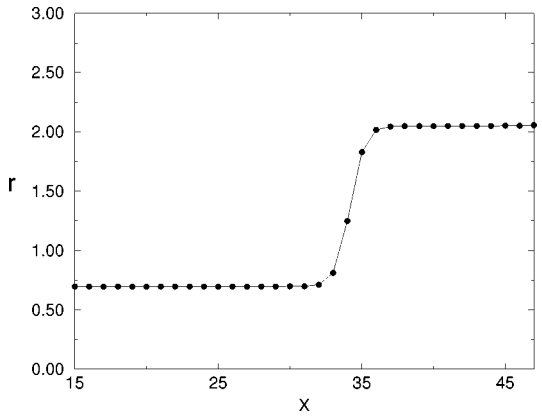

FIG. 3. Ratio r of successive distances among two consecutive points for one of the trajectories shown in Fig. 2. The x coordinate denotes the precedence of each point along the curve.

$\nu_\perp = 1.0972 \pm 0.0005$ [10] and the most accurate value for the critical point p_2, obtained by dynamical numerical simulations (see the following sections), is $p_2 \approx 0.7513$. Although our value of the critical exponent $\nu_\perp$ is still inaccurate, it is worth at this point comparing it with the one obtained using the same approximation (the simple mean field) in [1]. In that work the authors obtained $\nu_\perp = 0.931 \pm 0.005$, so that our result is a better approximation for the actual $\nu_\perp$. Our scheme is able to take into account correlations in a more accurate way than the original one.

By increasing the order of the approximation, results improve, as shown in Eq. (23). The value we found using the two-site approximation was $\nu_\perp = 1.013 \pm 0.001$, while the value found in the four-site approximation was $\nu_\perp = 1.015 \pm 0.001$, which is closer to the actual one. Below we show the critical value p_{cr} for the three approximations, as well as the corresponding value of Λ and $\nu_\perp$:

Approximation	p_{cr}	Λ	$\nu_\perp$
1	0.639825	2.050	0.963 ± 0.001
2	0.681490	1.982	1.013 ± 0.001
4	0.695017	1.979	1.015 ± 0.001

(23)

It should be noted that the present RG scheme leads to fairly good values for the exponent $\nu_\perp$ already within lower-order mean field approximations. This fact indicates that the introduction of new correlations in the transition probabilities plays a relevant role within the lower-order mean field approaches. For mean field approximations of order higher than 2, the convergence of the scheme becomes slower.

Reconsidering the ideas that led us to Eqs. (10) and (11) for the transition probabilities, one may think of obtaining better approximations for $\nu_\perp$ by allowing the transition probabilities to depend upon even more neighboring lattice sites. While this idea is clearly right, one can presently not overcome, in practice, the huge amount of computer time needed to obtain values that are accurate enough.

VI. SIMULATION TECHNIQUE

A. Spreading analysis

The basic rules governing the dynamical evolution of the system have been formulated in Sec. II. As in ordinary cellular automata, all lattice sites are updated simultaneously. Simulations were performed on lattices of size $L = 10\,000$, taking periodic boundary conditions. We briefly discuss here the scaling theory for directed percolation that supports the spreading analysis. A detailed treatment can be found elsewhere [9].

It should be stressed that for L finite, the steady state of the system is metastable since, due to fluctuations of the stochastic process, there is always a finite probability for the system to become empty. This probability increases when approaching the critical point. Consequently, it is very difficult to calculate critical points and critical exponents by means of numerical simulations. Furthermore, since the transition between the stationary regime and the absorbing state is second order, a mean field treatment is not adequate. These shortcomings can be avoided by evaluating critical exponents related to the dynamic critical behavior of the system. For this purpose one starts, at $t = 0$, with a particle at the center of the lattice otherwise empty, i.e., a configuration very close to the absorbing state. Then the following quantities are computed: (i) the survival probability $P(t)$, that is, the probability that at least a particle is still in the system at time t, (ii) the average number of particles $N(t)$, and (iii) the average mean distance $R(t)$ over which particles have spread. Averages are taken over 5×10^4 samples and runs are performed up to $t = 10^4$. Finite size effects are avoided since the epidemic disk never reaches the edge of the lattice during the simulation. Close to the critical point and for long enough times, the following scaling laws should hold [9]:

$$P(t) \propto t^{-\delta} \Phi\{\Delta t^{1/\nu_\parallel}\}, \tag{24}$$

$$N(t) \propto t^{\eta} \phi\{\Delta t^{1/\nu_\parallel}\}, \tag{25}$$

$$R(t) \propto t^{z/2} \Xi\{\Delta t^{1/\nu_\parallel}\}, \tag{26}$$

where $\Delta = |p - p_c|$, $\xi_t = \Delta^{-\nu_\parallel}$ gives the temporal correlation length close to p_c, $\nu_\parallel$ is the correlation length exponent (time direction), Φ, ϕ, and Ξ are suitable scaling functions, and δ, η, and z are critical exponents. In the absorbing state, $P(t)$ and $N(t)$ are expected to decay exponentially since correlations are short ranged. This can only happen if $\phi(\Delta, t) \propto \{\Delta t^{1/\nu_\parallel}\}^{-\eta\nu_\parallel} \exp(-\Delta^{\nu_\parallel} t)$ for $t \to \infty$. Therefore, one has from Eq. (25)

$$N(t) \propto \Delta^{-\eta\nu_\parallel} \exp(-\Delta^{\nu_\parallel} t), \quad t \to \infty. \tag{27}$$

At criticality, one expects that log-log plots of $P(t)$, $N(t)$, and $R(t)$ would give straight lines, while upward and downward deviations would occur even slightly off criticality. This behavior would allow a precise determination of the critical point and the critical exponents δ, η, and z. It should be noted that by means of Eq. (27) it would be also possible to calculate $\nu_\parallel$.

B. Finite size scaling analysis

As in standard second-order phase transitions it is assumed that in the supercritical region and close to the critical point, the system displays spatial correlations characterized by a typical length scale ξ_s, which diverges at criticality according to

$$\xi_s \propto \Delta^{-\nu_\perp}, \quad \Delta \to 0, \tag{28}$$

where $\nu_\perp$ is the correlation length exponent in the spatial direction. The natural order parameter of the model is the density of particles ρ, which at criticality depends on the system size L and Δ as

$$\rho(p,L) = L^{-\beta/\nu_\perp} f(\Delta L^{1/\nu_\perp}), \tag{29}$$

where f is a suitable scaling function and β is the order parameter critical exponent. For small positive Δ and $L \to \infty$, $f(x)$ should have the form

$$f(x) \propto x^{\beta} \tag{30}$$

in order to recover the well-known critical behavior of the order parameter in the thermodynamic limit

$$\rho \propto \Delta^{\beta}. \tag{31}$$

VII. SIMULATION RESULTS

Before presenting the simulation results we will briefly discuss the critical point at $p=0$. If we start at $t=0$ with a random initial configuration of density $\rho_0 = 0.5$, the stationary density of the system is $\rho \approx 1/8$. It should be noted that at $p=0$ the system reaches a static stationary state in one time step [see the evolution rules in Eq. (2)]. Taking into account this observation, the stationary density of the system can be obtained as follows. It is clear from the evolution rules that the probability of having an occupied site at $t=1$ is equal to the probability of finding an occupied site surrounded by empty sites at $t=0$. Since there are no correlations in the initial state, the stationary density of the system at $p=0$ can be determined as

$$\rho = \rho_0 (1-\rho_0)^2. \tag{32}$$

Then, since we used $\rho_0 = 0.5$, the expected value of the stationary density is $\rho = 1/8$.

For arbitrary small values of p, the stationary state is the vacuum state [5]. We now consider the relaxation process to the vacuum state. We take as starting configuration ($t=0$) any of the static stationary states at $p=0$. Then we follow the evolution of the density for p close to zero. It is clear from the evolution rules that after a particle is created, two particles are removed from the system at the next time step. Then the system stays in another static state until the next creation process occurs. Suppose that a creation process happens at position x_i. If no new particle is created in the neighborhood of x_i at the next time step, correlations cannot be generated [see the evolution rules in Eq. (2)]. The probability of two consecutive creation processes is $p^2 < p$. Then the system cannot develop long-range correlations and a mean field analysis should be appropriate. For p close to zero it is possible to consider the creation process as a creation-induced annihilation process. We then have

$$d\rho/dt = -p\rho. \tag{33}$$

Consequently, $p=0$ is a trivial critical point since the relaxation time behaves as $\tau = (1/p)^1$. This result is in disagreement with the one reported in Ref. [5], probably due to a poor statistics of the simulation data.

In the following, the results of the epidemics analysis at $p = p_2$ are presented. We measure the time evolution of $P(t)$, $N(t)$, and $R(t)$ for different values of the parameter p. Log-log plots of these quantities as a function of time are straight lines at the phase transition and show curvature away from the transition. It is important to mention that the epidemics analysis is a very sensitive method since it is possible to distinguish among supercritical and subcritical behavior for p values that differ in the fourth decimal. Our best estimation of the critical point is $p_2 = 0.7513 \pm 0.0002$ and the dynami-

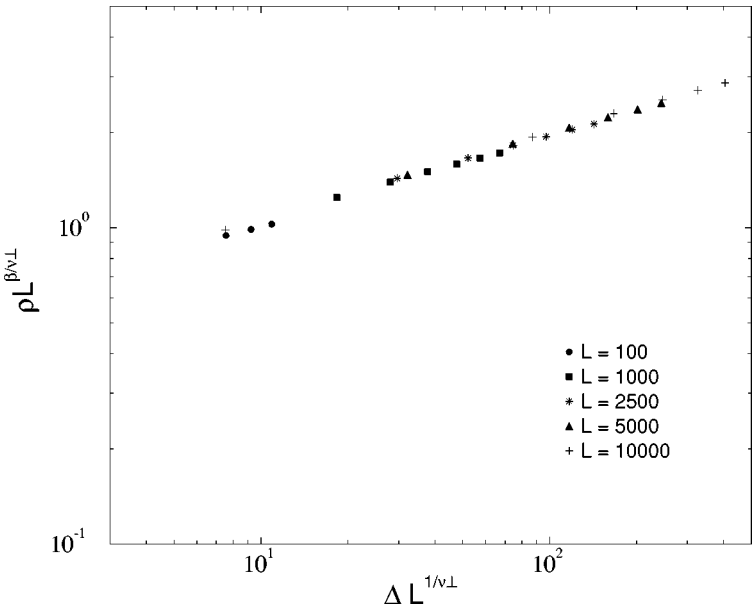

FIG. 4. Data collapse on a universal curve, according to Eq. (29).

cal critical exponents are $\delta = 0.162 \pm 0.0004$, $\eta = 0.304 \pm 0.0005$, and $z/2 = 0.643 \pm 0.0007$. It should be remarked that the error bars merely indicate the statistical error obtained from regressions. The values of the dynamical exponents are in good agreement with those corresponding to directed percolation in $1+1$ dimensions, as it was expected.

It is possible to calculate the exponent $\nu_\parallel$ from the analysis of the subcritical behavior [see Eq. (27)]. In fact, the decay constant $\lambda = \xi_t^{-1}$ governing the long-time behavior of $N(t)$ behaves according to

$$\lambda = \xi_t^{-1} = \Delta^{\nu_\parallel}, \tag{34}$$

so, if p_2 is known, we can calculate λ for different values of Δ. Then a log-log plot of λ vs Δ allows us to evaluate the exponent $\nu_\parallel$. This analysis gives an exponent $\nu_\parallel = 1.738 \pm 0.002$, which is quite close to the value $\nu_\parallel = 1.73$ corresponding to $(1+1)$-dimensional DP [10]. It should be pointed out that our value of $\nu_\parallel$ sharply differs from the one reported in Ref. [5] ($\nu_\parallel \approx 1.087$). The error in the last value of the exponent $\nu_\parallel$ is due to the fact that it was calculated taking into account not only subcritical but also supercritical curves.

We have also calculated the order parameter critical exponent measuring the density ρ as a function of Δ in the supercritical regime [see Eq. (31)]. We obtain $\beta = 0.277 \pm 0.002$, which is once again very close to $\beta = 199/720$, corresponding to $(1+1)$-dimensional DP [10]. It should be mentioned that the reported value of the order parameter critical exponent in Ref. [5] is $\beta \approx 0.32$, which differs around 16% from the theoretical value. This difference may be due again to the poor statistics of the data.

We finally present the finite size scaling analysis. Figure 4 shows a log-log plot of $\rho L^{\beta/\nu_\perp}$ vs $\Delta L^{1/\nu_\perp}$ for different values of p and lattice sizes L, where we have used $\beta = 199/720$ and $\nu_\perp = 1.0972$ [10] corresponding to $(1+1)$-dimensional DP. We obtain an excellent collapse of the data on an universal curve, as it is predicted by Eq. (29).

Right before submitting this manuscript for publication, the author of Ref. [5] improved some of his previous results [11]. Although no new results are reported for the trivial critical point p_1, the conclusions concerning the universality class of the model are in complete agreement with the ones found in this work.

VIII. CONCLUSIONS

We have introduced a renormalization group algorithm for probabilistic cellular automata with one absorbing state. The scheme introduces correlations in the RG procedure by allowing the transition probabilities to depend upon two neighboring lattice sites. Three different approximations for the stationary probability distribution have been used, namely, the simple mean field approximation, the pair mean field approximation, and the four-site mean field approximation.

The present RG scheme leads to fairly good values for $\nu_\perp$ even within mean field approximations of low order. This result shows that the introduction of spatial correlations in the transition probabilities is the relevant reason for the improvement of the results.

The critical exponents $\nu_\perp$, especially for low-order approximations, are better than the ones obtained with schemes that make use of an independent product of one-site transition probabilities [1,12]. Using very simple arguments, we have shown that $p_1 = 0$ is a trivial critical point since the time relaxation constant behaves as $\tau = (1/p)^1$. This behavior differs from the one reported in Ref. [5] [$\tau = (1/p)^{0.86}$].

We have also obtained, by means of numerical simulations, the critical point $p = p_2$ and the whole set of critical exponents. The value of the critical point $p_2 = 0.7513 \pm 0.0002$ is in agreement with [5]. However, we found very different values for exponents $\nu_\parallel$ and β. Our values of the exponents at $p = p_2$ are in good agreement with those corresponding to $(1+1)$-dimensional DP, as it was expected, since there is only one absorbing state for the system [13,14].

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