Angular momentum loss by a radiating toroidal dipole

E. E. Radescu
International Centre for Theoretical Physics, Trieste, Italy
and Institute of Atomic Physics, P.O. Box MG-6, Bucharest, Romania

D. H. Vlad
Department of Theoretical Physics and Mathematics, Faculty of Physics, University of Bucharest, P.O. Box MG-5211, Bucharest, Romania

(Received 12 June 1997)

If a system of charges and currents besides the usual electric and magnetic (time-dependent) multipole moments does possess also time varying toroidal moments and distributions, there will be, in general, an additional angular momentum loss by the system through the radiation emitted by the toroidal sources. The classical electrodynamics formula for the rate of angular momentum loss by a time-dependent toroidal dipole is derived and discussed in connection with a forced precession of the toroidal dipole arround a given axis. [S1063-651X(98)02104-7]

PACS number(s): 03.50.De, 41.20.−q

I. INTRODUCTION

The toroidal dipole was originally introduced by Zeldovich [1] (under the name of ''anapole'') as a specific electromagnetic moment of a dipole type, different, however, from the usual electric and magnetic dipoles, for systems in which parity is not conserved. In work summarized in the review articles [2,3], by clarifying and generalizing Zeldovich's idea, an entire class of toroidal multipoles was shown to be needed in order to achieve a correct and complete multipole characterization of the most general type of source in both classical and quantum electrodynamics. Nowadays toroidal moments are investigated in various contexts and research areas ranging from classical electrodynamics to elementary particle, nuclear, atomic, molecular, and solid state physics. References to previous work may be mainly found in the already mentioned reviews [2,3]. Among such investigations here we particularly note the following: work done in connection with parity nonconservation in atomic phenomena [4], the calculation done by Ginzburgh and Tsytovich for the Cherenkov radiation emitted by a classical pointlike toroidal dipole [5], electromagnetic properties of toroidal solenoids [6,7], toroidal electromagnetic structure of Majorana fermions, induced toroidal moments and toroidal polarizabilities [8,9], first experimental evidence of a nuclear spin-dependent contribution to atomic parity nonconservation claimed to agree with that predicted to arise from a nuclear toroidal dipole moment [10], detailed calculations of the intrinsic toroidal dipoles of hydrogenic atoms and positronium [11] (see also [12]), intrinsic toroidal moments of certain molecules arising even in the framework of the usual parity conserving electromagnetic interaction, on account of the particular internal structure of the molecule [13,14], recent work in condensed matter physics by Dubovik and collaborators [15], etc.

The toroidal dipole moment is defined [2] through the formula

$$\mathbf{T}(t)=\frac{1}{10c}\int[\mathbf{r}(\mathbf{r}\mathbf{j})-2\mathbf{r}^2\mathbf{j}]d\mathbf{r}, \tag{1}$$

where $\mathbf{j}(\mathbf{r},t)$ denotes the current density of the source. It is a vector having the same transformation properties as $\mathbf{j}(\mathbf{r},t)$; i.e., it is a polar vector odd under time reversal. The contribution of the toroidal dipole moment to the interaction energy with the external electromagnetic field $\mathbf{A}^{\text{ext}}$ is [1–3]

$$\begin{aligned}H_{\text{Toroidal dipole}}(t)&=-\mathbf{T}(t)\cdot[\nabla\times\nabla\times\mathbf{A}^{\text{ext}}(\mathbf{r},t)]_{\mathbf{r}=0}\\&=-\mathbf{T}(t)\cdot[\nabla\times\mathbf{H}^{\text{ext}}(\mathbf{r},t)]_{\mathbf{r}=0}\\&=-\mathbf{T}(t)\cdot\left[\frac{4\pi}{c}\mathbf{J}^{\text{ext}}(\mathbf{r},t)+\frac{1}{c}\dot{\mathbf{D}}^{\text{ext}}(\mathbf{r},t)\right]_{\mathbf{r}=0}\end{aligned} \tag{2}$$

and is responsible for everything regarding the manifestations and properties of the toroidal dipoles.

The purpose of this paper is to bring clarifications to the classical aspects of the theory. The main results are those stated in the abstract. This work contains also some parts essentially methodological in nature, devoted again to clarification purposes; some speculative remarks are also put forward.

The paper is organized as follows: In Sec. II we show in the framework of the classical electrodynamics how the notion of toroidal dipole moment arises quite naturally when investigating the behavior of a system of charges moving in an external magnetic field configuration $\mathbf{H}^{\text{ext}}$ chosen such that $\nabla\times\mathbf{H}^{\text{ext}}$ is constant in space and time. When computing the average force and moment of forces acting on the system, one finds that while the former is expressed through the usual average magnetic dipole moment, the latter turns out to be given just by the average of the less familiar toroidal dipole (both the force and moment of forces are, of course, vector products of the corresponding dipoles with $\nabla\times\mathbf{H}^{\text{ext}}$). In Sec. III we compute the angular momentum loss (through radiation of electromagnetic waves) of a time-dependent toroidal dipole and find a formula analogous with the known one for the case of the electric dipole, with the (expected) difference, however, that in the toroidal case the vector product of the second and third time derivatives of the toroidal dipole will enter rather than the first and second derivatives, as for the electric dipole. The obtained formula is particularized and discussed in connection with the toroidal dipoles rotating around a fixed axis. Section IV is devoted to conclusions and the discussion of the results, as well as to some speculative remarks put forward when commenting on how far the analogy with the Larmor precession in constant magnetic fields can be pushed for toroidal dipoles immersed in a constant current density. While in general there is no analog of the Larmor theorem in the toroidal case, there seem to be a lot of situations when a precession frequency similar to Larmor's one can be introduced, and even, perhaps, fruitfully exploited.

II. AVERAGE FORCE AND MOMENT OF FORCES ACTING ON A SYSTEM OF CHARGES IN AN EXTERNAL FIELD CONFIGURATION WITH $\nabla\times\mathbf{H}^{\mathrm{ext}}$ HOMOGENEOUS AND TIME INDEPENDENT

Let us consider a classical system of charges in an external field configuration described by the vector potential

$$A_i^{\mathrm{ext}}(\mathbf{r})=\frac{1}{10}(x_i x_j-2\mathbf{r}^2\delta_{ij})\frac{4\pi}{c}I_j\,; \tag{3}$$

I_j is a constant vector; summation over repeated indices $(i,j=1,2,3)$ will be understood throughout this paper. In other words, we are dealing with a system of charges moving in a (time-independent) magnetic field

$$\mathbf{H}^{\mathrm{ext}}=\nabla\times\mathbf{A}^{\mathrm{ext}}=\frac{2\pi}{c}\mathbf{I}\times\mathbf{r} \tag{4}$$

with $\mathbf{I}$ playing the role of a constant external current density:

$$\nabla\times\mathbf{H}^{\mathrm{ext}}=\frac{4\pi}{c}\mathbf{I}. \tag{5}$$

The gauge in Eq. (3) is such that

$$\partial_i A_i^{\mathrm{ext}}=0. \tag{6}$$

What are the average force $\mathbf{F}$ and the average moment of forces $\mathbf{K}$ acting on the system in the presence of the constant external current density $\mathbf{I}$, i.e., in the external field specified by Eq. (3) or (4)? The answer is the following (the bars below denote time averages):

$$\overline{\mathbf{F}}=\frac{2\pi}{c}\overline{\mathbf{m}}\times\mathbf{I}, \tag{7}$$

$$\overline{\mathbf{K}}=\frac{4\pi}{c}\overline{\mathbf{T}}\times\mathbf{I}; \tag{8}$$

$\mathbf{m}$ denotes the system's magnetic dipole moment

$$\mathbf{m}=\frac{1}{2c}\sum e\mathbf{r}\times\mathbf{v}, \tag{9}$$

while $\mathbf{T}$ is the system's toroidal dipole moment

$$\mathbf{T}=\frac{1}{10c}\sum e[(\mathbf{r}\cdot\mathbf{v})\mathbf{r}-2\mathbf{r}^2\mathbf{v}] \tag{10}$$

($\mathbf{v}=\dot{\mathbf{r}}=$ the velocity of the charge, the dot means differentiation with respect to time and the sum extends over all charges, summation indices over charges being, for simplicity, dropped out). So, it is seen from Eq. (8) that the toroidal dipole moment [Eq. (10)] arises quite naturally in connection with the particular external magnetic field configuration [Eq. (4)] considered here; the notion of this first element (dipole) of the less familiar class of toroidal multipoles could even have been introduced by means of such arguments, more related to particular physical problems rather than to formal multipole expansions.

The derivation of Eqs. (7) and (8) is straightforward and goes along the same lines as in connection with the analogous but simpler problem of the movement in a constant magnetic field, found in textbooks (see, e.g., [16]). We first treat the more general case in which *instantaneous* rather than time averaged expressions of the total force and moment of forces are looked for, since it is quite instructive to look for the interplay between various multipole characteristics of the system in controlling the instantaneous values of $\mathbf{F}$ and $\mathbf{K}$ and see directly what multipoles will drop out in the final expressions when time averages are taken. Starting with the Lorentz force expression one has

$$\mathbf{F}=\sum\frac{e}{c}\mathbf{v}\times\mathbf{H}^{\mathrm{ext}}, \tag{11}$$

$$\mathbf{K}=\sum\frac{e}{c}\mathbf{r}\times\mathbf{v}\times\mathbf{H}^{\mathrm{ext}}. \tag{12}$$

Introducing for $\mathbf{H}^{\mathrm{ext}}$ the form of Eq. (4) and using relations like

$$\mathbf{r}(\mathbf{v}\cdot\mathbf{I})=\frac{d}{dt}[\mathbf{r}(\mathbf{r}\cdot\mathbf{I})]-\mathbf{v}(\mathbf{r}\cdot\mathbf{I}), \tag{13}$$

after simple manipulations one finds

$$\mathbf{F}=\frac{2\pi}{c}\mathbf{m}\times\mathbf{I}+\frac{\pi}{c^2}\left[\frac{2}{3}\dot{Q}^{n=1}\mathbf{I}-\dot{\hat{Q}}_{l=2}\|\mathbf{I}\right], \tag{14}$$

$$\mathbf{K}=\frac{4\pi}{c}\mathbf{T}\times\mathbf{I}+\frac{4\pi}{5c^2}\dot{\mathbf{Q}}^{n=1}\times\mathbf{I}; \tag{15}$$

$\mathbf{m}$ and $\mathbf{T}$ have been already defined by Eqs. (9) and (10),

$$Q^{n=1}=\sum e\mathbf{r}^2 \tag{16}$$

is the squared radius of the charge distribution,

$$\mathbf{Q}^{n=1}=\sum e\mathbf{r}^2\mathbf{r} \tag{17}$$

is the squared radius of the electric dipole moment distribution, while

$$Q_{ij}=\sum e\left(x_i x_j-\frac{r^2}{3}\delta_{ij}\right) \tag{18}$$

is the electric quadrupole moment and the following notation for the tensor contraction has been used:

$$Q_{ij}I_j \equiv (\hat{Q}_{l=2}\|\mathbf{I})_i . \tag{19}$$

Recall that dots on multipoles in Eqs. (14) and (15) denote time derivatives. For convenience we also recall here the corresponding expressions for **F** and **K** in the ''normal'' simpler case of movement in a constant external magnetic field $\mathbf{H}^{\rm ext}$:

$$\mathbf{F}=\frac{1}{c}\,\dot{\mathbf{Q}}\times\mathbf{H}^{\rm ext}, \tag{20}$$

$$\mathbf{K}=\mathbf{m}\times\mathbf{H}^{\rm ext}+\frac{1}{2c}\left[\dot{\hat{Q}}_{l=2}\|\mathbf{H}^{\rm ext}-\frac{2}{3}\dot{Q}^{n=1}\mathbf{H}^{\rm ext}\right]; \tag{21}$$

$$\overline{\mathbf{F}}=0, \quad \overline{\mathbf{K}}=\overline{\mathbf{m}}\times\mathbf{H}^{\rm ext}. \tag{22}$$

In the formulas above

$$\mathbf{Q}=\sum e\mathbf{r} \tag{23}$$

is the electric dipole moment.

The average force $\overline{\mathbf{F}}$ and moment of forces $\overline{\mathbf{K}}$ displayed in Eqs. (7) and (8) follow now immediately from the corresponding instantaneous values expressed by Eqs. (14) and (15), by remembering that the average value of any time derivative of a quantity varying within finite limits vanishes.

Equation (8) tells us that if a system of charges possesses a nonvanishing average toroidal dipole moment, it will experience an average moment of forces when immersed in a constant current density and the latter will tend to rotate the system and align it on the current lines.

The above considerations represent a formal proof of this assertion within classical electrodynamics; the assertion itself was, of course, known in the literature since the beginning of the toroidal moments domain, but it was merely based on the specific contribution of the toroidal dipole to the Hamiltonian [Eq. (2)] and analogy with the usual electric and magnetic dipole interactions.

We end this section with the remark that although in our case (movement in an external constant current density) there is no direct analog with the Larmor theorem for systems of charges exposed to a constant magnetic field, something can still be said when the system is such that its toroidal dipole moment vector for some reasons lies along the system's total angular momentum vector. This is actually the case for some quantum systems (elementary particles, nuclei, atoms, etc.) and then an analog for the Larmor precession and Larmor frequency for toroidal dipole moments immersed in a constant current density can indeed be formulated. But we defer this discussion to the concluding Sec. IV.

III. LOSS OF ANGULAR MOMENTUM OF A TIME-DEPENDENT TOROIDAL DIPOLE THROUGH RADIATION OF ELECTROMAGNETIC WAVES

In this section we shall derive the toroidal analog of the following well-known classical electrodynamics formula expressing the rate of the loss of angular momentum of a time-dependent electric dipole $\mathbf{d}(t)$ through radiation:

$$\frac{d\mathbf{M}}{dt}=-\frac{2}{3c^3}\dot{\mathbf{d}}\times\ddot{\mathbf{d}}. \tag{24}$$

The derivation of Eq. (24) above may be recalled, for instance, from the work of Landau and Lifchitz [16], where it appears as problem No. 2 following paragraph 72, Chapter IX. Anticipating the result, if instead of the electric dipole $\mathbf{d}(t)$ (which is a multipole characteristic referring to the charge density distribution within the system) one has a toroidal dipole $\mathbf{T}(t)$ (related, this time, to the current density), then the analogous formula reads:

$$\frac{d\mathbf{M}}{dt}=-\frac{2}{3c^5}\ddot{\mathbf{T}}\times\dddot{\mathbf{T}}. \tag{25}$$

(Three dots means third derivative with respect to time.) We shall obtain Eq. (25) following closely the arguments presented in [16] in deriving Eq. (24). The results of this section may be viewed as a completion of the known ones regarding the usual electric (and magnetic) dipoles, extending them to the third dipole-type characteristic, the toroidal dipole.

Now, we start sketching the derivation of Eq. (25). With the definition of the toroidal dipole moment

$$\mathbf{T}(t)=\frac{1}{10c}\int [\mathbf{r}(\mathbf{j}\cdot\mathbf{r})-2r^2\mathbf{j}]d\mathbf{r} \tag{26}$$

one sees that for a pointlike toroidal dipole $\mathbf{T}(t)$ situated in the origin the corresponding current density is

$$\mathbf{j}(\mathbf{r},t)=\nabla\times\nabla\times\mathbf{T}(t)\,\delta^3(\mathbf{r}) \tag{27}$$

(this formula verifies identically the previous relation). The fields of the source given by Eq. (27) may be computed straightforwardly from the retarded vector potential

$$A_i(\mathbf{r},t)=\frac{1}{c}\int\frac{j_i(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}', \tag{28}$$

$$\mathbf{E}=-\frac{1}{c}\dot{\mathbf{A}}, \quad \mathbf{H}=\nabla\times\mathbf{A}. \tag{29}$$

One finds, outside the origin,

$$A_i(\mathbf{r},t)=-\frac{1}{rc^2}\ddot{T}_i(t_0)-\frac{1}{r^2c}\dot{T}_i(t_0)-\frac{1}{r^3}T_i(t_0) +n_in_j\left[\frac{1}{rc^2}\ddot{T}_j(t_0)+\frac{3}{r^2c}\dot{T}_j(t_0)+\frac{3}{r^3}T_j(t_0)\right]$$

$$t_0\equiv t-r/c, \quad n_i=x_i/r. \tag{30}$$

To calculate the rate of angular momentum loss through radiation one starts with the known formula [16]

$$\frac{d\mathbf{M}}{dt}=\lim_{R_0\to\infty}\frac{R_0^3}{4\pi}\int[(\mathbf{n}\times\mathbf{E})(\mathbf{n}\cdot\mathbf{E})+(\mathbf{n}\times\mathbf{H})(\mathbf{n}\cdot\mathbf{H})]d\Omega, \tag{31}$$

where the integration has to be done over a spherical surface of large radius R_0 and the limit $R_0\to\infty$ must subsequently be taken with due care concerning various contributions from the electric and magnetic fields **E**, **H**, as given by Eqs. (30). We recall that Eq. (31) follows from the fact that the total angular momentum lost by the system per unit time is equal to the flux of angular momentum of the radiation field through a spherical surface of radius R_0:

$$-\frac{dM_a}{dt}=\int\epsilon_{abc}x_b\sigma_{cd}n_d dS,\quad dS=R_0^2 d\Omega, \tag{32}$$

σ_{ab} is the three-dimensional Maxwell stress tensor

$$\sigma_{ab}=\frac{1}{4\pi}[-E_aE_b-H_aH_b+\tfrac{1}{2}\delta_{ab}(E^2+H^2)], \tag{33}$$

and **n** is a unit vector in the direction of $\mathbf{R}_0$. Calculating **E**, **H** with the aid of the vector potential (30) and keeping track of the relevant contributions to the limit $R_0\to\infty$, one finds

$$\frac{d\mathbf{M}}{dt}=-\frac{1}{2\pi c^5}\int(\mathbf{n}\cdot\ddot{\mathbf{T}})(\mathbf{n}\times\dddot{\mathbf{T}})d\Omega, \tag{34}$$

which after averaging over **n** using

$$\int n_in_jd\Omega=\frac{4\pi}{3}\delta_{ij} \tag{35}$$

leads to the already stated result displayed in Eq. (25).

In getting Eq. (34) one makes use of the fact that in our case $\mathbf{n}\cdot\mathbf{H}=0$, while

$$E_i=\frac{1}{rc^3}\dddot{T}_i+\frac{1}{r^2c^2}\ddot{T}_i+\frac{1}{r^3c}\dot{T}_i-n_in_j\left[\frac{1}{rc^3}\dddot{T}_j+\frac{3}{r^2c^2}\ddot{T}_j+\frac{3}{r^3c}\dot{T}_j\right] \tag{36}$$

and therefore

$$\mathbf{n}\cdot\mathbf{E}=-\frac{2}{r^2c^2}(\mathbf{n}\cdot\ddot{\mathbf{T}})-\frac{2}{r^3c}(\mathbf{n}\cdot\dot{\mathbf{T}}), \tag{37}$$

$$\mathbf{n}\times\mathbf{E}=\mathbf{n}\times\left[\frac{1}{rc^3}\dddot{\mathbf{T}}+\frac{1}{r^2c^2}\ddot{\mathbf{T}}+\frac{1}{r^3c}\dot{\mathbf{T}}\right]. \tag{38}$$

(The argument of **T** in these equations is $t_0=t-r/c$.)

At this point we find it worth recalling that the total intensity of the radiation emitted by a time-dependent toroidal dipole $\mathbf{T}(t)$ is [2]

$$I=\frac{2}{3c^5}|\dddot{\mathbf{T}}|^2. \tag{39}$$

If besides the toroidal dipole there is also the usual electric dipole $\mathbf{d}(t)$, the above formula goes into [2]

$$I=\frac{2}{3c^3}\left|\ddot{\mathbf{d}}-\frac{1}{c}\dddot{\mathbf{T}}\right|^2. \tag{40}$$

From Eq. (25) one sees that just as in the case of the usual electric dipole, for a linear oscillating toroidal dipole $\mathbf{T}=\mathbf{T}_0\cos\omega t$ with a constant amplitude $\mathbf{T}_0$, there will be no loss of angular momentum through radiation of electromagnetic vawes. By comparing Eqs. (25) and (24), one notes also that higher time derivatives enter the formula for $d\mathbf{M}/dt$ in the toroidal case, reflecting the situation that albeit a dipole characteristic, the toroidal dipole moment comes along with the usual electric and magnetic quadrupole moments in the multipole expansion.

If the toroidal dipole performs a *forced* precession around a given axis (specified by the unit vector **e**) with a constant angular velocity ω, one will have

$$\dot{\mathbf{T}}=\boldsymbol{\omega}\times\mathbf{T},\quad|\mathbf{T}|=T_0=\text{const},\quad\boldsymbol{\omega}=\omega\mathbf{e}. \tag{41}$$

We speak of a *forced* precession since unlike the case of a magnetic dipole *automatically* precessing in a constant external magnetic field, in general there is no Larmor theorem analog for a toroidal dipole in a constant current density, unless for some other reason **T** stays along the angular momentum vector. *Forced* precession means that one rotates the toroidal dipole ''by hand,'' i.e., on account of some other external factors. Anyway, if Eq. (41) is verified, a simple calculation shows that the angular momentum loss will be given by the formula

$$\frac{d\mathbf{M}}{dt}=-\frac{2\omega^2}{3c^5}(\boldsymbol{\omega}\times\mathbf{T})^2\cdot\boldsymbol{\omega}=-\frac{2\omega^5}{3c^5}(\mathbf{e}\times\mathbf{T})^2\cdot\mathbf{e}. \tag{42}$$

If the toroidal dipole rotates in a plane, $(\boldsymbol{\omega}\times\mathbf{T})^2=\omega^2T^2$ and Eq. (42) particularizes to

$$\frac{d\mathbf{M}}{dt}=-\frac{2\omega^5T^2}{3c^5}\mathbf{e}. \tag{43}$$

In the case of a nonuniform (forced) precession arround a *fixed* axis specified by the unit vector **e**, going with a time-dependent angular velocity $\omega(t)$,

$$\dot{\mathbf{T}}=\omega(t)\mathbf{e}\times\mathbf{T}, \tag{44}$$

the rate of angular momentum loss will be

$$\frac{d\mathbf{M}}{dt}=-\frac{2}{3c^5}(\mathbf{e}\times\mathbf{T})^2(\omega^5+3\omega\dot{\omega}^2-\ddot{\omega}\omega^2)\mathbf{e}. \tag{45}$$

We have displayed this result in order to stress the difference with respect to the corresponding known expression for the electric dipole case, when the angular momentum loss is given by

$$\frac{d\mathbf{M}}{dt}=-\frac{2}{3c^3}(\mathbf{e}\times\mathbf{d})^2\omega^3\mathbf{e} \tag{46}$$

irrespective of whether the precession arround the fixed axis **e** is uniform ($\omega=$const) or not $[\omega=\omega(t)]$. As seen from the equations above, in the toroidal dipole case the angular momentum loss depends, in general, on the derivatives $\dot{\omega},\ddot{\omega}$, while for the electric dipole there is no analogous sensitivity. We note further that there is a certain particular nonuniform precession, namely, that given by

$$\overline{\omega}(t)=\omega_0\left(\frac{t}{T_0}+1\right)^{-1/2} \tag{47}$$

with ω_0,T_0 constants, for which the last two terms in the right-hand side (rhs) of Eq. (45) cancel since $\overline{\omega(t)}$ as given by Eq. (47) is the general solution of the differential equation

$$3\dot{\omega}^2=\ddot{\omega}\omega. \tag{48}$$

Therefore in this case one has

$$\frac{d\mathbf{M}}{dt}=-\frac{2}{3c^5}(\mathbf{e}\times\mathbf{d})^2\overline{\omega}^5\mathbf{e} \tag{49}$$

which formally coincides with the expression corresponding to the uniform case Eq. (43), but with the obvious difference, however, that $\overline{\omega}$ is now time dependent, as given by Eq. (47).

Moreover, it is, perhaps, worth noting that there are particular nonuniform precessions for which there is no loss at all of angular momentum through radiation and this happens on account of more profound arguments rather than because of purely kinematical reasons. Indeed, if the nonuniform precession is such that $\omega(t)$ verifies the nonlinear second-order differential equation

$$\ddot{\omega}\omega-3\dot{\omega}^2-\omega^4=0, \tag{50}$$

then the whole rhs of Eq. (45) vanishes altogether and $d\mathbf{M}/dt=0$. Equation (50) can be immediately integrated by rewriting it in the form

$$\frac{d}{dt}\left(\frac{\dot{\omega}}{\omega^3}\right)=1, \tag{51}$$

wherefrom the general solution comes out as

$$\omega(t)=\omega_0\left(1-\frac{t}{\tau}-\omega_0^2t^2\right)^{-1/2} \tag{52}$$

with the two integration constants ω_0,τ denoting the initial angular velocity and a certain time scale, respectively. The fact that a rotating toroidal dipole in certain unusual circumstances may not lose angular momentum through radiation distinguishes it neatly from the electric dipole and may have some consequences.

In this section we have insisted on the comparison with the known situation for the case of the electric dipole since it is a general fact that toroidal multipoles (although related to the current density distribution inside the source) emit electric-type radiation, like the usual electric multipoles (but with a different frequency and phase content) [2,3]. To avoid confusion, we recall again (see Refs. [2,3,8]) that while there are three types of sources (electric, magnetic, toroidal), there are, of course, only two types of radiation (the usual electric El and magnetic Ml waves).

IV. DISCUSSION

As is well known, Larmor's theorem essentially says that a system of charges, all of them with the same e/m ratio and experiencing finite movements in, e.g., a centrally symmetric electric field and in a weak constant magnetic field $\mathbf{H}^{\text{ext}}$, in the nonrelativistic case, behaves just as the same system will do in the absence of the magnetic field, but in a coordinate system rotating uniformly with the angular velocity (Larmor frequency)

$$\mathbf{\Omega}=\frac{e\mathbf{H}}{2mc}. \tag{53}$$

This mainly happens because under the mentioned conditions the system's total angular momentum

$$\mathbf{M}=\sum \mathbf{r}\times\mathbf{p} \tag{54}$$

is proportional with the system's magnetic dipole moment Eq. (9)

$$\mathbf{m}=\frac{e}{2mc}\mathbf{M}. \tag{55}$$

Thus, after time averaging, one easily *proves* that both $\overline{\mathbf{M}}$ and $\overline{\mathbf{m}}$ satisfy

$$\frac{d\overline{\mathbf{m}}}{dt}=\mathbf{\Omega}\times\overline{\mathbf{m}} \tag{56}$$

i.e., the vector **m** undergoes a precession arround $\mathbf{H}^{\text{ext}}$, with the period $2\pi/\Omega$ ($|\overline{\mathbf{m}}|$ and the angle between $\overline{\mathbf{m}}$ and $\mathbf{H}^{\text{ext}}$ remain constants).

In turn, when one deals with movements in a magnetic field configuration with $\nabla\times\mathbf{H}^{\text{ext}}$ constant rather than $\mathbf{H}^{\text{ext}}$, in general one cannot straightforwardly get for the average toroidal moment of the system a time evolution equation like Eq. (56), although now the averaged toroidal moment expresses, according to Eq. (8), the average momentum of forces acting on the system in the same way as did the magnetic moment in the constant magnetic field case [Eq. (22)]. This occurs because one lacks, in general, the proportionality between the toroidal dipole moment **T** and the mechanical angular momentum **M** analogous to Eq. (55) for the magnetic moment. That is why in the previous section, when using a precessionlike formula for **T** of the type of Eq. (56), we spoke of a ''forced'' precession. Although the analogy with Larmor's precession, in general, cannot be pushed any further, this does not mean that there are no interesting particular cases when one can. Among them we note that of elementary quantum objects. Due to angular momentum quantization, the toroidal moment, if the object possesses one as an intrinsic electromagnetic characteristic, must lie on the spin direction. For elementary particles, nuclei and atoms, small electroweak parity violating interactions give rise to very small intrinsic toroidal moments, mostly still undetected, with the possible exception of that for the nucleus ^{133}Cs of Ref. [10]; for molecules (chiral [13] or even heteronuclear polar diatomic [14]) toroidal moments can appear on account of the usual parity conserving electromagnetic interaction because of the structural intricacy of the system and are more orders of magnitude larger than those previously mentioned, albeit still small enough. Despite the smallness of the effect we find it useful to dwell below on the possibility of introducing a new Larmor-type frequency analogous to the usual Larmor frequency Eq. (53), but associated this time with constant current densities, or, more generally, with constant $\nabla\times\mathbf{H}^{\mathrm{ext}}$, instead of constant magnetic fields. So, whenever the physical system under consideration is such that its toroidal dipole moment $\mathbf{T}$ is proportional to its angular momentum $\mathbf{M}$,

$$\mathbf{T}=\lambda\mathbf{M}, \tag{57}$$

because of the relationship between the time variation of $\mathbf{M}$ and the total moment of forces $\mathbf{K}$ acting on the system

$$\frac{d\mathbf{M}}{dt}=\mathbf{K}, \tag{58}$$

it follows [on account of Eq. (8) obtained in Sec. II] that in the presence of a magnetic field of constant curl [Eq. (5)]

$$\nabla\times\mathbf{H}^{\mathrm{ext}}\equiv\frac{4\pi}{c}\mathbf{I} \tag{59}$$

(in the static case $\mathbf{I}$ is just a constant conduction current density; otherwise, $\mathbf{I}$ may include a displacement current density too, i.e., an external electric field with a linear time dependence) one will have

$$\frac{d\mathbf{T}}{dt}=\lambda\mathbf{K}=\frac{4\pi}{c}\lambda\mathbf{T}\times\mathbf{I}. \tag{60}$$

In other terms, the toroidal dipole moment $\mathbf{T}$ will precess around the direction of the total (constant) current density $\mathbf{I}$ with an angular velocity $\mathbf{\Omega}_T$ given by

$$\mathbf{\Omega}_T=\frac{4\pi\lambda}{c}\mathbf{I}. \tag{61}$$

In getting the precession formula

$$\frac{d\mathbf{T}}{dt}=\mathbf{T}\times\mathbf{\Omega}_T \tag{62}$$

we have omitted the bars denoting time averages when making an appeal to Eq. (8) of Sec. II because of the irrelevance of this point in the discussion presented in this section. Now, while the usual Larmor frequency [Eq. (53)] is related to the constant magnetic field through a proportionality factor $e/2mc$ basically expressed in terms of essentially fundamental constants, the new frequency Ω_T of Eq. (61) above loses much of the generality of Larmor's precession. However, we would like to stress here that a certain amount of generality is still contained in Eq. (61) in the following sense and in the following cases.

Since Zeldovich's pioneering paper [1] in which the ‘‘anapole’’ (i.e., essentially the toroidal dipole we are dealing with) was introduced, it was known that a typical scale for the toroidal moment of any particle should be

$$\tau_0\equiv\frac{eG_F}{4\pi\hbar c}\approx 0.36\times 10^{-33}e\ \mathrm{cm}^2 \tag{63}$$

since the new type of electromagnetic interaction envisaged by Zeldovich (parity violating but preserving invariance under time reversal) should have been, in a way, ‘‘half’’ electromagnetic and ‘‘half’’ weak, wherefrom the appearance of the electron's charge e and Fermi constant G_F in Eq. (63) was just what one had to expect. The utility of the universal scale set (essentially) by Eq. (61) was recently reemphasized in Ref. [12]. Nowadays, on the basis of the Glashow-Salam-Weinberg theory one not only knows that indeed each elementary particle must have such an intrinsic toroidal dipole, but, moreover, one has at hand the necessary means to calculate it. Estimates have indeed been obtained in various cases, which do confirm the scale provided by τ_0 [Eq. (63)], which we shall call (quite improperly, perhaps, but useful to spare words) a ‘‘toroidon,’’ by analogy with the word ‘‘magneton.’’ A superficial argument in favor of such a name is that τ_0 fixes the value of λ in the proportionality relation (57) if one sets the angular momentum $M=\hbar$, with the result

$$\lambda=\frac{\tau_0}{\hbar}, \tag{64}$$

in the same way as the gyromagnetic relation Eq. (55) sets the value of the Bohr's magneton $\mu_0=e\hbar/2mc$ when the angular momentum quantum unit $\hbar$ is used. All proportions preserved regarding the ‘‘toroidon’’ (unlike the magneton, it only serves as a scale to measure toroidal moments of actual physical systems), one cannot refrain from mentioning that it nonetheless has an advantage over the magneton since it does not contain the mass of the particle (which leads to Bohr's magnetons, nuclear magnetons, etc.) but is expressed entirely in terms of the system-independent fundamental constants $e,G_F,\hbar,c$. Note also that while for magnetons the Planck constant $\hbar$ appears in the numerator, for the ‘‘toroidon’’ $\hbar$ appears in the denominator; however, this occurrence could be only formal in nature and devoid of any physical relevance, since G_F is still a phenomenological constant, so that room may be left for other alternatives.

The precession frequency for the ‘‘toroidon’’ is given by

$$\mathbf{\Omega}_{\tau_0}=\frac{eG_F}{\hbar^2c^2}\mathbf{I}=\frac{\tau_0}{\hbar}\frac{4\pi}{c}\mathbf{I}=\frac{\tau_0}{\hbar}\nabla\times\mathbf{H}^{\rm ext}. \tag{65}$$

To substantiate the usefulness of the ''toroidon'' notion as a unity to measure the toroidal moments of elementary particles, nuclei, atoms, etc., we mention some typical numerical estimates obtained so far in the literature. We take them from the review given in Ref. [12] to which we send the reader for some details on the methods used in their derivation and corresponding references. Denoting by τ (with adequate labels) the various toroidal moments of interest, one has in terms of τ_0

$$\tau_{\rm electron}\sim\tau_0,\quad \tau_{\rm nucleon}\sim 0.2\tau_0,$$

$$\tau_{\rm nucleus}\sim 0.2A^{2/3}\tau_0,\quad \tau_{\rm deuterium(2S\,state)}\sim 644\tau_0,$$

$$\tau_{\rm cesium\ atom\ (ground\ state)}\sim 8400\tau_0. \tag{66}$$

To avoid confusion, we note the difference in the definition of the toroidal dipole moment $\mathbf{T}$ as it appears in Eq. (1) of our paper and the one in Ref. [12], for instance, where one uses the ''anapole'' moment defined as

$$\mathbf{a}=-(\pi/c)\int r^2\mathbf{J}(\mathbf{r})d^3\mathbf{r}. \tag{67}$$

In the static case (i.e., when there is no time dependence in the charge (ρ) and current ($\mathbf{j}$) densities), they are equivalent (up to a factor of 4π),

$$\mathbf{T}=\frac{1}{4\pi}\mathbf{a}, \tag{68}$$

because in the static case $\nabla\mathbf{j}=0$ and the piece $\mathbf{x}(\mathbf{xj})$ under the integral sign in the definition of $\mathbf{T}$ can be brought to $r^2\mathbf{j}$ using

$$\int d^3r\frac{\partial}{\partial x_m}(x_ix_kx_lj_m)=0. \tag{69}$$

The anapole definition ($\mathbf{a}$, above) presents, in our view, two drawbacks: One is serious, because in the nonstationary case (when the continuity relation $\partial\rho/\partial t+\nabla\mathbf{j}=0$ holds) $\mathbf{a}$ does not posess a clear-cut multipole content and effects from various types of multipoles may get mixed; the other less serious but still inconvenient drawback is that of introducing unnatural factors of 4π with respect to the other usual electric and magnetic multipoles.

It is worth noting that unlike magnetons, the ''toroidons'' of the electron, nucleons, and nuclei are more or less of the same order of magnitude, despite the large differences in mass (which did set the scale, e.g., for Bohr's and nuclear magnetons).

In this respect we note the strange coincidence found in Ref. [11] between the values of the ground-state toroidal dipole moment for positronium and the part of the toroidal dipole moment of the ground state in hydrogen induced by the spin-independent part of the parity nonconserving interaction, which also hints to a certain kind of mass insensitivity.

We recall that in the case of Majorana particles the toroidal moments and distributions are the only intrinsic electromagnetic characteristics left [8] and for spin 1/2, the toroidal dipole moment could still be nonvanishing even for zero mass particles [17]. Thus the results of the present paper might be perhaps of some relevance for neutrinos and in some astrophysical considerations.

As shown in Refs. [13,14], toroidal moments much larger than the ''toroidon'' scale could arise in chiral molecules and even in heteronuclear diatomic (polar) molecules. This time the effect is not due to parity violation at the fundamental level (in the Hamiltonian) but comes out as a result of the usual electromagnetic forces in electronic systems possessing certain types of complexity such that, e.g., a pseudoscalar can be formed with the parameters available in the problem. The appearing toroidal moments (now of pure electromagnetic origin) are of the order $\tau\sim(4\pi)^{-1}\alpha ea_0^2$ (α is the fine structure constant, a_0 is the Bohr radius) and are, of course, many orders of magnitude larger than τ_0. Two ways of detecting them have been sugested in Ref. [13]. The immersion in a current density would be realized either by slow electron scattering (when an asymmetry is to be expected in the scattering of electrons from a sample of polarized molecules), or in conducting solutions of chiral radicals by an applied voltage. As noted in Ref. [13] the effects are still very small, but apparently not entirely hopeless. Anyway, the results derived in our paper may be of some help in such studies, as well as in the different context of toroidal moments in condensed matter physics, where new investigations have been recently undertaken (toroidal excitation of nuclear magnetic resonance, toroidal moments in aggregate magnetic fluids, etc. [15]).

Note: For clarity purposes some comments about the form of the vector potential given in Eq. (3) may be in order. As shown in the last of Ref. [8] (since we deal with sources nonvanishing at infinity) Eq. (6) is still not fixing uniquely the gauge. We could have worked as well, for instance, with the new vector potential

$$A_i'=A_i^{\rm ext}-\frac{\partial\chi}{\partial x_i} \tag{70}$$

with χ taken, for instance, as

$$\chi=\frac{4\pi}{c}|\mathbf{I}|\frac{\mathbf{nr}}{20}\left[r^2-\frac{5}{3}(\mathbf{nr})^2\right], \tag{71}$$

$$\mathbf{n}=\frac{\mathbf{I}}{|\mathbf{I}|},\quad \Delta\chi=0, \tag{72}$$

when

$$A_i' = -\frac{4\pi}{c}|\mathbf{I}|\frac{n_i}{4}[r^2 - (\mathbf{nr})^2], \tag{73}$$

$$\nabla\times\mathbf{H}^{\mathrm{ext}} = \nabla\times\nabla\times\mathbf{A}' = \frac{4\pi}{c}\mathbf{I}. \tag{74}$$

Now the vector potential $\mathbf{A}'$ and the constant current density are collinear. The results found in Sec. II of this paper were obtained operating only with the magnetic field and the gauge freedom still existing in Eq. (3) has, of course, no influence on them, as it must.

[1] Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **33**, 1531 (1957) [Sov. Phys. JETP **6**, 1184 (1958)].

[2] V. M. Dubovik and A. A. Tcheshkov, Fiz. Elem. Chastits At. Yadra **5**, 791 (1974) [Sov. J. Part. Nucl. **5**, 318 (1974)]; V. M. Dubovik and L. A. Tosunian, *ibid.* **14**, 1193 (1983) [*ibid.* **14**, 504 (1983)].

[3] V. M. Dubovik and V. V. Tugushev, Phys. Rep. **187**, 145 (1990).

[4] I. B. Khriplovich, *Parity Nonconservation in Atomic Phenomena* (Nauka, Moskow, 1988; Gordon and Breach, New York, 1991).

[5] V. L. Ginzburg and V. N. Tsytovich, Zh. Eksp. Teor. Fiz. **88**, 84 (1985) [Sov. Phys. JETP **61**, 48 (1985)].

[6] G. N. Afanasev, Fiz. Elem. Chastits At. Yadra **21**, 172 (1990) [Sov. J. Part. Nucl. **21**, 74 (1990)]; *ibid.* **23**, 1264 (1992) [*ibid.* **23**, 552 (1992)].

[7] G. N. Afanasev and V. M. Dubovik, in *New Trends in Theoretical and Experimental Nuclear Physics* (World Scientific, Singapore, 1992).

[8] E. E. Radescu, Phys. Rev. D **32**, 1266 (1985); JINR-Dubna Communication Report No. E4-85-165 (1985); Rev. Roumaine Physique **31**, 139 (1986); **31**, 143 (1986); **31**, 145 (1986); A. Costescu and E. E. Radescu, Phys. Rev. D **35**, 3496 (1987); Ann. Phys. (N.Y.) **209**, 13 (1991).

[9] E. E. Radescu, in *New Trends in Theoretical and Experimental Nuclear Physics*, (World Scientific, Singapore, 1992).

[10] M. C. Noecker, B. P. Masterson, and C. E. Wieman, Phys. Rev. Lett. **61**, 310 (1988); C. S. Wood, S. C. Bennett, D. Cho, B. P. Masterson, J. L. Roberts, C. E. Tanner, and C. E. Wieman, Science **275**, 1759 (1997).

[11] E. R. Boston and P. G. H. Sandars, J. Phys. B **23**, 2663 (1990); E. R. Boston A. C. Hartley, and P. G. H. Sandars, J. Phys. B **24**, 2877 (1991).

[12] R. R. Lewis, Phys. Rev. A **48**, 4107 (1993).

[13] I. B. Khriplovich and M. E. Pospelov, Z. Phys. D **17**, 81 (1990).

[14] R. R. Lewis, Phys. Rev. A **49**, 3376 (1994).

[15] V. M. Dubovik *et al.*, JINR-Dubna Report No. P14-92-321 (1992); Report No. P6-93-244 (1993); Report No. P17-93-455, 456 (1993); Report No. E14-93-245 (1993).

[16] L. D. Landau and E. M. Lifschitz, *The Classical Theory of Fields*, Course of Theoretical Physics Vol. 2 (Pergamon Press, New York, 1983), 4th revised English ed.

[17] F. Boudjema, C. Hamzaoui, V. Rahal, and H. C. Ren, Phys. Rev. Lett. **62**, 852 (1989).