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Cylindrical solitary waves and their interaction in Bénard-Marangoni layers

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We consider a motionless Newtonian and incompressible shallow liquid layer bounded below by a bottom plate where temperature is held fixed and above with a free deformable surface whose tension is linearly temperature dependent and on which the heat flux is fixed. When the dimensionless surface tension gradient, measured by a Marangoni number, slightly exceeds its critical value, radially symmetric, long-wavelength excitations obey a dissipative cylindrical Korteweg–de Vries (DCKdV) equation. A dissipative cylindrical Kadomtsev-Petviashvili (DCKP) equation is also derived for nearly radially symmetric disturbances. Exact solitary wave solutions of the DCKP equation are found and the solitary wave solutions of the DCKdV equation are also discussed. Finally the head-on collision between two concentric cylindrical solitary waves is considered and its solitonic character is displayed. [S1063-651X(98)10304-5]

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I. INTRODUCTION

Solitons in physical systems without dissipation and effects of geometrical distortion have been intensively studied. It was shown that the evolution of weakly nonlinear, long shallow water waves satisfies the Korteweg–de Vries (KdV) equation [1] or the Kadomtsev-Petviashvili (KP) equation [2]. When the effects of geometrical distortion on propagating gravity waves, capillary waves, or gravity-capillary waves are comparable with those of amplitude nonlinearity and phase dispersion, it is not uncommon to find that its mathematical description can be reduced to some form of variable-coefficient KdV equation [3]. For radially propagating waves, a cylindrical Korteweg–de Vries (CKdV) equation was derived by Maxon and Viecelli [4] for ion-acoustic waves in collisionless plasma and by Miles [5] for free-surface gravity waves in an ideal, viscous-free fluid. Cylindrical solitons of the CKdV type have been realized in a number of collisionless plasma and water wave experiments, and the agreement with theory is generally good [6]. The cylindrical Kadomtsev-Petviashvili (CKP) and its higher-order generalization also have been obtained in Refs. [7,8] for inviscid surface waves in shallow water. An internal-wave CKdV equation for an ideal, density-stratified fluid has also been derived recently [9]. Cylindrical solitons of KdV type in the vortex dynamics of an ultraclean type-II superconductor have also been defined [10]. Here we explore the possibility of cylindrical solitons excited in *driven-dissipative* systems.

In recent years, attention has been paid to oscillatory instabilities in fluid layers. One of the interesting examples is Bénard-Marangoni convection [11–18]. When heating an open horizontal liquid layer from the air side, as a first instability one expects oscillatory Marangoni-Bénard convection while in the opposite case we expect a stationary convective pattern (Bénard cells) [13–17]. The excitations generated by such an instability may be capillary-gravity waves with long wavelength sustained by the Marangoni thermocapillary effect [15]. In Cartesian geometry, it has been shown that the nonlinear evolution of the surface waves in shallow liquid layers obeys a dissipative KdV equation when the Marangoni number of the system slightly exceeds the instability threshold [16–18]. These studies predicted the existence of KdV-like solitary waves in driven-dissipative systems and already have promising qualitative agreement with experimental results [19–22]. Figure 1 is an illustration of the time evolution of a cylindrical solitary wave in a Bénard-Marangoni geometry [23]. The wave is obtained by adsorbing and subsequently absorbing pentane vapor on a shallow toluene liquid layer. The experimental setup for observing such a wave can be found in Ref. [22]. Unfortunately there is not enough quantitative analysis to allow comparison with theory. As experiments are under way, in this paper we shall address theoretical questions about cylindrical solitary wave

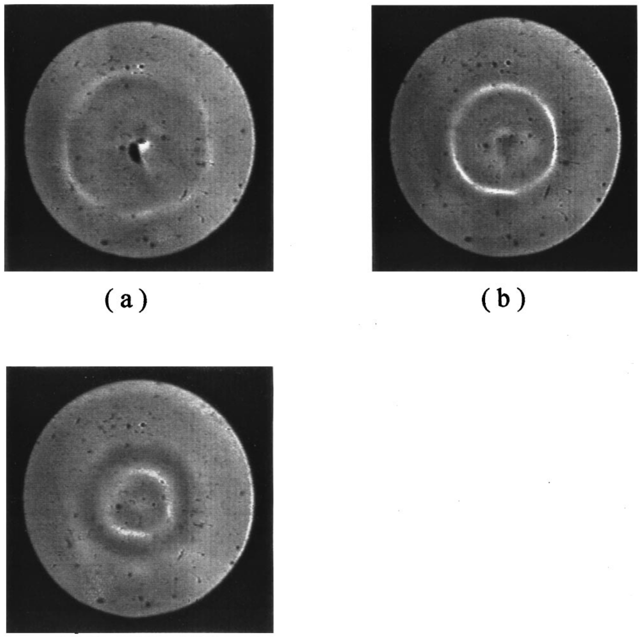

FIG. 1. Surface tension gradient-driven (Marangoni stress) cylindrical solitary wave traveling inward in a Bénard layer where pentane vapor is absorbed on liquid toluene. The white area (ring) represents the amplitude peak (surface deformation and pentane concentration) of the solitary wave on the surface of the toluene layer. The time difference between the images from (a) to (c) is 0.25 s. Pictures obtained by shadowgraph method shining light from below.

in Bénard-Marangoni convection that hopefully would be addressed by the experiments in the near future.

We extend the analysis used in Refs. [4,8] to a driven-dissipative system to investigate the radially or nearly radially propagating capillary-gravity waves in the Marangoni-Bénard convection. In Sec. II we give the formulation of the problem, including the basic equations and the boundary conditions in cylindrical coordinates. In Sec. III we introduce an asymptotic expansion for long-wavelength excitations and derive a dissipative cylindrical Korteweg–de Vries (DCKdV) and a dissipative cylindrical Kadomtsev-Petviashvili (DCKP) equation for cylindrical and nearly cylindrical symmetric disturbances, respectively. In Sec. IV we give some exact solitary wave solutions of the DCKP equation and the qualitative feature of solitary wave solutions for the DCKdV equation is also discussed. In Sec. V we study the head-on collision of two concentric cylindrical solitary waves. Finally, in the last section a summary of our results and a discussion of their relevance to experiment is provided.

II. FORMULATION OF THE PROBLEM

Let us consider an infinitely extended horizontal layer of a Newtonian incompressible liquid open to air in a cylindrical geometry. The bottom of the layer ($z=0$) is flat while its upper surface $[z=d+h(r,\phi,t)]$ is deformable and is subjected to a constant heat flux, i.e., $(\mathbf{n}\cdot\boldsymbol{\nabla})T=\beta=\text{const}$ on the surface. Here z, r, ϕ, and t are vertical, radial, polar angle, and time coordinates, respectively; d is the depth of the liquid at rest; h is the time- and space-dependent surface deformation; T is temperature and $\mathbf{n}$ is the normal outward unit vector to the upper surface. We assume that d is small, which means that we have a *shallow* liquid layer, hence the possible motion at the open air-liquid surface is dominated by the surface tension gradient (Marangoni effect). Thus we neglect buoyancy but not gravity kept at the air-liquid interface and take all material properties of the liquid, like its density ρ, viscosity μ, and thermal diffusivity χ as constants (Boussinesq approximation), except for the surface tension σ which linearly varies with the temperature: $\sigma=\sigma_0+\gamma(T-T_0)(\gamma<0)$.

We start with a motionless conducting base state $\mathbf{v}_s=\mathbf{0}$, $T_s(z)=T_0+\beta(z-d)$, and $p_s(z)=p_0-\rho g(z-d)$, where T_0 and p_0 are reference values of the temperature and pressure, respectively. We introduce dimensionless quantities using suitable scales: d for length, χ/d for velocity, d^2/χ for time, $\mu\chi/d^2$ for pressure, and βd for temperature. Then in cylindrical coordinates the equations of motion governing disturbances upon the base state are

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\phi}{\partial\phi}+\frac{\partial v_z}{\partial z}=0, \tag{1}$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial v_r}{\partial t}-\frac{v_\phi^2}{r}\right)=-\frac{\partial p}{\partial r}+\Delta v_r-\frac{2}{r^2}\frac{\partial v_\phi}{\partial\phi}-\frac{v_r}{r^2}, \tag{2}$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial v_\phi}{\partial t}+\frac{1}{r}v_r v_\phi\right)=-\frac{1}{r}\frac{\partial p}{\partial\phi}+\Delta v_\phi+\frac{2}{r^2}\frac{\partial v_r}{\partial\phi}-\frac{v_\phi}{r^2}, \tag{3}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial v_z}{\partial t}=-\frac{\partial p}{\partial z}+\Delta v_z, \tag{4}$$

$$\frac{\partial T}{\partial t}=\Delta T-v_z, \tag{5}$$

where $\mathbf{v}=(v_r,v_\phi,v_z)$ represents the dimensionless fluid velocity. Expecting no confusion in the reader, we have used the same notations for the scaled dimensionless variables. The boundary conditions on the open surface $z=1+h(r,\phi,t)$ are

$$\frac{\partial h}{\partial t}+v_r\frac{\partial h}{\partial r}+\frac{v_\phi}{r}\frac{\partial h}{\partial\phi}=v_z, \tag{6}$$

$$\frac{\partial h}{\partial r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial h}{\partial\phi}\frac{\partial T}{\partial\phi}-\frac{\partial T}{\partial z}=1-N, \tag{7}$$

$$\begin{aligned}p=Gh&+\frac{2}{N^2}\Bigg[\left(\frac{\partial h}{\partial r}\right)^2\frac{\partial v_r}{\partial r}+\frac{1}{r^2}\left(\frac{\partial h}{\partial\phi}\right)^2\left(\frac{1}{r}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r}{r}\right)+\frac{\partial v_z}{\partial z}\\&+\frac{1}{r}\frac{\partial h}{\partial r}\frac{\partial h}{\partial\phi}\left(\frac{1}{r}\frac{\partial v_r}{\partial\phi}+\frac{\partial v_\phi}{\partial r}-\frac{v_\phi}{r}\right)\\&-\frac{1}{r}\frac{\partial h}{\partial\phi}\left(\frac{\partial v_\phi}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial\phi}\right)-\frac{\partial h}{\partial r}\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)\Bigg]\\&-[K^{-1}-M(T+h)]\Bigg[\frac{\partial}{\partial r}\left(\frac{\partial N}{\partial(\partial h/\partial r)}\right)+\frac{\partial}{\partial\phi}\left(\frac{\partial N}{\partial(\partial h/\partial\phi)}\right)\\&+\frac{1}{r}\frac{\partial N}{\partial(\partial h/\partial r)}\Bigg],\end{aligned} \tag{8}$$

$$2\frac{\partial h}{\partial r}\left(\frac{\partial v_z}{\partial z}-\frac{\partial v_r}{\partial r}\right)-\frac{1}{r}\frac{\partial h}{\partial \phi}\left(\frac{1}{r}\frac{\partial v_r}{\partial \phi}+\frac{\partial v_\phi}{\partial r}-\frac{v_\phi}{r}\right)-\frac{1}{r}\frac{\partial h}{\partial r}\frac{\partial h}{\partial \phi}\left(\frac{\partial v_\phi}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \phi}\right)+\left[1-\left(\frac{\partial h}{\partial r}\right)^2\right]\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)+MN\left(\frac{\partial T}{\partial r}+\frac{\partial T}{\partial z}\frac{\partial h}{\partial r}+\frac{\partial h}{\partial r}\right)=0, \quad (9)$$

$$\frac{2}{r}\frac{\partial h}{\partial \phi}\left(\frac{\partial v_z}{\partial z}-\frac{1}{r}\frac{\partial v_\phi}{\partial \phi}-\frac{v_r}{r}\right)-\frac{\partial h}{\partial r}\left(\frac{1}{r}\frac{\partial v_r}{\partial \phi}+\frac{\partial v_\phi}{\partial r}-\frac{v_\phi}{r}\right)-\frac{1}{r}\frac{\partial h}{\partial r}\frac{\partial h}{\partial \phi}\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)+\left[1-\frac{1}{r^2}\left(\frac{\partial h}{\partial r}\right)^2\right]\times\left(\frac{\partial v_\phi}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \phi}\right)+MN\frac{1}{r}\left(\frac{\partial T}{\partial \phi}+\frac{\partial T}{\partial z}\frac{\partial h}{\partial \phi}+\frac{\partial h}{\partial \phi}\right)=0, \quad (10)$$

where $N=[1+(\partial h/\partial r)^2+(1/r^2)(\partial h/\partial \phi)^2]^{1/2}$.

The lower boundary is taken as a good conducting solid support where temperature is held fixed and, for simplicity, stress-free for the liquid. Thus on $z=0$ we have

$$v_z=0, \quad \frac{\partial v_r}{\partial z}=0, \quad \frac{\partial v_\phi}{\partial z}=0, \quad T=0. \quad (11)$$

From Eqs. (1)–(11) we see that there are four dimensionless parameters involved in the problem. They are the Prandtl number $\mathrm{Pr}=\nu/\chi$, the Galileo number $G=gd^3/\nu\chi$, the capillary number $K=\mu\chi/\sigma_0 d$, and the Marangoni number $M=-\gamma\beta d^2/\mu\chi$. Note that the choice in the sign of β coupled to $-\gamma$ or simply to $|\gamma|$ leads to the consideration of *positive* Marangoni numbers when heating (cooling) the fluid layer from above (below).

III. ASYMPTOTIC EXPANSION, DISSIPATIVE CKdV AND CKP EQUATIONS

The linear stability analysis [13,14,16,17] shows that the motionless base state is unstable to long-wavelength surface oscillations if $M>M^{(0)}$ where $M^{(0)}$ depends on the boundary condition (B.C.). In order to obtain the nonlinear evolution of the wave excitations, first we consider a disturbance with a radial symmetry, i.e., we let $\partial/\partial\phi=0$, and $v_\phi=0$. We introduce the multiple-scale variables $\xi=\epsilon(r-ct)$ and $r_3=\epsilon^3 r$, and the following scalings:

$$v_r=\epsilon^2(u^{(0)}+\epsilon u^{(1)}+\cdots), \quad v_z=\epsilon^3(w^{(0)}+\epsilon w^{(1)}+\cdots),$$

$$p=\epsilon^2(p^{(0)}+\epsilon p^{(1)}+\cdots), \quad T=\epsilon^3(\theta^{(0)}+\epsilon\theta^{(1)}+\cdots),$$

$$h=\epsilon^2(h^{(0)}+\epsilon h^{(1)}+\cdots), \quad c=c^{(0)}+\epsilon^2 c^{(2)}+\cdots,$$

$$M=M^{(0)}+\epsilon^2 M^{(2)}+\cdots,$$

where ϵ is a small, ordering parameter that later on we shall identify with the deviation of M from its critical value $M^{(0)}$. With these expansions, Eqs. (1)–(11) can then be solved order by order in ϵ. The leading order [$O(1)$ order] gives

$$u^{(0)}=c^{(0)}h^{(0)}(\xi,r_3), \quad w^{(0)}=-c^{(0)}h_\xi^{(0)}z, \quad (12)$$

$$p^{(0)}=Gh^{(0)}, \quad \theta^{(0)}=-c^{(0)}h_\xi^{(0)}T_0(z), \quad (13)$$

where $h^{(0)}(\xi,r_3)$ is a function to be determined in a higher order. The subscripts now represent partial derivatives. The definition of the functions $T_j(z)$ and $P_j(z)$ $(j=0,1,2,\ldots)$ appearing in this and the next orders (see below) has been given in Appendix A.

At order $O(\epsilon)$, we find

$$u^{(1)}=c^{(0)}h^{(1)}-M^{(0)}h_\xi^{(0)}[\tfrac{1}{3}+P_0(z)], \quad (14)$$

$$w^{(1)}=[-c^{(0)}h_\xi^{(1)}+\tfrac{1}{3}M^{(0)}h_{\xi\xi}^{(0)}]z+M^{(0)}h_{\xi\xi}^{(0)}T_0(z), \quad (15)$$

$$p^{(1)}=Gh^{(1)}-2c^{(0)}h_\xi^{(0)}, \quad (16)$$

$$\theta^{(1)}=(-c^{(0)}h_\xi^{(1)}+\tfrac{1}{3}M^{(0)}h_{\xi\xi}^{(0)})T_0(z)+[M^{(0)}+(c^{(0)})^2]h_{\xi\xi}^{(0)}T_1(z), \quad (17)$$

where $h^{(1)}=h^{(1)}(\xi,r_3)$ is left undetermined at this order. The solvability condition $u_z^{(1)}(1)=\int_0^1 dz\, u_{zz}^{(1)}(z)$ determines the critical (phase) wave velocity

$$c^{(0)}=\pm\sqrt{\mathrm{Pr}(G+M^{(0)})}. \quad (18)$$

Clearly, the excitation may propagate outward or inward.

At the $O(\epsilon^2)$ order, the solvability condition $u_z^{(2)}(1)=\int_0^1 dz\, u_{zz}^{(2)}(z)$ gives the critical Marangoni number $M^{(0)}=12$ which shows the quantitatively drastic limitation of the stress-free B.C. (11). The solution in this order is

$$u^{(2)}=c^{(0)}h^{(2)}+c^{(2)}h^{(0)}-c^{(0)}(h^{(0)})^2-c^{(0)}\int d\xi(h_{r_3}^{(0)}+h^{(0)}/r_3)-12h_\xi^{(1)}\left[\frac{1}{3}+P_0(z)\right]-\frac{c^{(0)}}{\mathrm{Pr}}\left\{(3\mathrm{Pr}-4)\left[\frac{1}{3}+P_0(z)\right]+12\left[\frac{2}{15}-P_1(z)\right]\right\}h_{\xi\xi}^{(0)}, \quad (19)$$

$$w^{(2)}=\left[-c^{(0)}h_\xi^{(2)}+4h_{\xi\xi}^{(1)}-c^{(2)}h_\xi^{(0)}+2c^{(0)}h^{(0)}h_\xi^{(0)}+c^{(0)}\left(1+\frac{4}{15\mathrm{Pr}}\right)h_{\xi\xi\xi}^{(0)}\right]z+[12h_{\xi\xi}^{(1)}+c^{(0)}(3-4\mathrm{Pr}^{-1})h_{\xi\xi\xi}^{(0)}]T_0(z)-12c^{(0)}\mathrm{Pr}^{-1}h_{\xi\xi\xi}^{(0)}T_1(z), \quad (20)$$

$$p^{(2)}=Gh^{(2)}-2c^{(0)}h_\xi^{(1)}+(8-K^{-1})h_{\xi\xi}^{(0)}-Gh_{\xi\xi}^{(0)}P_0(z), \quad (21)$$

$$\theta^{(2)}=c^{(0)}h^{(0)}h_{\xi}^{(0)}z+\left[-c^{(0)}h_{\xi}^{(2)}+4h_{\xi\xi}^{(1)}-c^{(2)}h_{\xi}^{(0)}+2c^{(0)}h^{(0)}h_{\xi}^{(0)}+c^{(0)}\left(1+\frac{4}{15\mathrm{Pr}}\right)h_{\xi\xi\xi}^{(0)}\right]T_0(z)+[(12+(c^{(0)})^2)h_{\xi\xi}^{(1)}-4c^{(0)}\mathrm{Pr}^{-1}h_{\xi\xi\xi}^{(0)}]T_1(z)-c^{(0)}\left(12+(c^{(0)})^2+\frac{12}{\mathrm{Pr}}\right)h_{\xi\xi\xi}^{(0)}T_2(z), \quad (22)$$

where $h^{(2)}=h^{(2)}(\xi,r_3)$ is an undetermined function at this order.

At the $O(\epsilon^3)$ order, the solvability condition $u_z^{(3)}(1)=\int_0^1 dz\, u_{zz}^{(3)}(z)$ yields the equation controlling the evolution of the function $h^{(0)}(\xi,r_3)$:

$$\frac{\partial h^{(0)}}{\partial r_3}+\frac{1}{2r_3}h^{(0)}+\alpha_1 h^{(0)}\frac{\partial h^{(0)}}{\partial \xi}+\alpha_2\frac{\partial^3 h^{(0)}}{\partial \xi^3}=0, \quad (23)$$

with

$$\alpha_1=\frac{3(G+8)}{2(G+12)}, \quad (24)$$

$$\alpha_2=\frac{1}{30(G+12)}[24\mathrm{Pr}G+5G+288\mathrm{Pr}+168-5K^{-1}]. \quad (25)$$

Equation (23) is the well known cylindrical Korteweg–de Vries equation. The appearance of such an integrable equation at the threshold of the instability is due to the fact that the energy released by the gradient of surface tension (Marangoni stress) balances exactly the amount of kinetic energy dissipated by heat and viscosity. The solutions for $u^{(3)}$, $w^{(3)}$, $p^{(3)}$, and $\theta^{(3)}$ can be obtained but their concrete expressions are omitted here.

With the above solutions we can go up to $O(\epsilon^4)$ order. In this order, the solvability condition $u_z^{(4)}(1)=\int_0^1 dz\, u_{zz}^{(4)}(z)$ yields the evolution equation for $h^{(1)}$,

$$\frac{\partial h^{(1)}}{\partial r_3}+\frac{1}{2r_3}h^{(1)}+\alpha_1\frac{\partial}{\partial \xi}(h^{(0)}h^{(1)})+\alpha_2\frac{\partial^3 h^{(1)}}{\partial \xi^3}+\alpha_3\frac{\partial^2 h^{(0)}}{\partial \xi^2}+\alpha_4\frac{\partial^4 h^{(0)}}{\partial \xi^4}+\alpha_5\frac{\partial}{\partial \xi}\left(h^{(0)}\frac{\partial h^{(0)}}{\partial \xi}\right)=0, \quad (26)$$

with

$$\alpha_3=\frac{\sqrt{\mathrm{Pr}}M^{(2)}}{6(G+12)}, \quad (27)$$

$$\alpha_4=\frac{1}{105\sqrt{\mathrm{Pr}}(G+12)}[268\mathrm{Pr}+34\mathrm{Pr}^2G+408\mathrm{Pr}^2+44], \quad (28)$$

$$\alpha_5=\frac{4\sqrt{\mathrm{Pr}}}{\sqrt{G+12}}, \quad (29)$$

and α_1 and α_2 have been given in Eqs. (24) and (25). For the surface displacement $h=h^{(0)}+\epsilon h^{(1)}$, by recombining Eqs. (23) and (26), we obtain

$$\frac{\partial h}{\partial r_3}+\frac{1}{2r_3}h+\alpha_1 h\frac{\partial h}{\partial \xi}+\alpha_2\frac{\partial^3 h}{\partial \xi^3}+\epsilon\left[\alpha_3\frac{\partial^2 h}{\partial \xi^2}+\alpha_4\frac{\partial^4 h}{\partial \xi^4}+\alpha_5\frac{\partial}{\partial \xi}\left(h\frac{\partial h}{\partial \xi}\right)\right]=0, \quad (30)$$

which is a dissipative cylindrical Korteweg–de Vries equation. It is a combination of the original cylindrical KdV equation [4,5] and a *cylindrical Kuramoto-Sivashinsky equation*, with the additional term $(hh_{\xi})_{\xi}$, nonlinear consequence of the Marangoni effect [17].

If we look for excitations with a nearly cylindrical symmetry, we can choose the multiple-scale variables as $\xi=\epsilon(r-ct)$, $r_3=\epsilon^3 r$, and $\phi=\Phi_0\Phi$ with $\Phi_0=1/\epsilon$. The dimensionless azimuthal velocity can be expanded as $v_{\phi}=\epsilon^3(v^{(0)}+\epsilon v^{(1)}+\cdots)$. Then up to $O(\epsilon^4)$, for the surface displacement h we have

$$\frac{\partial}{\partial \xi}\left\{\frac{\partial h}{\partial r_3}+\frac{1}{2r_3}h+\alpha_1 h\frac{\partial h}{\partial \xi}+\alpha_2\frac{\partial^3 h}{\partial \xi^3}+\epsilon\left[\alpha_3\frac{\partial^2 h}{\partial \xi^2}+\alpha_4\frac{\partial^4 h}{\partial \xi^4}+\alpha_5\frac{\partial}{\partial \xi}\left(h\frac{\partial h}{\partial \xi}\right)\right]\right\}+\frac{c^{(0)}}{2r_3^2}\frac{\partial^2 h}{\partial \Phi^2}=0, \quad (31)$$

where α_j $(j=1,2,3,4,5)$ are the same as in Eq. (30). Equation (31) is a dissipative cylindrical Kadomtsev-Petviashvili equation [18]. When $\epsilon=0$, i.e., at the threshold of instability, Eq. (31) reduces to the cylindrical Kadomtsev-Petviashvili equation given by Johnson for an ideal, viscous-free fluid [7].

In the (ξ,τ) version, Eq. (31) takes the form

$$\frac{\partial}{\partial \xi}\left\{\frac{\partial h}{\partial \tau}+\frac{1}{2\tau}h+c^{(0)}\alpha_1 h\frac{\partial h}{\partial \xi}+c^{(0)}\alpha_2\frac{\partial^3 h}{\partial \xi^3}+\epsilon\left[c^{(0)}\alpha_3\frac{\partial^2 h}{\partial \xi^2}+c^{(0)}\alpha_4\frac{\partial^4 h}{\partial \xi^4}+c^{(0)}\alpha_5\frac{\partial}{\partial \xi}\left(h\frac{\partial h}{\partial \xi}\right)\right]\right\}+\frac{1}{2\tau^2}\frac{\partial^2 h}{\partial \Phi^2}=0. \quad (32)$$

When deriving Eqs. (30)–(32) we have assumed a stress-free boundary condition for the velocity field on the bottom $z=0$ [see Eq. (11)]. One can replace the stress-free condition on $z=0$ by

$$v_z=0, \quad \frac{\partial v_r}{\partial z}=\delta v_r, \quad \frac{\partial v_0}{\partial z}=\delta v_{\theta}. \quad (33)$$

Equation (33) yields a no-slip condition for $\delta=\infty$ and a stress-free condition for $\delta=0$ [24]. In some cases we can use a slightly perturbed stress-free boundary condition by assuming $\delta\neq 0$, but much smaller than unity. This kind of assumption is still for simplicity in theory. However, besides understanding of relevant qualitative features, even quantitative comparison with an experiment may be possible [24].

When taking $\delta=\delta(\epsilon)=\epsilon^4 b$ with $b=O(1)$, for the surface displacement h we obtain

$$\frac{\partial}{\partial \xi}\left\{\frac{\partial h}{\partial \tau}+\frac{1}{2\tau}h+c^{(0)}\alpha_1 h\frac{\partial h}{\partial \xi}+c^{(0)}\alpha_2\frac{\partial^3 h}{\partial \xi^3}+\epsilon\left[c^{(0)}\alpha_3\frac{\partial^2 h}{\partial \xi^2}+c^{(0)}\alpha_4\frac{\partial^4 h}{\partial \xi^4}+c^{(0)}\alpha_5\frac{\partial}{\partial \xi}\left(h\frac{\partial h}{\partial \xi}\right)+c^{(0)}\alpha_6 h\right]\right\}+\frac{1}{2\tau^2}\frac{\partial^2 h}{\partial \Phi^2}=0, \quad (34)$$

with $\alpha_6=b\sqrt{\mathrm{Pr}}/[2(G+12)]$, i.e., a new term $\epsilon c^{(0)}\alpha_6\partial h/\partial\xi$ is added into Eq. (32).

IV. SOLITARY WAVE SOLUTIONS OF THE DCKP AND THE DCKdV EQUATIONS

The CKdV and the CKP equations in dissipation-free systems admit exact solitary wave solutions and they are shown to be completely integrable in the sense that there is a transformation (e.g., inverse scattering transform) which would convert them to an uncoupled set of ordinary differential equations for the amplitudes and phases of normal modes. Naturally, one can pose the problem of existence of solitons (or nonlinear coherent structures) for the DCKdV equation (30) and the DCKP equation (32) or (34) in order to look for a possible relevance to experiment [22,23]. Unfortunately, solving these equations analytically is exceedingly difficult because they involve nonlinearity, dissipation, dispersion, and variable coefficients. Although we have been unable to obtain an exact solution for Eqs. (30) and (34), we have, however, been able to find exact solitary wave solutions for the DCKP equation (32) by using a variable transformation and the tanh-function ansatz developed in Ref. [25].

In Eq. (32) there are two terms with a variable coefficient: $h_\xi/(2\tau)$ and $h_{\phi\phi}/(2\tau^2)$. We seek a variable transformation which can make these terms cancel each other. This becomes possible if we assume (see Appendix B for the derivation)

$$\zeta=\xi-\tfrac{1}{2}\Phi^2\tau, \quad h=h(\zeta,\tau). \quad (35)$$

Then the DCKP equation (32) is transformed to

$$\frac{\partial h}{\partial \tau}+c^{(0)}\alpha_1 h\frac{\partial h}{\partial \zeta}+c^{(0)}\alpha_2\frac{\partial^3 h}{\partial \zeta^3}+\epsilon\left[c^{(0)}\alpha_3\frac{\partial^2 h}{\partial \zeta^2}+c^{(0)}\alpha_4\frac{\partial^4 h}{\partial \zeta^4}+c^{(0)}\alpha_5\frac{\partial}{\partial \zeta}\left(h\frac{\partial h}{\partial \zeta}\right)\right]=0. \quad (36)$$

Equation (36) is just the dissipative KdV equation [17,18]. The transformation (35) is similar to that used by Johnson for inviscid fluids [7]. The exact solitary wave solutions of Eq. (36) can be obtained by using the tanh-function method (see Appendix B for details). Thus we get the exact solitary wave solutions of the DCKP equation (32) as follows.

(i) The case $\alpha_2\alpha_5\neq\alpha_1\alpha_4$. We have

$$h=A_0+A_1\tanh[k\xi+(\omega-\tfrac{1}{2}k\Phi^2)\tau]+A_2\tanh^2[k\xi+(\omega-\tfrac{1}{2}k\Phi^2)\tau], \quad (37)$$

with

$$A_0=-\frac{\alpha_3}{\alpha_5}+\frac{1}{25\epsilon^2\alpha_4\alpha_5^3}(\alpha_2\alpha_5-\alpha_1\alpha_4)(26\alpha_1\alpha_4-\alpha_2\alpha_5), \quad (38)$$

$$A_1=\pm\frac{6(\alpha_2\alpha_5-\alpha_1\alpha_4)^2}{25\epsilon^2\alpha_4\alpha_5^3}, \quad (39)$$

$$A_2=-\frac{6(\alpha_2\alpha_5-\alpha_1\alpha_4)^2}{5\epsilon^2\alpha_4\alpha_5^3}, \quad (40)$$

$$k=\pm\frac{\alpha_2\alpha_5-\alpha_1\alpha_4}{10\epsilon\alpha_4\alpha_5}, \quad (41)$$

$$\omega=\pm\frac{\alpha_1(\alpha_2\alpha_5-\alpha_1\alpha_4)}{40\epsilon^3 c^{(0)}\alpha_4\alpha_5^4}\times\left[4\alpha_3+\epsilon\frac{(c^{(0)})^2}{5\alpha_4}(\alpha_2\alpha_5-\alpha_1\alpha_4)(2\alpha_2\alpha_5-15\alpha_1\alpha_4)\right]. \quad (42)$$

In this case we see that, on the one hand, *the amplitude and wave velocity of the kink-type solitary wave (37) are uniquely determined by the parameters of the system*. The solitary waves excited in integrable systems have the amplitude (and velocity) solely depending on initial conditions. Here, however, as a nonequilibrium driven system although there is no energy conservation yet there is a balance in the energy input-output at a steady state. Energy enters in the long-wave range thus helping create the solitary wave, and leaks out by viscous and heat dissipation in the short-wave range [18]. On the other hand, the solitary wave (37) is ϵ dependent with $A_0\sim\epsilon^{-2}$, $A_1\sim\epsilon^{-2}$, $A_2\sim\epsilon^{-2}$, $k\sim\epsilon^{-1}$, and $\omega\sim\epsilon^{-3}$. Thus the *solitary wave solutions obtained here are nonperturbative*. Furthermore, from Eq. (37) we see that the phase velocity of the solitary wave is angle dependent in the phase. This means that the cylindrical wave will slightly deform as time goes on.

(ii) The case $\alpha_2\alpha_5=\alpha_1\alpha_4$. We get

$$h=A_0+A_2\tanh^2[k\xi+(\omega-\tfrac{1}{2}k\Phi^2)\tau]=A_0+A_2-A_2\,\mathrm{sech}^2[k\xi+(\omega-\tfrac{1}{2}k\Phi^2)\tau], \quad (43)$$

with

$$A_0=-\frac{1}{\alpha_5}(\alpha_3-8\alpha_4k^2), \quad (44)$$

$$A_2=-\frac{12}{\alpha_5}\alpha_4k^2=-\frac{12}{\alpha_1}\alpha_2k^2, \quad (45)$$

$$\omega=\frac{c_0\alpha_2\alpha_3}{\alpha_4}, \quad (46)$$

where k is an arbitrary constant. In this case we have a *bell-type, hump solitary wave*. The solitary wave solution (43) provides a possible explanation of the cylindrical solitary wave observed in experiment (Fig. 1). We see from Fig. 1 that the shape of the cylindrical solitary wave is slightly angle dependent. The exact conoidal wave solutions of the

DCKP equation (32) can also be obtained by using the Weiss-Tabor-Carnevale method [26], but we shall not do it here.

For Eq. (34) we can also use the same transformation (35) to transfer it into

$$\frac{\partial h}{\partial \tau}+c^{(0)}\alpha_1 h\,\frac{\partial h}{\partial \zeta}+c^{(0)}\alpha_2\,\frac{\partial^3 h}{\partial \zeta^3}+\epsilon\left[c^{(0)}\alpha_3\,\frac{\partial^2 h}{\partial \zeta^2}+c^{(0)}\alpha_4\,\frac{\partial^4 h}{\partial \zeta^4}+c^{(0)}\alpha_5\,\frac{\partial}{\partial \zeta}\left(h\,\frac{\partial h}{\partial \zeta}\right)+c^{(0)}\alpha_6 h\right]=0. \tag{47}$$

It is just the dissipative KdV equation found by Rednikov *et al.* from a Bénard-Marangoni convection for Cartesian geometry in the case of a slightly perturbed stress-free boundary condition on the bottom [24,27]. Although we have been unable to obtain an exact solution of Eq. (47) yet, there exist several numerical studies for its solitary wave solutions [27]. The results show that the slightly perturbed stress-free condition (i.e., $\alpha_6\neq 0$) although it helps appearing periodic wave trains does not destroy the solitary wave solutions. Thus we can safely conclude that a cylindrical solitary wave exists when $\alpha_6\neq 0$.

Now we turn our discussion to the solitary wave solutions of the DCKdV equation (30). Using a suitable Galilean transformation and taking appropriate scales for coordinate and time variables, Eq. (30) can be written in the form

$$\frac{\partial h}{\partial t}+h\,\frac{\partial h}{\partial x}+\frac{\partial^3 h}{\partial x^3}+\frac{1}{2t}\,h+\epsilon\left[\frac{\partial^2 h}{\partial x^2}+\frac{\partial^4 h}{\partial x^4}+D\,\frac{\partial^2 (h^2)}{\partial x^2}\right]=0, \tag{48}$$

which is the DCKdV equation in the (x,t) version. Since an exact solution of Eq. (48) has also been found, we turn to a qualitative discussion of its solitary wave solutions.

Consider the time evolution of a solitary wave from $t=-\infty$ to 0. Thus the cylindrical solitary wave moves inward towards the center. When $|t|\gg 1$, the term $h/(2t)$ in Eq. (48) can be neglected. Thus in this case the time evolution of the solitary wave can be depicted by the planar dissipation-modified KdV equation for which some results on solitary wave solutions are already known [18,25–27]. Such flat, stationary solitary waves have the unique amplitude A_s which is related to the parameter D as

$$A_s=\frac{21}{(5-24D)}. \tag{49}$$

Stationary solitary waves exist only if $D<\frac{5}{24}$. When $|t|\sim O(1)$, the cylindrical term in Eq. (48) begins to be essential and makes the amplitude of the solitary waves increase. As the amplitude increases, the velocity of the solitary wave also increases due to the term $h\partial h/\partial x$. The expected behavior has been confirmed by our numerical simulation. At variance with the solitary waves of the standard, dissipation-free CKdV equation, the amplitude, and corresponding velocity and width, of the solitary waves of the DCKdV equation (48) are selected by the fixed physical parameters, and the energy input-output balance in the system.

V. HEAD-ON COLLISION BETWEEN TWO DISSIPATIVE CONCENTRIC CYLINDRICAL SOLITARY WAVES

In conservative systems, one of the striking properties of solitons is their asymptotic preservation of form following a collision as first remarked by Zabusky and Kruskal [28]. This led them to the coinage of soliton. Two distinct types of one-dimensional interaction have been studied. One is the *overtaking* collision and the other one is the *head-on* collision [29,30]. For two-dimensional water waves, the oblique interaction of two Cartesian KdV solitary waves has also been investigated by Miles [31]. The interaction of the Cartesian solitary waves in oscillatory Bénard convection has been recently considered [18,32]. In this section we employ an extended Poincaré-Lighthill-Kuo (PLK) method [32,33] to investigate the head-on collision between two concentric cylindrical surface tension gradient-driven solitary waves.

We consider that two concentric cylindrical solitary waves, of small but finite amplitudes, R and L, have been excited when the Marangoni number of the system slightly exceeds its critical value. The solitary wave $R(L)$ is traveling outward (inward) from the initial point of the coordinate system. The initial position (at time $t=0$) of the solitary wave $R(L)$ is at $r=r_R(r_L)$ with $r_L\gg r_R$. After some time they interact, following a *head-on* collision, and then separate away. We expect that the *head-on* collision will result in phase shifts in their postcollision trajectories. Hence we introduce the following transformation of wave-frame coordinates with the phase functions:

$$\xi=\epsilon(r-c_R t-r_R)+\epsilon^2\Theta^{(0)}(\eta,r_3)+\epsilon^3\Theta^{(1)}(\xi,\eta,r_3)+\cdots, \tag{50}$$

$$\eta=\epsilon(r+c_L t-r_L)+\epsilon^2\Psi^{(0)}(\xi,r_3)+\epsilon^3\Psi^{(1)}(\xi,\eta,r_3)+\cdots, \tag{51}$$

where $r_3=\epsilon^3 r$, and c_R and c_L are constants. $\Theta^{(j)}$ and $\Psi^{(j)}$ $(j=0,1,2,\ldots)$ are functions yet to be determined. Thus for the spatial and temporal derivatives we have

$$\frac{\partial}{\partial r}=\epsilon\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)+\epsilon^3\left[\frac{\partial}{\partial r_3}+\Theta_\eta^{(0)}\,\frac{\partial}{\partial \xi}+\Psi_\xi^{(0)}\,\frac{\partial}{\partial \eta}\right]+\cdots, \tag{52}$$

$$\frac{\partial}{\partial t}=\epsilon\left(-c_R\,\frac{\partial}{\partial \xi}+c_L\,\frac{\partial}{\partial \eta}\right)+\epsilon^3\left[c_L\Theta_\eta^{(0)}\,\frac{\partial}{\partial \xi}-c_R\Psi_\xi^{(0)}\,\frac{\partial}{\partial \eta}\right]+\cdots. \tag{53}$$

Introducing the asymptotic expansion $v_r=\epsilon^2(u^{(0)}+\epsilon u^{(1)}+\cdots)$, $v_z=\epsilon^3(w^{(0)}+\epsilon w^{(1)}+\cdots)$, $p=\epsilon^2(p^{(0)}+\epsilon p^{(1)}+\cdots)$, $T=\epsilon^3(\theta^{(0)}+\epsilon\theta^{(1)}+\cdots)$, $h=\epsilon^2(h^{(0)}+\epsilon h^{(1)}+\cdots)$, $M=M^{(0)}+\epsilon^2 M^{(2)}+\cdots$, and

$$c_R=c^{(0)}(1+\epsilon^2 R^{(2)}+\cdots), \tag{54}$$

$$c_L=c^{(0)}(1+\epsilon^2 L^{(2)}+\cdots), \tag{55}$$

where $R^{(j)}$ and $L^{(j)}$ $(j=2,3,\ldots)$ are constants to be determined in the next orders. With these expansions Eqs. (1)–(11) with $\partial/\partial\phi=0$ and $v_\phi=0$ yield a hierarchy of equations by equating the powers of ϵ. To the first-order approximation $[O(1)]$ we obtain the solution

$$u^{(0)}=c^{(0)}[H_1^{(0)}(\xi,r_3)-H_2^{(0)}(\eta,r_3)], \tag{56}$$

$$w^{(0)}=-c^{(0)}(H_{1,\xi}^{(0)}-H_{2,\eta}^{(0)})T_{-1}(z), \tag{57}$$

$$p^{(0)}=G(H_1^{(0)}+H_2^{(0)}), \tag{58}$$

$$\theta^{(0)}=-c_0(H_{1,\xi}^{(0)}-H_{2,\eta}^{(0)})T_0(z), \tag{59}$$

$$h^{(0)}=H_1^{(0)}+H_2^{(0)}, \tag{60}$$

where $H_1^{(0)}$ and $H_2^{(0)}$ are two functions to be determined in a higher-order approximation. Thus in the leading order we have two cylindrical waves, one of which, $H_1^{(0)}(\xi,r_3)$, is traveling outward, and the other one, $H_2^{(0)}(\eta,r_3)$, is traveling inward.

To the second order, we find

$$u^{(1)}=c^{(0)}(\partial_\xi+\partial_\eta)^{-1}(\partial_\xi-\partial_\eta)h^{(1)}-M^{(0)}(H_{1,\xi}^{(0)}+H_{2,\eta}^{(0)})\times[\tfrac{1}{3}+P_0(z)], \tag{61}$$

$$w^{(1)}=-c^{(0)}(\partial_\xi-\partial_\eta)h^{(1)}T_{-1}(z)+M^{(0)}(H_{1,\xi\xi}^{(0)}+H_{2,\eta\eta}^{(0)})\times[\tfrac{1}{3}T_{-1}(z)+T_0(z)], \tag{62}$$

$$p^{(1)}=Gh^{(1)}-2c^{(0)}(H_{1,\xi}^{(0)}-H_{2,\eta}^{(0)}), \tag{63}$$

$$\theta^{(1)}=-c^{(0)}(\partial_\xi-\partial_\eta)h^{(1)}T_0(z)+(H_{1,\xi\xi}^{(0)}+H_{2,\eta\eta}^{(0)})\times\{\tfrac{1}{3}M^{(0)}T_0(z)+[M^{(0)}+(c^{(0)})^2]T_1(z)\}, \tag{64}$$

where $(\partial_\xi+\partial_\eta)^{-1}$ is the inverse operator of $(\partial_\xi+\partial_\eta)$ and $h^{(1)}(\xi,\eta,r_3)$ is an undetermined function. The solvability condition $u_z^{(1)}(1)=\int_0^1 dz\ u_{zz}^{(1)}(z)$ yields $c^{(0)}=\sqrt{\Pr(G+M^{(0)})}$ (critical wave speed).

At order $O(\epsilon^2)$, by solving the corresponding approximate equations we obtain $u^{(2)}$, $w^{(2)}$, and $p^{(2)}$. The solvability condition $u_z^{(2)}(1)=\int_0^1 dz\ u_{zz}^{(2)}(z)$ yields

$$h^{(1)}=\frac{(12-M^{(0)})\Pr}{12c^{(0)}}\int d\xi\int d\eta(H_{1,\xi\xi\xi}^{(0)}-H_{2,\eta\eta\eta}^{(0)})+H_1^{(1)}(\xi,r_3)+H_2^{(1)}(\eta,r_3), \tag{65}$$

where $H_1^{(1)}$ and $H_2^{(1)}$ are undetermined functions at this order. From Eq. (65) we see that if $h^{(1)}$ is not to be divergent we must set $M^{(0)}=12$ as expected for stress-free B.C. (11). Then we have

$$h^{(1)}=H_1^{(1)}(\xi,r_3)+H_2^{(1)}(\eta,r_3). \tag{66}$$

To order $O(\epsilon^3)$, the solvability condition $u_z^{(3)}(1)=\int_0^1 dz\ u_{zz}^{(3)}(z)$ results in

$$\left(\frac{\partial}{\partial r_3}+\frac{1}{2r_3}\right)+\alpha_1 H_1^{(0)}\frac{\partial}{\partial\xi}H_1^{(0)}+\alpha_2\frac{\partial^3}{\partial\xi^3}H_1^{(0)}=0, \tag{67}$$

$$\left(\frac{\partial}{\partial r_3}+\frac{1}{2r_3}\right)+\alpha_1 H_2^{(0)}\frac{\partial}{\partial\xi}H_2^{(0)}+\alpha_2\frac{\partial^3}{\partial\xi^3}H_2^{(0)}=0, \tag{68}$$

$$\frac{\partial}{\partial\eta}\Theta^{(0)}=\frac{1}{4}\left[1-\frac{\Pr}{(c^{(0)})^2}\left(30-\frac{7}{2}M^{(0)}\right)\right]H_2^{(0)}(\eta,r_3), \tag{69}$$

$$\frac{\partial}{\partial\xi}\Psi^{(0)}=\frac{1}{4}\left[1-\frac{\Pr}{(c^{(0)})^2}\left(30-\frac{7}{2}M^{(0)}\right)\right]H_1^{(0)}(\xi,r_3), \tag{70}$$

$$R^{(2)}=L^{(2)}=\frac{\Pr M^{(2)}}{2(c^{(0)})^2}, \tag{71}$$

where α_1 and α_2 are the same as Eqs. (24) and (25). The function $h^{(2)}$ is given by

$$h^{(2)}=\frac{\Pr}{2(c^{(0)})^2}\left(G+\frac{9}{2}-30\right)H_1^{(0)}(\xi,r_3)H_2^{(0)}(\eta,r_3)+H_1^{(2)}(\xi,r_3)+H_2^{(2)}(\eta,r_3), \tag{72}$$

where $H_1^{(2)}$ and $H_2^{(2)}$ are two functions yet to be determined in higher-order approximations.

In order to obtain a clear physical picture of the collision, we use the asymptotic solutions of the CKdV equations (67) and (68), rather than their exact solutions which involve the Airy functions. For large r_3 we obtain the quasisolitary wave solutions of Eqs. (67) and (68) [6],

$$H_1^{(0)}=A_R\left(\frac{r_{3R}}{r_3}\right)^{2/3}\operatorname{sech}^2\left\{\left(\frac{\alpha_1 A_R}{12\alpha_2}\right)^{1/2}\left(\frac{r_{3R}}{r_3}\right)^{1/3}\times\left[\xi-\alpha_1 A_R\left(\frac{r_{3R}}{r_3}\right)^{2/3}r_3\right]\right\}, \tag{73}$$

$$H_2^{(0)}=A_L\left(\frac{r_{3L}}{r_3}\right)^{2/3}\operatorname{sech}^2\left\{\left(\frac{\alpha_1 A_L}{12\alpha_2}\right)^{1/2}\left(\frac{r_{3L}}{r_3}\right)^{1/3}\times\left[\eta-\alpha_1 A_L\left(\frac{r_{3L}}{r_3}\right)^{2/3}r_3\right]\right\}, \tag{74}$$

where $r_{3R}=\epsilon^3 r_R$ and $r_{3L}=\epsilon^3 r_L$. A_R (A_L) is the amplitude of the cylindrical solitary wave R (L) at the initial position $r=r_R$ $(r=r_L)$. Using Eqs. (69) and (70) we obtain the phase change of the solitary waves due to the collision:

$$\begin{aligned}\Theta^{(0)}&=\Lambda\int_{\eta|_{t=0}}^{\eta}H_2^{(0)}(\eta',r_3)d\eta'\\&=\Lambda\left(\frac{12\alpha_2 A_L}{\alpha_1}\right)^{1/2}\left(\frac{r_{3L}}{r_3}\right)^{1/3}\Bigg(\tanh\left\{\left(\frac{\alpha_1 A_L}{12\alpha_2}\right)^{1/2}\left(\frac{r_{3L}}{r_3}\right)^{1/3}\times\left[\eta-\alpha_1 A_L\left(\frac{r_{3L}}{r_3}\right)^{2/3}r_3\right]\right\}-\tanh\left\{\left(\frac{\alpha_1 A_L}{12\alpha_2}\right)^{1/2}\left(\frac{r_{3L}}{r_3}\right)^{1/3}\times\left[\eta|_{t=0}-\alpha_1 A_L\left(\frac{r_{3L}}{r_3}\right)^{2/3}r_3\right]\right\}\Bigg),\end{aligned} \tag{75}$$

$$\Psi^{(0)}=\Lambda\int_{\eta|_{t=0}}^{\eta} H_1^{(0)}(\xi',r_3)d\xi'$$
$$=\Lambda\left(\frac{12\alpha_2 A_R}{\alpha_1}\right)^{1/2}\left(\frac{r_{3R}}{r_3}\right)^{1/3}\Bigg(\tanh\left\{\left(\frac{\alpha_1 A_R}{12\alpha_2}\right)^{1/2}\left(\frac{r_{3R}}{r_3}\right)^{1/3}\right.$$
$$\left.\times\left[\xi-\alpha_1 A_R\left(\frac{r_{3R}}{r_3}\right)^{2/3} r_3\right]\right\}-\tanh\left\{\left(\frac{\alpha_1 A_R}{12\alpha_2}\right)^{1/2}\left(\frac{r_{3R}}{r_3}\right)^{1/3}\right.$$
$$\left.\times\left[\xi|_{t=0}-\alpha_1 A_R\left(\frac{r_{3R}}{r_3}\right)^{2/3} r_3\right]\right\}\Bigg), \tag{76}$$

where $\xi|_{t=0}=-\eta|_{t=0}=\epsilon(r_L-r_R)$ and

$$\Lambda=\frac{1}{4}\left[1-\frac{\text{Pr}}{(c^{(0)})^2}\left(30-\frac{7}{2}M^{(0)}\right)\right]. \tag{77}$$

From Eqs. (50), (51), (75), and (76) we now estimate the phase shifts in the *head-on* collision process. The phase shift $\Delta_R(\Delta_L)$ for solitary wave $R(L)$ is given by

$$\Delta_R=-\epsilon^2\Lambda\left(\frac{12\alpha_2 A_L}{\alpha_1}\right)^{1/2}\left(\frac{r_L}{r}\right)^{1/3}\Bigg(\tanh\left\{\left(\frac{\alpha_1 A_L}{12\alpha_2}\right)^{1/2}\left(\frac{r_L}{r}\right)^{1/3}\right.$$
$$\left.\times\left[\epsilon(2c^{(0)}t+r_R-r_L)-\epsilon^3\alpha_1 A_L\left(\frac{r_L}{r}\right)^{2/3} r\right]\right\}$$
$$-\tanh\left\{\left(\frac{\alpha_1 A_L}{12\alpha_2}\right)^{1/2}\left(\frac{r_L}{r}\right)^{1/3}\left[\epsilon(r_R-r_L)\right.\right.$$
$$\left.\left.-\epsilon^3\alpha_1 A_L\left(\frac{r_L}{r}\right)^{2/3} r\right]\right\}\Bigg), \tag{78}$$

$$\Delta_L=-\epsilon^2\Lambda\left(\frac{12\alpha_2}{\alpha_1}\right)^{1/2}\left(\frac{r_R}{r}\right)^{1/3}\Bigg(\tanh\left\{\left(\frac{\alpha_1 A_R}{12\alpha_2}\right)^{1/2}\left(\frac{r_R}{r}\right)^{1/3}\right.$$
$$\left.\times\left[\epsilon(-2c^{(0)}t+r_L-r_R)-\epsilon^3\alpha_1 A_R\left(\frac{r_R}{r}\right)^{2/3} r\right]\right\}$$
$$-\tanh\left\{\left(\frac{\alpha_1 A_R}{12\alpha_2}\right)^{1/2}\left(\frac{r_R}{r}\right)^{1/3}\left[\epsilon(r_L-r_R)\right.\right.$$
$$\left.\left.-\epsilon^3\alpha_1 A_R\left(\frac{r_R}{r}\right)^{2/3} r\right]\right\}\Bigg), \tag{79}$$

when returning to the original variables.

If the initial distance between the two solitary waves R and L is large enough, i.e., $r_L-r_R\gg 1$, and the observation time $t\gg t_C=\frac{1}{2}(r_L-r_R)$ (t_C is the collision time), from Eqs. (78) and (79) we have

$$\Delta_R=-\epsilon^2 2\Lambda\left(\frac{12\alpha_2 A_L}{\alpha_1}\right)^{1/2}\left(\frac{r_L}{r}\right)^{1/3}, \tag{80}$$

$$\Delta_L=\epsilon^2 2\Lambda\left(\frac{12\alpha_2 A_R}{\alpha_1}\right)^{1/2}\left(\frac{r_R}{r}\right)^{1/3}, \tag{81}$$

which satisfy

$$\frac{1}{\sqrt{A_L}}\left(\frac{1}{r_L}\right)^{1/3}\Delta_R+\frac{1}{\sqrt{A_R}}\left(\frac{1}{r_R}\right)^{1/3}\Delta_L=0. \tag{82}$$

Equations (80) and (81) show that, asymptotically, *the phase shifts of two colliding concentric cylindrical solitary waves in a head-on collision are proportional to* $r^{-1/3}$ *and depend on their initial positions*, r_R *and* r_L. In the collision process, a phase-conserving relation, Eq. (82), is preserved. The higher-order corrections, not considered here, may give some secondary structures for the wave forms and phase shifts.

VI. DISCUSSION AND SUMMARY

In this paper we have derived a DCKdV equation and a DCKP equation, describing radial and nearly radial symmetric, long-wavelength oscillatory disturbances excited and sustained by Marangoni stresses due to the nonuniform distribution of surface tension along the open surface of a shallow horizontal Bénard liquid layer heated from the air side. Exact solitary wave solutions of the DCKP equation have been found and the solitary wave solutions of the DCKdV equation have also been discussed. Furthermore, the head-on collisions between two concentric cylindrical solitary waves have been considered and their solitonic character is displayed.

For a Newtonian, incompressible shallow liquid layer bounded below by a solid support where temperature is held constant and above with a free, deformable surface, a long-wavelength oscillatory instability occurs when the Marangoni number reaches a critical value. We have shown that radially symmetric, weakly nonlinear excitations satisfy the CKdV equation. When the Marangoni number of the system slightly exceeds its critical value past the instability threshold, the surface displacement is found to obey the DCKdV equation (30). It is a combination of the CKdV equation and a cylindrical Burgers-Kuramoto-Sivashinsky equation with an additional term $(hh_\xi)_\xi$, nonlinear consequence of the Marangoni effect. Nearly radially symmetric solutions have been obtained. These exact solutions are ϵ dependent with negative powers thus being nonperturbative.

We have considered the head-on collision of two concentric cylindrical solitary waves by using an extended Poincaré-Lighthill-Kuo method. The results show that there are some new geometric and dynamic effects, given by Eqs. (80)–(82). The phase shifts of the solitary waves in head-on collisions are shown to be proportional to $r^{-1/3}$ and also depend on their initial positions.

Cylindrical solitons in dissipation-free systems have been widely studied both theoretically and experimentally. The CKdV and CKP equations are nonautonomous generalizations of the standard KdV and KP equations and they are also completely integrable. The DCKdV equation (30) as well as the DCKP equations (32) and (34) derived in this paper include nonlinearity, dispersion, effect of geometrical distortion, instability, and dissipation. They are the natural dissipative generalizations of the CKdV and CKP equations incorporating an input-output energy balance, hence dissipation in the Bénard-Marangoni problem. Our theoretical study of the dissipative cylindrical solitary waves and their head-on interactions given here is promising for guiding new experimental findings about the solitary waves in driven-dissipative systems. Linde and co-workers [19–23] have observed solitonlike behavior with solitary waves and wave trains in a Bénard-Marangoni convecting system with and without cylindrical geometry. Unfortunately, there is still not enough quantitative information in Refs. [22,23] about the cylindrical case (Fig. 1) to allow comparison with our theoretical predictions. However, work still in progress [22,23] supports our findings.

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APPENDIX A

The functions $T_j(z)$ and $P_j(z)$ $(j=0,1,2,\dots)$ used in the main text are defined by

$$T_j(z)=\int_0^z dz_1\int_1^{z_1}dz_2T_{j-1}(z_2)=\int_0^z dz_1P_j(z_1), \quad \text{(A1)}$$

$$P_j(z)=\int_1^z dz_1T_{j-1}(z), \quad \text{(A2)}$$

$$T_{-1}(z)=z. \quad \text{(A3)}$$

From Eqs. (A1)–(A3) it follows that

$$T_0(z)=\frac{1}{3!}(z^3-3z), \quad \text{(A4)}$$

$$T_1(z)=\frac{1}{5!}(z^5-10z^3+25z), \quad \text{(A5)}$$

$$T_2(z)=\frac{1}{7!}(z^7-21z^5+175z^3-427z), \quad \text{(A6)}$$

$$\cdots$$

and

$$P_0(z)=\frac{1}{2!}(z^2-1), \quad \text{(A7)}$$

$$P_1(z)=\frac{1}{4!}(z^4-6z^2+5), \quad \text{(A8)}$$

$$P_2(z)=\frac{1}{6!}(z^6-15z^4+75z^2-61), \quad \text{(A9)}$$

$$\cdots .$$

APPENDIX B

For convenience and generalization for our analysis we extend the DCKP equation (32) here to the more general form

$$\frac{\partial}{\partial x}\left[\frac{\partial h}{\partial t}+\frac{1}{2t}h+\beta_1 h\frac{\partial h}{\partial x}+\beta_2\frac{\partial^3 h}{\partial x^3}+\beta_3\frac{\partial^2 h}{\partial x^2}+\beta_4\frac{\partial^4 h}{\partial x^4}+\beta_5\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)\right]+\frac{1}{2t^2}\frac{\partial^2 h}{\partial y^2}=0, \quad \text{(B1)}$$

where β_j $(j=1,\dots,5)$ are constants depending on the corresponding physical system. For traveling-wave solutions we anticipate that the velocity of the waves may have a possible y dependence due to the last term in Eq. (B1). Thus we assume

$$z=x-f(y)t, \quad h=h(z,t), \quad \text{(B2)}$$

where $f(y)$ is a function yet to be determined. Then Eq. (B1) is transformed to

$$\frac{\partial}{\partial z}\left[\frac{\partial h}{\partial t}+\beta_1 h\frac{\partial h}{\partial z}+\beta_2\frac{\partial^3 h}{\partial z^3}+\beta_3\frac{\partial^2 h}{\partial z^2}+\beta_4\frac{\partial^4 h}{\partial z^4}+\beta_5\frac{\partial}{\partial z}\left(h\frac{\partial h}{\partial z}\right)\right]-\frac{1}{2t}\left(\frac{d^2 f}{dy^2}-1\right)\frac{\partial h}{\partial z}+\left[\frac{1}{2}\left(\frac{df}{dy}\right)^2-f(y)\right]\frac{\partial^2 h}{\partial z^2}=0. \quad \text{(B3)}$$

In order to eliminate the variable-coefficient terms of Eq. (B3), we set

$$\frac{d^2 f}{dy^2}-1=0, \quad \frac{1}{2}\left(\frac{df}{dy}\right)^2-f(y)=0. \quad \text{(B4)}$$

It is easy to get the solution for $f(y)$ as

$$f(y)=\tfrac{1}{2}y^2. \quad \text{(B5)}$$

Thus for Eq. (B1) we have

$$\frac{\partial h}{\partial t}+\beta_1 h\frac{\partial h}{\partial z}+\beta_2\frac{\partial^3 h}{\partial z^3}+\beta_3\frac{\partial^2 h}{\partial z^2}+\beta_4\frac{\partial^4 h}{\partial z^4}+\beta_5\frac{\partial}{\partial z}\left(h\frac{\partial h}{\partial z}\right)=0, \quad \text{(B6)}$$

with $z=x-\frac{1}{2}y^2t$. Equation (B6) is just the dissipative KdV equation [17,18].

We use the tanh-function ansatz [25] to solve Eq. (B6). For the traveling-wave solution $h(z,t)=h(\eta)=h(kz+\omega t)$, Eq. (B6) becomes

$$\omega h+\tfrac{1}{2}\beta_1 kh^2+\beta_2k^3h_{\eta\eta}+\beta_3k^2h_\eta+\beta_4k^4h_{\eta\eta\eta}+\beta_5k^2hh_\eta=C, \quad \text{(B7)}$$

after integrating once with respect to ξ, where C is an integration constant. Based on the consideration of the balance between the highest-order derivative and the highest-order nonlinearity in Eq. (B7), we make the assumption

$$h(z,t)=h(\eta)=A_0+A_1\tanh\eta+A_2\tanh^2\eta, \quad \text{(B8)}$$

with A_j $(j=0,1,2)$ constants yet to be determined. By substituting Eq. (B8) into Eq. (B7) and equating the coefficients of $\tanh^j\eta\,(j=0,1,\ldots,5)$, we obtain a set of algebraic equations for A_j $(j=0,1,2)$, k, ω, and C. Solving these equations we obtain their solutions in the following.

(i) When $\beta_2\beta_5\neq\beta_1\beta_4$, we have

$$A_0=-\frac{\beta_3}{\beta_5}+\frac{1}{25\beta_4\beta_5^3}(\beta_2\beta_5-\beta_1\beta_4)(26\beta_1\beta_4-\beta_2\beta_5), \tag{B9}$$

$$A_1=\pm\frac{6(\beta_2\beta_5-\beta_1\beta_4)^2}{25\beta_4\beta_5^3}, \tag{B10}$$

$$A_2=-\frac{6(\beta_2\beta_5-\beta_1\beta_4)^2}{5\beta_4\beta_5^3}, \tag{B11}$$

$$k=\pm\frac{\beta_2\beta_5-\beta_1\beta_4}{10\beta_4\beta_5}, \tag{B12}$$

$$\omega=\pm\frac{\beta_1(\beta_2\beta_5-\beta_1\beta_4)}{40\beta_4\beta_5^4}\left[4\beta_3+\frac{1}{5\beta_4}(\beta_2\beta_5-\beta_1\beta_4)\times(2\beta_2\beta_5-15\beta_1\beta_4)\right]. \tag{B13}$$

The expression of C is not needed and is not given explicitly here.

(ii) If $\beta_2\beta_5=\beta_1\beta_4$, one has

$$A_0=-\frac{1}{\beta_5}(\beta_3-8\beta_4k^2), \tag{B14}$$

$$A_1=0, \tag{B15}$$

$$A_2=-\frac{12}{\beta_5}\beta_4k^2=-\frac{12}{\beta_1}\beta_2k^2, \tag{B16}$$

$$\omega=\frac{\beta_2\beta_3}{\beta_4}, \tag{B17}$$

with k being arbitrary in this case.

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