Breaking the symmetry in bimodal frequency distributions of globally coupled oscillators

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The mean-field Kuramoto model for synchronization of phase oscillators with an *asymmetric* bimodal frequency distribution is analyzed. Breaking the reflection symmetry facilitates oscillator synchronization to rotating wave phases. Numerical simulations support the results based on bifurcation theory and high-frequency calculations. In the latter case, the order parameter is a linear superposition of parameters corresponding to rotating and counterrotating phases. [S1063-651X(98)02805-0]

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Collective synchronization and incoherence in large populations of nonlinearly coupled oscillators received a great deal of attention in recent years. The motivation for this can be found in the broad variety of phenomena that can be modeled in this framework. Indeed, synchronous flashing in swarms of fireflies [1], crickets that chirp in unison [2], epilectic seizures in the brain [3], electrical synchrony among cardiac pacemaker cells [4], arrays of Josephson junctions [5], chemical processes [6], some models of charge-density waves in quasi-one-dimensional metals [7], and some neural networks used to model dynamic learning processes [8] all seem to be described in these terms.

The mathematical model conceived first as a large collection of elementary nonlinear phase oscillators, each with a globally attracting limit cycle, goes back to Winfree [9]. It was later formulated as a system of nonlinearly coupled differential equations by Kuramoto [10] in the mean-field coupling case, and as a system of Langevin equations (adding external white-noise sources) by Sakaguchi [11],

$$\dot{\theta}_i=\omega_i+\xi_i(t)+\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i),\quad i=1,\ldots,N. \tag{1}$$

Here $\theta_i(t)$ denotes the ith oscillator phase, ω_i is its natural frequency [picked up from a given distribution $g(\omega)$], $K>0$ represents the coupling strength, and the ξ_i's are independent identically distributed white noises. Consequently, the one-phase oscillator probability density $\rho(\theta,t,\omega)$ obeys the following nonlinear Fokker-Planck equation, in the thermodynamic limit $N\to\infty$:

$$\frac{\partial\rho}{\partial t}=D\frac{\partial^2\rho}{\partial\theta^2}-\frac{\partial}{\partial\theta}(v\rho), \tag{2}$$

where $D>0$ comes from the noise terms in Eq. (1) and

$$v(\theta,t,\omega)=\omega+Kr\,\sin(\psi-\theta). \tag{3}$$

Here the complex-valued order parameter $re^{i\psi}$ is defined by

$$re^{i\psi}=\int_0^{2\pi}\int_{-\infty}^{+\infty}e^{i\theta}\rho(\theta,t,\omega)g(\omega)d\omega\,d\theta. \tag{4}$$

It is understood that Eq. (2) must be accompanied by the prescription of the initial value $\rho(\theta,0,\omega)=\rho_0(\theta,\omega)$, 2π-periodic boundary conditions, and normalization $\int_0^{2\pi}\rho(\theta,t,\omega)d\theta=1$.

The fundamental phenomenon of transition from incoherence ($\rho\equiv 1/2\pi$, $r\equiv 0$) to collective synchronization ($r\neq 0$) is similar to phase transitions in statistical physics and has been analyzed rigorously by Strogatz and Mirollo [12]. They studied the linear stability of incoherence of populations characterized by unimodal frequency distributions. In [13], a nonlinear stability analysis was accomplished and bimodal frequency distributions [$g(\omega)$ with two peaks] were also considered. In the latter case, different bifurcations were discovered, showing the existence of a rich phenomenology, such as subcritical spontaneous stationary synchronization, supercritical time-periodic synchronization, bistability, and hysteretic phenomena. A large amount of information was obtained in [12,13], adopting as models of unimodal and bimodal frequency distributions $g(\omega)=\delta(\omega)$ and $g(\omega)=\frac{1}{2}[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$. It may be surprising now to realize that the *asymmetric bimodal* distribution

$$g(\omega)=\alpha\delta(\omega-\omega_0)+(1-\alpha)\delta(\omega+\omega_0),\quad 0<\alpha<1, \tag{5}$$

entails essentially different features with respect to the symmetric case $\alpha=\frac{1}{2}$, even for α close to $\frac{1}{2}$. The case $\alpha=0$, as well as $\alpha=1$, yields the unimodal distribution already considered elsewhere [12,13]. These cases, in fact, may be reduced to the case $\omega_0=0$, going to a rotating frame. This *cannot* be done, however, for any α with $0<\alpha<1$. Since,

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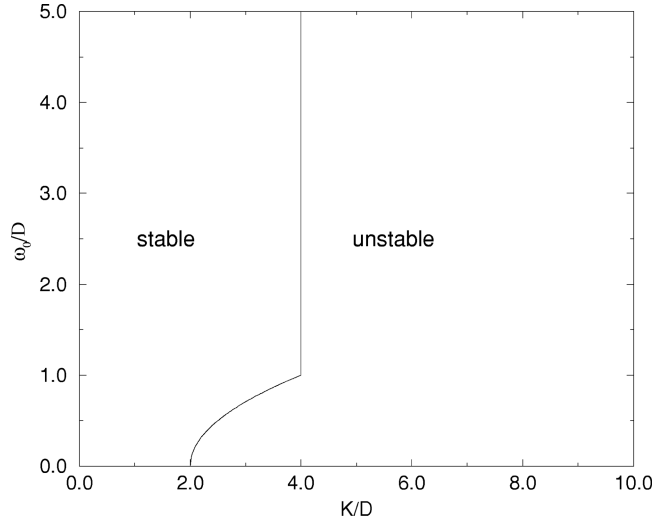

FIG. 1. Stability boundaries for the incoherent solution for the *symmetric* bimodal frequency distribution. Incoherence is linearly stable in the region to the left of the solid line.

due to unavoidable imperfections, possibly small deviations from symmetry are most likely in nature, the *asymmetric* case should be rather ubiquitous. The purpose of this paper is to illustrate the distinctive features of an asymmetric oscillator frequency distribution. The main qualitative effect of asymmetry is that no synchronized stationary phase is possible. Synchronized phases branch off from incoherence as traveling waves (TWs, see below) and their structure becomes richer as the strength of the coupling increases. The asymmetry of the frequency distribution changes the stability boundaries of the incoherence (see the phase diagrams in Figs. 1 and 2), rendering it less stable and, consequently, rendering the partially synchonized solution (whose order parameter, however, now always depends on time) more stable.

The stability boundaries for the incoherent solution $\rho_0(\theta,\omega)\equiv 1/2\pi$ can be calculated by setting to zero the greatest of the $\mathrm{Re}(\lambda)$'s, where $\rho=\rho_0+\epsilon e^{\lambda t}\eta(\theta,\omega)(\epsilon\to 0)$ and λ are the eigenvalues of the linearized problem. They are given by [12]

$$\frac{x}{2}\int_{-\infty}^{+\infty} d\omega \frac{g(\omega)}{Z+i\dfrac{\omega}{D}}=1, \tag{6}$$

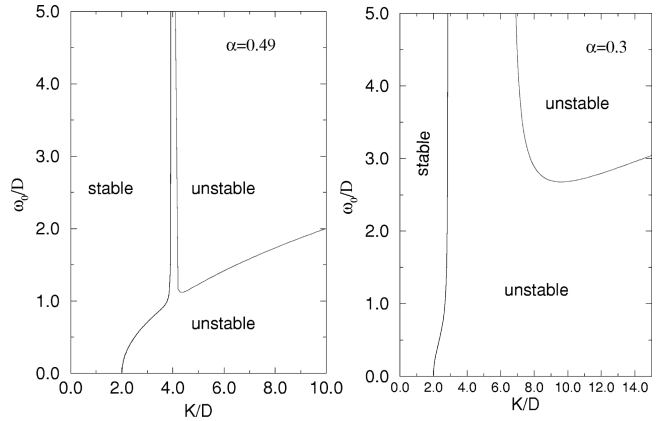

FIG. 2. Stability boundaries for the incoherent solution for the *asymmetric* bimodal frequency distribution $\alpha=0.49$ and 0.3.

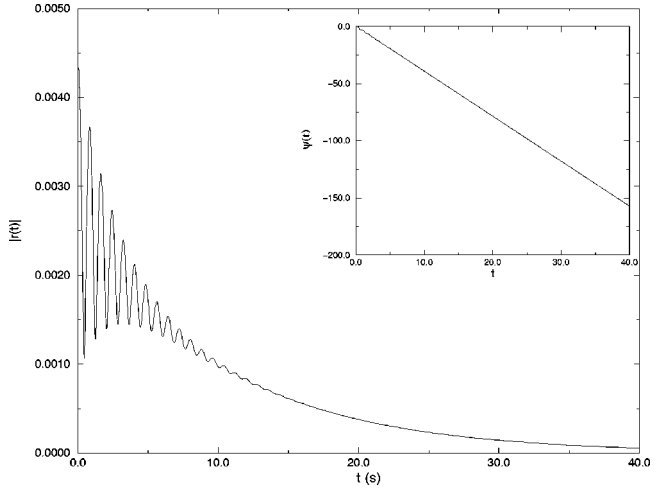

FIG. 3. Time evolution of the amplitude $|r(t)|$ and phase $\psi(t)$ of the order parameter for $K=3$, $D=1$, and $\omega_0=4$, with $\alpha=0.4$. Note the stability of the incoherent solution.

where we have defined $x=K/D$ and $Z=1+\lambda/D$. For the *asymmetric* bimodal distribution (5) we find

$$Z_{1,2}=\frac{x}{4}\pm\left[\frac{x^2}{16}-y^2+i\frac{xy}{2}(2\alpha-1)\right]^{1/2}, \tag{7}$$

where $y=\omega_0/D$. The stability regions in Figs. 1 and 2 are then determined by the condition $\max \mathrm{Re} Z_{1,2}\leqslant 1$.

The branch on the right of the asymptote in Fig. 2 is not completely unexpected. Indeed, its counterpart in the symmetric case is a parabolic profile continuing that in Fig. 1 (see [13]). In the latter case, however, such a branch is not as important as in Fig. 2 since it does not separate different stability regions. The behavior depicted in Figs. 1 and 2 is confirmed by direct numerical simulation [14,15] of the Kuramoto-Sakaguchi equation (2); see the evolution of the amplitude and phase of the order parameter in Figs. 3–5. Observe that the phase $\psi(t)$ is always time dependent rather than constant as in the case of the symmetric bimodal distribution [13,16]. The new synchronized phases are described

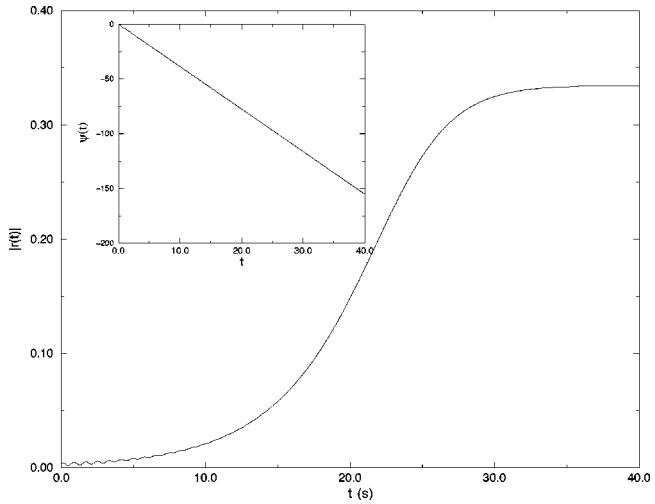

FIG. 4. Time evolution of the amplitude $|r(t)|$ and phase $\psi(t)$ of the order parameter for $K=4$, $D=1$, and $\omega_0=4$, with $\alpha=0.4$. This is the region of stability of the TW solution.

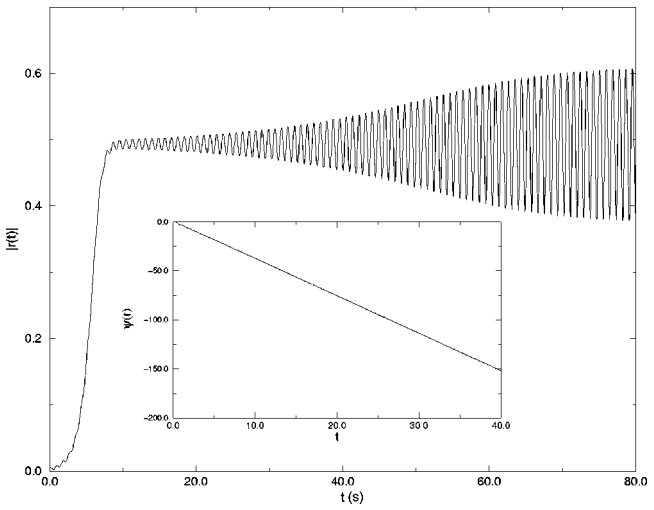

FIG. 5. Time evolution of the amplitude $|r(t)|$ and phase $\psi(t)$ of the order parameter for $K=6$, $D=1$, and $\omega_0=4$, with $\alpha=0.4$. Another TW solution has bifurcated from incoherence, resulting in a more complex behavior of the order parameter.

by a bifurcation analysis near the line in the parameter space where the incoherence loses stability:

$$\frac{\omega_0}{D}=\frac{2\left(1-\frac{K_c}{4D}\right)\sqrt{\frac{K_c}{2D}-1}}{\sqrt{\alpha(1-\alpha)}\sqrt{\left(\frac{2}{\alpha}-\frac{K}{D}\right)\left(\frac{2}{1-\alpha}-\frac{K}{D}\right)}}. \quad (8)$$

This is obtained by setting the largest real part of the eigenvalues to zero and corresponds to $K_c=4D$, $\omega_0>D$ of the symmetric case (see Fig. 1). The two-time asymptotic analysis conducted in [13] may be used unchanged for bifurcations at the line (8) with the asymmetric frequency distribution, taking into account that now $\Omega^2=\omega^2+D^2-K_cD/2$ and that $g(\omega)$ is the asymmetric frequency distribution in Eq. (5). In fact, the symmetric case possesses the reflection symmetry $\omega_0\to-\omega_0$, $\theta\to-\theta$, which causes the eigenvalues to be doubly degenerated [17], whereas this is *not* the case for the asymmetric frequency distribution. Then the simple analysis of Ref. [13] (which overlooked eigenvalue multiplicity, as pointed out in [17]; see also [16]) can be directly used for the asymmetric case.

The result is that a branch of stable synchronized phases bifurcates from incoherence at the point given by Eq. (8). Near the bifurcation line, these solutions have the form of TWs rotating counterclockwise [13]:

$$\rho(\theta,t,\omega)=\frac{1}{2\pi}+\frac{Re^{i\Psi_0(K-K_c)(t-t_0)}}{2\pi[D+i(\omega+\Omega)]}e^{i(\Omega t+\theta)}+\text{c.c.} \quad (9)$$

$$+O(K-K_c),$$

where c.c. denotes the complex conjugate of the preceding term and

$$R=\sqrt{\frac{(K-K_c)\mathrm{Re}\lambda_1}{\mathrm{Re}\gamma}}, \quad \Psi_0=\mathrm{Im}\lambda_1-\mathrm{Im}\gamma\frac{\mathrm{Re}\lambda_1}{\mathrm{Re}\gamma}. \quad (10)$$

See Ref. [13] for the explicit expressions of the parameters γ and $\lambda_1=(\partial\lambda/\partial K)|_{K=K_c}$. In the symmetric case, another solution corresponding to waves rotating clockwise has to be added to Eq. (9). This results in a stable standing wave solution, whose order parameter has a constant phase and an oscillatory amplitude [16,17].

In the high-frequency limit $\omega_0\to\infty$ a different perturbation analysis provides expressions for the evolution of the probability density, either near or far from bifurcation points [14]. The main result is that the frequency distribution decomposes into as many phases as peaks of the oscillator frequency distribution in such a limit. Each phase rotates with the frequency corresponding to its respective peak. Then the order parameter may be written as a linear superposition of the order parameters of the different phases. For the asymmetric bimodal distribution, the overall order parameter evolves (except by a constant phase shift) to

$$re^{i\psi}=\alpha R_+e^{i(\omega_0t+\Psi_+)}+(1-\alpha)R_-e^{-i(\omega_0t+\Psi_-)}, \quad (11)$$

where $R_\pm$ and $\Psi_\pm$ correspond to phases rotating with angular speeds $\pm\omega_0$. They can be calculated with the stationary formulas (2.1) and (1.7) of Ref. [13], with zero frequency [14]. Let $\alpha<1/2$ to be specific.

We have the following possibilities depending on the value of the coupling constant.

(i) If $0<K<2D/(1-\alpha)$, the incoherent solution $\rho_0\equiv1/2\pi$ is stable and it is the only possible stationary solution.

(ii) If $2D/(1-\alpha)<K<2D/\alpha$, a globally stable partially synchronized solution branches off incoherence at $K=2D/(1-\alpha)$. It has $R_+=0$, $\psi=\Psi_--\omega_0t$, and $r=(1-\alpha)R_-$. Its component $\rho_+\equiv1/2\pi$ is incoherent, while its component ρ_- is synchronized. The overall effect is having a TW solution (rotating clockwise).

(iii) If $K>2D/\alpha$, the component ρ_+ becomes partially synchronized too. The probability density then has TW components rotating clockwise and counterclockwise. Their order parameters have different strengths and $R_->R_+$ if $\alpha<1/2$.

Let us now compare the analytical results obtained in the high-frequency limit with those obtained by means of bifurcation theory. As $\omega_0\to\infty$, the parameters λ_1, γ, and K_c in Eq. (10) become (cf. [13])

$$\lambda_1=\frac{2D^2}{(1-\alpha)K_c^2}, \quad \gamma=\frac{1}{2D}, \quad K_c=\frac{2D}{1-\alpha}.$$

We can now calculate the order parameter in Eq. (4) by using Eqs. (9) and (10) and the previous expression:

$$re^{i\psi}\approx(1-\alpha)\left\{\frac{K(1-\alpha)}{D}-2\right\}^{1/2}e^{-i\omega_0t}. \quad (12)$$

Equation (12) agrees exactly with the results of the high-frequency limit (11) in [14]: The amplitude of the order parameter is constant and its phase decreases linearly in time. Of course, for larger values of the coupling constant another branch of oscillatory solutions (TW rotating clockwise) bifurcates from incoherence. Then the overall probability den-

sity is richer, with an order parameter whose amplitude and phase both vary with time as in Fig. 5.

In conclusion, we have analyzed the mean-field Kuramoto-Sakaguchi model of oscillator synchronization with an *asymmetric bimodal* frequency distribution. In this case, reflection symmetry is broken, which results in stable synchronized phases that have the form of TWs (rotating clockwise or counterclockwise). These waves have order parameters with constant amplitude and phases that depend linearly on time in the limit of high frequencies and almost linearly for ω_0 finite [14]. As the strength of the coupling constant increases, such a synchronized phase bifurcates from incoherence. Larger values of the coupling strength result in a new bifurcation, which contributes to another TW. Then both the phase and amplitude of the order parameter become time dependent. Numerical simulations of the model favorably agree with the results of bifurcation theory and of high-frequency perturbation expansions [14]. One of the main consequences of the asymmetry considered in the present model is that *no* stationary synchronized solution exists, which contradicts what is claimed in ideally symmetric models, as it can be found in the currently available literature. Extensions of our analysis to the case of a multimodal frequency distribution [i.e., a discrete or a continuous $g(\omega)$ having m peaks] are worth considering in future works.

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[1] J. Buck, Q. Rev. Biol. **63**, 265 (1988).
[2] T. J. Walker, Science **166**, 891 (1969).
[3] R. D. Traub, R. Miles, and R. K. S. Wong, Science **243**, 1319 (1989).
[4] D. C. Michaels, E. P. Matyas, and J. Jalife, Circ. Res. **61**, 704 (1987).
[5] K. Wiesenfeld, P. Colet, and S. H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996).
[6] M. Scheutzow, Prob. Theory Related Fields **72**, 425 (1986).
[7] L. L. Bonilla, Phys. Rev. B **35**, 3637 (1987).
[8] C. J. Pérez Vicente, A. Arenas, and L. L. Bonilla, J. Phys. A **29**, L9 (1996).
[9] A. T. Winfree, *Geometry of Biological Time* (Springer, New York, 1990).
[10] Y. Kuramoto, in *International Symposium of Mathematical Problems in Theoretical Physics*, edited by H. Araki, Lecture Notes in Physics Vol. 39 (Springer, New York, 1975).
[11] H. Sakaguchi, Prog. Theor. Phys. **79**, 39 (1988).
[12] S. H. Strogatz and R. E. Mirollo, J. Stat. Phys. **63**, 613 (1991).
[13] L. L. Bonilla, J. C. Neu, and R. Spigler, J. Stat. Phys. **67**, 313 (1992).
[14] J. A. Acebrón and L.L. Bonilla, Physica D **114**, 296 (1998).
[15] C. J. Pérez Vicente and F. Ritort, J. Phys. A **30**, 8095 (1998).
[16] L. L. Bonilla, C. J. Pérez Vicente, and R. Spigler, Physica D **113**, 79 (1998).
[17] J. D. Crawford, J. Stat. Phys. **74**, 1047 (1994).