Characterizations of natural patterns

Gemunu H. Gunaratne*
Department of Physics, The University of Houston, Houston, Texas 77204

David K. Hoffman†
Department of Chemistry and Ames Laboratory, Iowa State University, Ames, Iowa 50011

Donald J. Kouri‡
Department of Chemistry, The University of Houston, Houston, Texas 77204

(Received 3 September 1997)

Labyrinthine patterns are observed in systems as diverse as animal coats, slime mold colonies, fish scales, and cloud streets. However, even under well-controlled conditions, repetition of an experiment gives structures with vastly different details. A theoretical analysis of ''universal'' aspects of patterns requires a quantitative description that gives similar values for distinct configurations. We introduce a function to characterize the ''disorder'' of labyrinthine patterns that is the same for structures generated under identical control parameters. Furthermore, patterns with different visual characteristics are described by distinct functions. [S1063-651X(98)10805-X]

PACS number(s): 05.70.Ln, 82.40.Ck, 47.54.+r

Reaction-diffusion systems [1], convecting fluids [2], and ferrofluids [3] are a few examples of the many uniform, well-controlled experimental systems generating complex labyrinthine structures. Even though patterns resulting from the repetition of an experiment are vastly different in detail, it is easy to discern ''features'' common to them. Furthermore, patterns from distinct experiments share many similar characteristics. These observations suggest the necessity for system-independent quantitative characterizations; at a minimum, these measures should be similar for distinct structures (e.g., Fig. 1) generated under the same external conditions.

Figure 1 shows two patterns that are generated by evolving distinct random initial fields via the Swift-Hohenberg equation [4]

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = D[\epsilon - (1 + k_0^{-2}\nabla^2)^2]u - \nu u^3, \tag{1}$$

constrained by periodic boundary conditions. The intensity of the initial random field, and the control parameters D, k_0, and ν are the same for the two cases. Each pattern is a complex labyrinthine array that consists of patches of stripes of an (almost) uniform width. Globally, domain walls and defects are distributed throughout the structure. Even though the patterns shown are vastly different in detail, it is easy to recognize several common ''signatures;'' e.g., the mean size of a domain, the density of defects, etc. The issue we address is, given a scalar field $v(\mathbf{x})$ [e.g., $u(\mathbf{x},t_0)$ from Eq. (1) at a given time t_0] describing a pattern, how one can determine the form of quantitative measures of this ''commonality,'' and how they can be estimated from $v(\mathbf{x})$.

Measures motivated by statistical physics and dynamical systems theory have been introduced to characterize patterns. Correlation length can either be estimated directly [5] or deduced from the decay of the power spectrum [6]. The complexity of a labyrinthine pattern is reflected in the density of the power spectrum, and the ''spectral entropy'' [7] is a possible measure of the disorder. For patterns exhibiting persistent evolution, the dynamics of the field at a point can be used as a measure of complexity through the use of ''Lyapunov dimension density'' [8]. These order parameters do not take advantage of the local structure of the pattern.

It is difficult to determine, *a priori*, a set of measures that will yield equal values for patterns generated under similar conditions. We instead impose a weaker requirement on our characterizations, that they remain invariant under all *rigid* motions of the pattern, i.e., translations, rotations, and reflections [9]. Rather surprisingly, the measures so defined are identical for distinct patterns such as those of Fig. 1.

The most significant feature of labyrinthine patterns is that they are locally striped, in a suitable neighborhood $v(\mathbf{x}) \sim \sin(\mathbf{k}\cdot\mathbf{x})$, where the modulus k_0 of the wave vector does not vary significantly over the pattern. Structures generated in experiments and model systems include higher harmonics due to the presence of nonlinearities. They play no role in the structure of the pattern, but only contribute to the shape of the cross section of stripes. In order to use the simplest characterization of patterns, we eliminate the second and higher harmonics by a suitable use of a window function in Fourier space.

The simplest local field that is derived from $v(\mathbf{x})$ and whose value remains the same under all rigid motions is its Laplacian $\Delta v(\mathbf{x})$. Terms such as $\Delta^n v^m(\mathbf{x})$, though invariant, are difficult to extract from an incompletely sampled field (typically given on a lattice). The requirement that perfect stripes be assigned a null measure (they are not disordered), coupled with the local sinusoidal form of the (filtered) pattern, implies that the lowest-order field relevant for our pur-

*Electronic address: gemunu@uh.edu
†Electronic address: hoffman@qdynamic.ameslab.gov
‡Electronic address: kouri@uh.edu

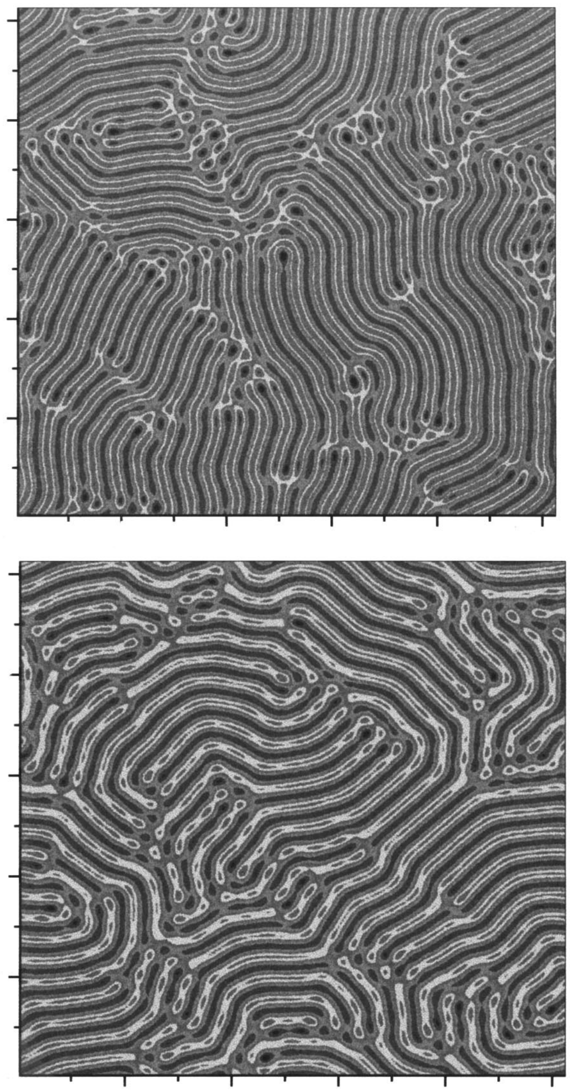

FIG. 1. Two patterns generated by evolving random initial states via the Swift-Hohenberg equation for 1600 time units. The parameters used were $D=0.1$, $\epsilon=0.2$, $\nu=2$, and $k_0=1$. The initial states consisted of white noise whose intensity varied between 0 and 10^{-3}. Periodic boundary conditions were imposed on the square domain of 256×256 lattice points, the length of whose sides are $(48\pi/k_0)$.

pose is $(\Delta+k_0^2)v(\mathbf{x})$. The family of measures, referred to as the "disorder function," is defined by

$$\delta(\beta)=(2-\beta)\frac{\int da|(\Delta+k_0^2)v(\mathbf{x})|^\beta}{k_0^{2\beta}\langle|v(\mathbf{x})|\rangle^\beta},\tag{2}$$

where $\langle|v(\mathbf{x})|\rangle$ denotes the mean of $|v(\mathbf{x})|$, and $\delta(\beta)$ has been normalized so that the "intensive variables" $\overline{\delta(\beta)}=\delta(\beta)/\int da$ are scale invariant. The moment β is restricted to lie between 0 and 2 for reasons discussed below. Local deviations of the patterns from stripes (due to curvature of the contour lines [9,10]) contribute to $\delta(\beta)$ through the Laplacian, while variations of the width of the stripes contribute via the choice of a "global" k_0.

The critical requirement for a good estimate of $\delta(\beta)$ is a sufficiently accurate determination of the Laplacian. Calculation of derivatives of a field from values given on a lattice is a delicate task, especially in the presence of noise. A technique, referred to as distributed approximating functionals (DAFs), has been introduced recently, to analytically fit or approximate a continuous function from known values on a discrete grid [11]. Unlike typical grid methods, it estimates derivatives using a range of neighboring points (~40 in our case); consequently it is much less sensitive to noise.

The most useful for applications have been a class of DAFs for which the order of accuracy of the fit is the same both on and off the grid. (This is in contrast to interpolation, which forces the fit to be exact on the grid, but always leads to interwining about the function off the grid, thus leading to inaccurate estimation of derivatives.) Their most general derivation is via a variational principle [12], yielding $g_{\mathrm{DAF}}(x)=\Sigma_k I(x,x_k)g(x_k)$, where k labels the grid points, $g(x_k)$ are the known input values of the function (which may contain noise). For suitable $I(x,x_k)$, the function and its derivatives are evaluated to a comparable accuracy [12]. This proves crucial for the estimation of the disorder function.

The calculations presented are carried out using the "Hermite DAF" [13], defined by

$$I(x,x_k)=I(x-x_k)=\frac{\Delta}{\sigma}\frac{e^{-z^2}}{\sqrt{2\pi}}\sum_{j=0}^{M/2}\left(-\frac{1}{4}\right)^j\frac{1}{j!}H_{2j}(z),\tag{3}$$

where $z=(x-x_k)/\sigma\sqrt{2}$, Δ is the lattice spacing, and $H_n(z)$ is the nth Hermite polynomial. The Gaussian weight (of width σ) in Eq. (3) makes $I(x_l-x_k)$ highly banded, reducing the computational cost of applying the DAF to data. The DAF representation of derivatives of a function known only on a grid is given by

$$\left(\frac{d^lg}{dx^l}\right)_{\mathrm{DAF}}(x)=\sum_k\frac{d^l}{dx^l}I(x,x_k)g(x_k),\tag{4}$$

which can be evaluated either on or off the grid. In the continuum limit, the derivative of the DAF equals (exactly) the DAF of the derivative [13]. The calculations reported were done on a grid with ~10 points per stripe, with $\sigma/\Delta=4.0$, and $M=12$, while the sum of Eq. (4) ran over 20 grid points on each side of x. With this choice of parameters, the required derivatives are obtained to similar accuracy as the DAF approximation of the function itself [13].

The DAF approximation to a function that is sampled on a square grid (x_m,y_n) can be obtained using the two-dimensional extension $I((x,y),(x_m,y_n))=I_X(x,x_m)I_Y(y,y_n)$ of the approximating identity kernel [14]. Thus to estimate (say) d^2v/dx^2, Eq. (4) needs to be applied in the y direction (with $l=0$) and along the x direction (with $l=2$). (The application of the DAF operators in the two directions commute and can be carried out in any order.)

For a perfect set of stripes the function $\delta(\beta)=0$. A domain wall contains curvature of the contour lines and variations of the stripe width; consequently it will contribute to disorder. $\delta(\beta)$ for a single domain wall is a monotonically increasing function of the angle θ between the stripes of the two domains. Thus $\delta(\beta)$ provides information absent in characterizations such as the correlation length. The disorder function for a target pattern $v(\mathbf{x})=a\cos(k_0r)$ is known [9], and is used to estimate the accuracy of the numerical algo-

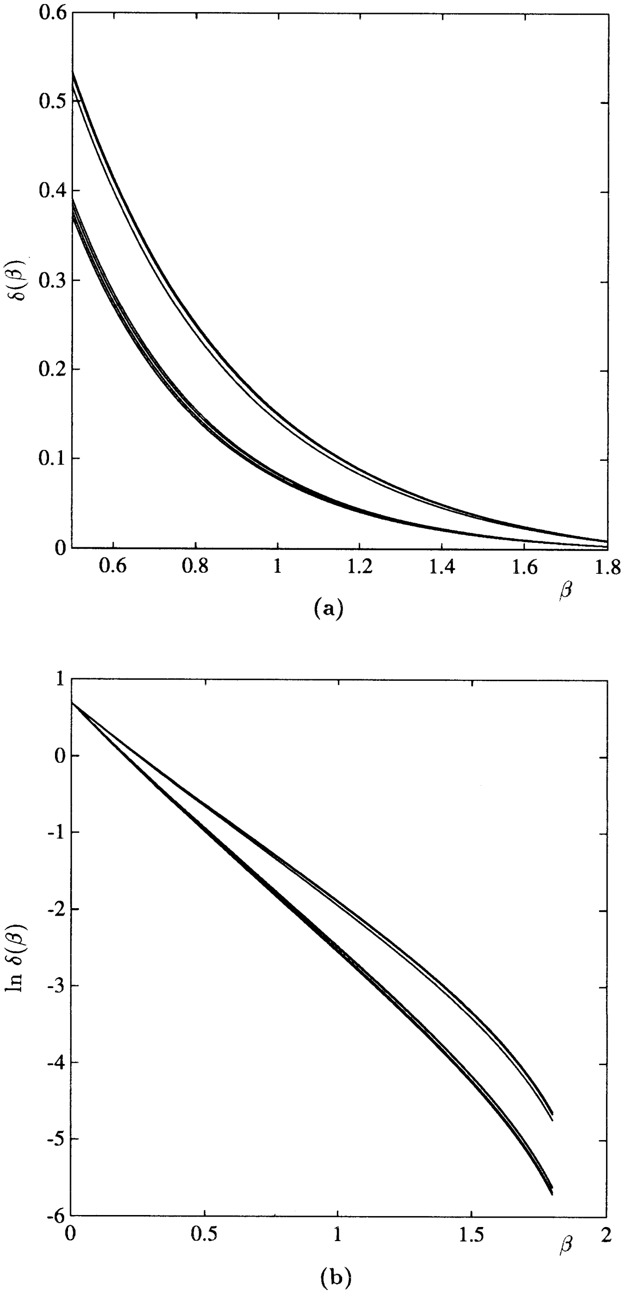

FIG. 2. The curves $\overline{\delta}(\beta)$ for patterns generated at two different sets of control parameters. The lower bunch consists of curves for four patterns at the first set of control parameters (Fig. 1) while the upper bunch consist of those for a second set of control parameters (Fig. 3). (b) shows the same plots with a logarithmic vertical scale.

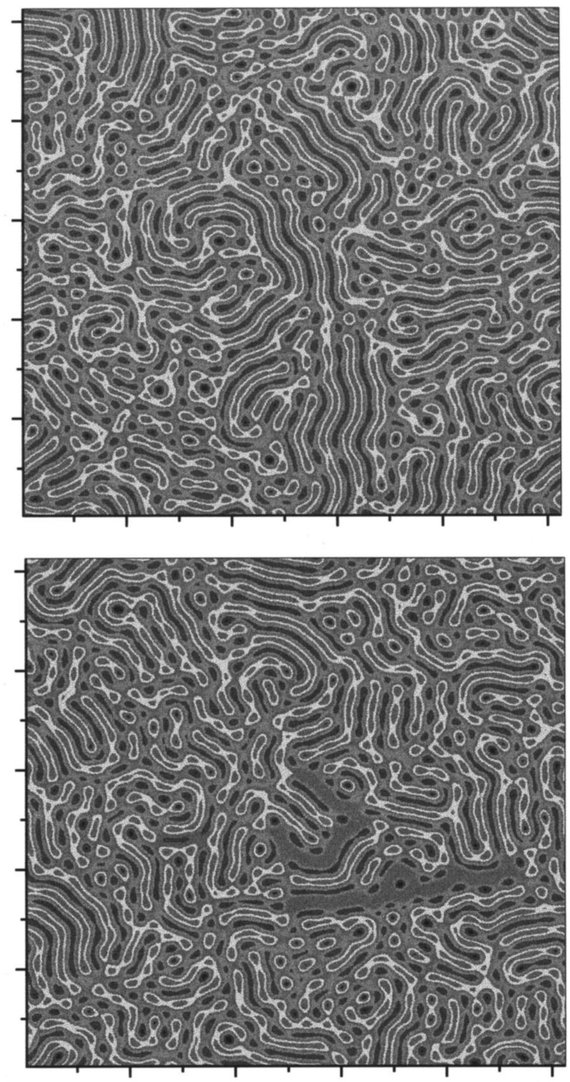

FIG. 3. Two patterns generated by evolving a random initial state via the Swift-Hohenberg equation for 2400 time units. The parameters used for the integration were $D=0.01$, $\epsilon=0.4$, $\nu=2$, and $k_0=1/3$. The initial states consisted of white noise whose intensity varied between $\pm 10^{-2}$. The length of each side of the square is $(48\pi/k_0)$.

rithms. For target patterns, the integral in the numerator diverges as $(2-\beta)^{-1}$, and leads to limiting the range of $\beta(<2)$, and to the introduction of the prefactor in Eq. (2). The effects of noise on the calculations are minimal. For example, addition of 10% white noise typically changes $\delta(\beta)$ by less than 2%. The effects of distinct characteristics of a pattern (e.g., domain walls, defects, variations of the stripe width, etc.) contributing to disorder are separated using distinct moments β.

The measures $\delta(\beta)$ were derived by insisting that they be invariant under rigid motions of a labyrinthine pattern. *Are these limited restrictions sufficient to yield characterizations that can delineate the observed ''commonality'' in distinct patterns generated under identical conditions?* Surprisingly, it appears to be the case. Figure 2 shows the disorder functions for several patterns. The curves bunched at the bottom show $\delta(\beta)$ for four structures (two of which are shown in Fig. 1) generated at fixed control parameters. $\delta(\beta)$ appears to have captured the commonality of these distinct patterns. Patterns generated in the Gray-Scott model [16] and in a vibrated layer of granular material [17] exhibit similar properties [15].

The next question is if $\delta(\beta)$ can differentiate between ''visually distinct'' patterns. Figure 3 shows two structures obtained from Eq. (1) for a second set of control parameters. They have characteristics that differ from patterns of Fig. 1; e.g., they contain smaller domains and a larger density of defects. $\delta(\beta)$ for four such patterns are bunched together on the upper curves in Fig. 2. The significant separation of the two sets of curves (e.g., the values of $\delta(1)$ between the two sets of curves is about 25 times larger than the average difference between curves within a set) confirms the ability of $\delta(\beta)$ to quantify the differences of the two groups of patterns.

The disorder function quantifies the characteristics of a labyrinthine pattern using the local curvature of the contour lines and the wavelength variations, which typically increase with the (visual) disorder of a pattern. Thus, $\delta(\beta)$ is able to quantify (Fig. 2) the observation that patterns of Fig. 3 are more disordered that those of Fig. 1.

Labyrinthine patterns are observed in a wide range of spatiotemporal systems. Normal regions in the intermediate state of a type I superconductor [18] and the speckle patterns formed by laser light reflected off a metal surface [19] are two examples. One may inquire if the former can be used to deduce properties of the superconductor, or if the latter can be used to quantify the roughness of a metal surface on a microscale. What is required to address such issues is characterizations that depend on external parameters but are ''configuration independent.'' Properties of labyrinthine patterns such as the observed organization of numerically generated patterns [6] and the onset of spatiotemporal dynamics in reaction-diffusion systems [9] have proven amenable to such analyses. The disorder function is a continuum of measures (analogous to generalized dimensions in chaotic systems [20]) to characterize labyrinthine patterns. Complex patterns whose local structure consists of other planforms such as hexagons or squares can also be characterized using the disorder function.

We have benefited from discussions with M. Golubitsky, M. Gorman, R. E. Jones, I. Melbourne, and H. L. Swinney. This research was supported by the Office of Naval Research (G.G.), the Ames Laboratory of the Department of Energy (D.H.), the National Science Foundation (D.K.), and the Energy Laboratory of the University of Houston (G.G. and D.K.).

[1] Q. Ouyang and H. L. Swinney, Nature (London) **352**, 610 (1991).
[2] M. S. Heutmaker and J. P. Gollub, Phys. Rev. A **35**, 242 (1987).
[3] R. E. Rosenweig, *Ferrohydrodynamics* (Cambridge University Press, Cambridge, 1985).
[4] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[5] Q. Ouyang and H. L. Swinney, Chaos **1**, 411 (1991).
[6] M. C. Cross, D. Meiron, and Y. Tu, Chaos **4**, 607 (1994).
[7] D. A. Egolf, I. V. Melnikov, and E. Bodenschatz (unpublished).
[8] D. A. Egolf and H. S. Greenside, Phys. Rev. Lett. **74**, 1751 (1995).
[9] G. H. Gunaratne, R. E. Jones, Q. Ouyang, and H. L. Swinney, Phys. Rev. Lett. **75**, 3281 (1995).
[10] Curvature has been used as a tool for characterizing labyrinthine patterns in Y. Hu, R. Ecke, and G. Ahlers, Phys. Rev. E **51**, 3263 (1995). The numerical methods used were significantly more complicated.
[11] D. K. Hoffman, M. Arnold, and D. J. Kouri, J. Phys. Chem. **96**, 6539 (1992).
[12] D. K. Hoffman, T. L. Marchioro, M. Arnold, Y. Huang, W. Zhu, and D. J. Kouri, J. Math. Chem. **20**, 117 (1996).
[13] D. K. Hoffman and D. J. Kouri, in *Proceedings of the Third International Conference on Mathematical and Numerical Aspects of Wave Propagation*, edited by G. Cohen (SIAM, Philadelphia, 1995), pp. 56–83.
[14] W. Zhu, Y. Huang, G. A. Parker, D. J. Kouri, and D. K. Hoffman, J. Phys. Chem. **98**, 12 516 (1994).
[15] R. E. Jones, Ph.D. thesis, University of Houston, 1997.
[16] P. Gray and S. K. Scott, Chem. Eng. Sci. **38**, 29 (1983); J. Phys. Chem. **89**, 22 (1985).
[17] F. Melo, P. Umbanhower, and H. L. Swinney, Phys. Rev. Lett. **72**, 172 (1993).
[18] M. Seul and D. Andelman, Science **267**, 476 (1995).
[19] J. C. Dainty, *Laser Speckle and Related Phenomena* (Springer-Verlag, Berlin, 1984).
[20] H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).