Critical lines of the Yang-Lee edge singularity of Ising ferromagnets on square, triangular, and honeycomb lattices

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(Received 29 October 1997)

We extend our previous approach to determine the critical lines of the Yang-Lee edge singularity of Ising ferromagnets on square, triangular, and honeycomb lattices by considering the zeros of the Ising partition function on elementary cycles of these lattices. It is found that the critical lines have the properties $h_0 \to t^{15/8}$ as $T \to T_c+$ and $\beta h_0 \to \pi/2$ as $T \to \infty$. Using the asymptotic formulas valid in the high-temperature limit and the zero-field limit we obtain the functional form of the critical line. [S1063-651X(98)02005-4]

PACS number(s): 05.50.+q, 75.40.Cx, 75.10.Hk, 64.60.Cn

I. INTRODUCTION

In 1952, Yang and Lee [1] opened a new way to the study of phase transitions. They called attention to the zeroes of the grand partition function in the complex fugacity plane. They showed that in the thermodynamic limit the zero distribution approaches the positive real axis and gives the transition point. In application to the ferromagnetic Ising model, they considered the zeroes of the partition function in the complex magnetic plane and proved the famous circle theorem. The Yang-Lee circle theorem states that the zeroes of the partition function in the complex magnetic plane are distributed on a unit circle. The theorem asserts that the critical line of an Ising ferromagnet is located at $h=0$ for $T<T_c$. Later this theorem was extended to many ferromagnetic systems, such as higher-order Ising model [2], Ising models with multiple spin interactions, the quantum Heisenberg model [3], the classical XY and Heisenberg model [4], and some continuous spin systems [5]. Ruelle [6] extended the theorem to noncircular regions. Lee [7] presented a generalized circle theorem to the asymmetric transitions and further to a continuum system.

Above the critical temperature, $T>T_c$, the zeroes do not come close to the real h axis in the thermodynamic limit and the free energy is not analytic in h. There exists a gap on the imaginary h axis, where zeroes are void. Since the gap size depends on the temperature, one can envision a critical line $h=ih_0(T)$ (here h_0 is real) along which the free energy becomes singular, $F\sim(h-ih_0)^\theta$ (here θ is a critical exponent) [8]. This singularity was termed the Yang-Lee edge singularity by Fisher. Fisher [9] proved that the edge singularities, representing the zeroes lying closest to the real axis of the field, are closely analogous to the conventional critical points and that the relevant scaling laws are applicable. Furthermore, the universality should hold for them too and the nature of these singularities is independent of the detailed lattice structure and depends only on the dimensionality and the symmetry property of the order parameter.

Since the Yang-Lee edge singularity has the most important influence on the equation of state of a ferromagnet, there have been many studies on it. These include the Yang-Lee edge singularity in the Ising model [10], in the hierarchical model [11], in the spherical model [12], in the classical n-vector models and quantum Heisenberg model [13], as well as in the relation with conformal invariance in two dimensions [14] and in the relation with the critical behavior of branched polymers [15], etc.

Unlike the critical exponents, the critical lines are not universal and depend on temperature, detailed lattice structure, and interaction strengths. Unfortunately, it is difficult to obtain the critical line and little is known about the actual form of $h_0(T)$ since the Ising model in a magnetic field has not been solved exactly so far except for the one-dimensional Ising ferromagnet and the 2D Ising model on the Kagome lattice [16]. Kurtze and Fisher [13] analyzed high-field and high-temperature series for ferromagnetic Ising models to obtain the asymptotic formula for $h_0(T)$, which is valid for all Ising models.

Recently, we introduced an approach [17] for two-dimensional Ising models. By considering the zeroes of the Ising partition functions on elementary cycles of square, triangular, and honeycomb lattices, we obtained the exact zero-field critical conditions. Making use of Griffiths' smoothness postulate [18], we extended the zero-field results to the nonzero-field case and obtained accurate closed-form approximations of the critical lines of isotropic and anisotropic Ising antiferromagnets on square and honeycomb lattices. Our results are in good agreement with the numerical results obtained by other means.

Since the Yang-Lee edge singularity behaves like an ordinary critical point there are good reasons for extending our approach to this case also. In this paper, we extend our approach to obtain closed-form approximations to the critical lines of the Yang-Lee edge singularities of Ising ferromagnets on square, triangular, and honeycomb lattices.

This paper is organized as follows. In Sec. II we discuss the Yang-Lee edge singularity of a ferromagnet with the Curie point. We use the Griffiths' equation of state to obtain the equation of edge singularity, valid near the zero-field critical temperature. In Sec. III we consider the exact solution of the one-dimensional Ising ferromagnet. From this we derive the equation of edge singularity and obtain its asymptotic form in the high-temperature limit. In Sec. IV we consider the exact solution of the two-dimensional Ising ferromagnet at $h=i(\pi/2)kT$ obtained by Yang and Lee. We derive the critical field in the high-temperature limit. In Sec. V we briefly review our basic approach developed previously and refine it for the 2D Ising ferromagnets. In Sec. VI we apply this approach to isotropic Ising ferromagnets on square, triangular, and honeycomb lattices and obtain closed-form approximations to the critical lines on these lattices. In Sec. VII we give a summary of the paper.

II. THE YANG-LEE EDGE SINGULARITY OF A FERROMAGNET

It is interesting to ask whether a ferromagnet with the Curie point has the Yang-Lee edge singularity. Here we show that a ferromagnet satisfying Griffiths' analyticity does have the Yang-Lee edge singularity.

If a ferromagnet has Griffiths' analyticity [19], then near the critical point, the equation of state can be written as

$$\frac{h(M,t)}{kT_c}=\sum_{n=0}^{\infty} a_n|t|^{\gamma-2\beta n}M^{2n+1}, \tag{1}$$

where β and γ are critical exponents, and t is the reduced temperature $t=(T-T_c)/T_c$. If we approach the critical point enough, we can consider only the lowest two terms:

$$\frac{h(M,t)}{kT_c}=a_0|t|^{\gamma}M+a_1|t|^{\gamma-2\beta}M^3, \tag{2}$$

and thus we obtain

$$\left(\frac{\partial h}{\partial M}\right)_T=(a_0|t|^{\gamma}+3a_1|t|^{\gamma-2\beta}M^2)kT_c. \tag{3}$$

Above the critical point, $a_0>0$ and $a_1>0$. At the phase transition point we must have $(\partial h/\partial M)_T=0$, whose nontrivial solution is

$$M=i\left(\frac{a_0}{3a_1}\right)^{1/2}|t|^{\beta}. \tag{4}$$

Substituting Eq. (4) into Eq. (2) we obtain the critical magnetic field,

$$h=ih_0=i\frac{2\sqrt{3}a_0^{3/2}}{9a_1^{1/2}}|t|^{\beta+\gamma}kT_c. \tag{5}$$

We identify Eq. (5) with the Yang-Lee edge singularity. Since Griffiths' analyticity is a general property of a ferromagnet with the Curie point, we see that *any ferromagnet with the Curie point has the Yang-Lee edge singularity*. The Yang-Lee edge singularity is a general aspect of critical phenomena and it is more general than the circle theorem.

Essam and Hunter [20] used the series expansion method and obtained a few coefficients of the $2n$th derivatives of the free energy with respect to magnetic field for ferromagnetic Ising models. Gaunt and Domb [21,22] used these coefficients to calculate some coefficients a_n for ferromagnetic Ising models with the aid of the inversion method. For a square lattice Ising ferromagnet, they obtained $a_0=1.0387$, $a_1=0.8479$, $a_2=0.7495$, $a_3=0.6801$, $a_4=1.1376$, and $a_5=6.7041$. The estimated errors are $E(a_0)=0.001$, $E(a_1)=0.004$, $E(a_2)=0.03$, and $E(a_3)>0.1$, respectively. The errors increase rapidly for higher-order coefficients a_n. Using only a_0 and a_1 in Eq. (5), we obtain $h_0=t^{15/8}0.4425kT_c$ in the limit $T\rightarrow T_c+$.

Using the numerical results of Essam and Hunter [20], we calculate a_0 and a_1 for some lattices: For a triangular lattice, $a_0=1.0818$ and $a_1=0.9143$, and $h_0=t^{15/8}0.4529kT_c$. For three-dimensional Ising models, $\beta=5/16$ and $\gamma=5/4$. For a simple cubic lattice, $a_0=0.9436$, $a_1=0.4851$, and $h_0=t^{25/16}0.5065kT_c$; for a face centered cubic lattice, $a_0=1.0248$, $a_1=0.5885$, and $h_0=t^{25/16}0.5205kT_c$; for a body centered cubic lattice, $a_0=1.0097$, $a_1=0.5758$, and $h_0=t^{25/16}0.5146kT_c$; for a diamond lattice, $a_0=0.8532$, $a_1=0.3895$, and $h_0=t^{25/16}0.4861kT_c$.

III. 1D ISING FERROMAGNET

The partition function of an Ising model in the presence of a magnetic field is given by

$$Z=\sum_{\{S_i\}}\exp\left[\beta\sum_{\langle ij\rangle}K_{ij}S_iS_j+\beta h\sum_i S_i\right], \tag{6}$$

where $S_i=\pm 1$, K_{ij} are the interaction strengths, and $\beta=1/kT$. The sum over $\langle ij\rangle$ runs over nearest neighbors on the lattices. We consider the ferromagnetic case $K_{ij}>0$.

In order to best illustrate the Yang-Lee edge singularity, let us consider the 1D Ising ferromagnet. The exact solution [23,24] gives the free energy,

$$F/N=-K-\beta^{-1}\ln[\cosh\beta h+(\sinh^2\beta h+e^{-4\beta K})^{1/2}], \tag{7}$$

and the magnetization,

$$M=\frac{\sinh\beta h}{(\sinh^2\beta h+e^{-4\beta K})^{1/2}}. \tag{8}$$

Thus we have

$$\left(\frac{\partial M}{\partial h}\right)_T=\frac{\beta\cosh\beta h}{(\sinh^2\beta h+e^{-4\beta K})^{1/2}}-\frac{\beta\sinh\beta h\cosh\beta h}{(\sinh^2\beta h+e^{-4\beta K})^{3/2}}. \tag{9}$$

The phase transition condition, $(\partial h/\partial M)_T=0$, requires that $\sinh^2\beta h+e^{-4\beta K}=0$. This equation does not have a real solution except at $h=0$ and $T=0$. So h must be complex and purely imaginary, namely, $h=ih_0$. The critical line is given by

$$\sin\beta h_0=e^{-2\beta K}. \tag{10}$$

Expanding Eq. (7) around ih_0 we obtain $F\sim(h-ih_0)^{1/2}$ near the critical line. It is easy to verify that in the limit $T\rightarrow\infty$, Eq. (10) yields

$$\beta h_0\rightarrow\pi/2-(4\beta K)^{1/2}+\tfrac{2}{3}(\beta K)^{3/2}+O(T^{-5/2}). \tag{11}$$

The critical line is plotted in Fig. 1.

IV. EXACT CRITICAL FIELD IN THE HIGH-TEMPERATURE LIMIT

Though the 2D Ising model in the absence of magnetic field was solved exactly by Onsager [25], its behavior in a

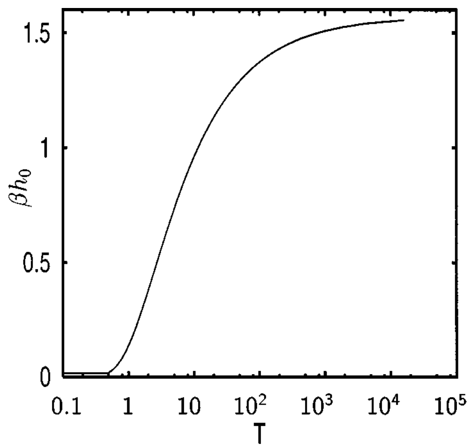

FIG. 1. The critical line of the 1D ferromagnet. The unit of T is k/K.

magnetic field is not known exactly except for the case of a Kagome lattice [16]. We do not have precise knowledge of its thermodynamic properties in general. Nevertheless, in their classic paper [1], Yang and Lee did find an exact solution for the square lattice Ising ferromagnet in a specific imaginary magnetic field $h=i(\pi/2)kT$. Here we give only the result. The free energy and magnetization are respectively given by

$$F(z=-1,x)=-\frac{i\pi}{2}\frac{kT}{4\pi^2}\int_0^{\pi}\int_0^{\pi}d\omega_1 d\omega_2 \ln\{(1-x^2)^2 \times[1+(6-4\cos^2\omega_1-4\cos^2\omega_2)x^2+x^4]\} \tag{12}$$

and

$$M(z=-1,x)=\left[\frac{(1+x^2)^2}{(1-x^2)(1+6x^2+x^4)^{1/2}}\right]^{1/4}, \tag{13}$$

where $x=e^{-2\beta K}$ and $z=e^{-2\beta h}$. Since β and K are real, in the limit $T\to\infty$, we have $x\to 1$. In this limit the above equations approach, respectively,

$$F\sim\ln(1-x) \tag{14}$$

and

$$M\sim(1-x)^{-1/4}. \tag{15}$$

Thus $x=1$ is a singularity and corresponds to the critical point. Since $h=i(\pi/2)kT$ is purely imaginary, we identify this critical point $T\to\infty$ as the Yang-Lee edge singularity. Then the critical field must be equal to $h_0=(\pi/2)kT$. This is consistent with the result obtained by Kurtze and Fisher [13].

V. BASIC APPROACH

The Ising partition function in the absence of a magnetic field on each elementary cycle of triangular, square, and honeycomb lattices can be written respectively as

$$z_t=2[e^{\beta(K_1+K_2+K_3)}+e^{\beta(-K_1-K_2+K_3)}+e^{\beta(-K_2-K_3+K_1)}+e^{\beta(-K_3-K_1+K_2)}], \tag{16}$$

$$z_s=2[e^{2\beta(K_1+K_2)}+e^{2\beta(K_1-K_2)}+e^{-2\beta(K_1-K_2)}+e^{-2\beta(K_1+K_2)}+4], \tag{17}$$

$$z_h=2[e^{2\beta(K_1+K_2+K_3)}+4e^{2\beta K_1}+4e^{2\beta K_2}+4e^{2\beta K_3}+4e^{-2\beta K_1}+4e^{-2\beta K_2}+4e^{-2\beta K_3}+e^{2\beta(K_1+K_2-K_3)}+e^{2\beta(K_2+K_3-K_1)}+e^{2\beta(K_3+K_1-K_2)}+e^{-2\beta(K_1+K_2-K_3)}+e^{-2\beta(K_2+K_3-K_1)}+e^{-2\beta(K_3+K_1-K_2)}+e^{-2\beta(K_1+K_2+K_3)}]. \tag{18}$$

Making a transformation $\exp(2\beta K_j)\to i\exp(2\beta K_j)$, we obtain

$$z_t'=2i^{3/2}e^{\beta(K_1+K_2+K_3)}[1-e^{-2\beta(K_1+K_2)}-e^{-2\beta(K_2+K_3)}-e^{-2\beta(K_3+K_1)}], \tag{19}$$

$$z_s'=2\zeta_1^{-1}\zeta_2^{-1}[(\zeta_1+\zeta_2)^2-(1-\zeta_1\zeta_2)^2], \tag{20}$$

$$z_h'=-2i\zeta_1^{-1}\zeta_2^{-1}\zeta_3^{-1}[(1-\zeta_1\zeta_2-\zeta_2\zeta_3-\zeta_3\zeta_1)^2-(\zeta_1+\zeta_2+\zeta_3-\zeta_1\zeta_2\zeta_3)^2], \tag{21}$$

where $\zeta_j\equiv\exp(-2\beta K_j)$. Thus the real solutions of $z'=0$ give the exact zero-field critical temperatures of Ising ferromagnets on triangular, square, and honeycomb lattices [26]: square: $\zeta_1\zeta_2+\zeta_1+\zeta_2=1$; triangular: $\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1=1$; honeycomb: $\zeta_1\zeta_2\zeta_3-\zeta_1\zeta_2-\zeta_2\zeta_3-\zeta_3\zeta_1-\zeta_1-\zeta_2-\zeta_3+1=0$. Thus we make the following observation:

Lemma 1: Let the Ising partition function on each elementary cycle of the square, triangular, and honeycomb lattices be $z=z(T,h=0)$. Make a transformation $\exp(2\beta K_j)\to i\exp(2\beta K_j)$ and thus $z\to z'$. Then the critical temperatures of the Ising ferromagnets on square, triangular, and honeycomb lattices in the absence of a magnetic field are given by the real solutions of $z'=0$.

Along the critical line of the Yang-Lee edge singularity $(\partial h/\partial M)_T(h=ih_0)=0$. Near the critical line, the magnetization exhibits a branch point of the form

$$M(h,T)\sim[h-ih_0(T)]^{\sigma}, \tag{22}$$

where σ is the critical exponent ($\sigma<1$). This suggests that near the critical line, the magnetization takes the form

$$M(T>T_c,h)=g(T,h)[P(T,h)]^{\sigma}, \tag{23}$$

where $g(T,h)$ and $P(T,h)$ are analytic functions of T and h. Let us consider

$$\left(\frac{\partial M}{\partial h}\right)_T=\frac{\partial g}{\partial h}P^{\sigma}+\sigma g\frac{\partial P}{\partial h}P^{\sigma-1}. \tag{24}$$

Since $g(T,h)$ and $P(T,h)$ and their derivatives with respect to h do not diverge for arbitrary h, along the critical line $(\partial h/\partial M)_T(h=ih_0)=0$ requires $P(T,h=ih_0)=0$, which gives the critical line.

On the other hand, for a square lattice Ising model, the spontaneous magnetization is given by [27]

$$M(T<T_c,h=0)=\left[1-\left(\frac{2\zeta_1}{1-\zeta_1^2}\frac{2\zeta_2}{1-\zeta_2^2}\right)^2\right]^{1/8}$$
$$\sim[z'(T,h=0)]^{1/8}. \tag{25}$$

Thus $P(T=T_c,h=ih_0=0)=z'(T=T_c,h=0)$. Therefore we might plausibly extend Lemma 1 to the case $h=ih_0$, $T>T_c$:

Conjecture: Let the Ising partition function on each elementary cycle of square, triangular, and honeycomb lattices be $z=z(T,h)$. Make the transformation

$$e^{2\beta K_j}\to ie^{2\beta K_j} \quad \text{and} \quad \beta h\to f(\beta h_0). \tag{26}$$

Thus $z\to z'$ with $f(0)=0$ and $P(T=T_c,h=ih_0=0)=z'(T=T_c,h=0)$. Here $f(\beta h_0)$ is assumed to be a real function of βh_0. Then *the critical line of the Yang-Lee edge singularity is given by* $P(T,h=ih_0)=z'=0$.

In the following, we will apply this approach to the isotropic Ising ferromagnets on square, triangular, and honeycomb lattices.

VI. ISOTROPIC ISING MODELS

A. Derivation of the fitting function $f(\beta h_0)$

In this case, the partition function on each elementary cycle of square, triangular, and honeycomb lattices is given by

$$z=\lambda_+^N+\lambda_-^N, \tag{27}$$

with

$$\lambda_\pm=e^{\beta K}[\cosh\,\beta h\pm(\sinh^2\beta h+e^{-4\beta K})^{1/2}], \tag{28}$$

where N is the number of edges of an elementary cycle. Thus $z=0$ yields

$$\frac{\cosh\,\beta h}{(\sinh^2\beta h+e^{-4\beta K})^{1/2}}=(-i)\cot\,\pi\frac{2n+1}{2N}$$
$$(n=0,1,\ldots,N-1). \tag{29}$$

Making the transformations (26), we obtain

$$e^{-4\beta K}=\tan^2\pi\frac{2n+1}{2N}\cosh^2 f(\beta h_0)+\sinh^2 f(\beta h_0)$$
$$(n=0,1,\ldots,N-1), \tag{30}$$

where only $n=N-1$ is allowed. Thus the critical line is given by

$$e^{-4\beta K}=\tan^2\left[\frac{\pi(q-2)}{4q}\right]\cosh^2 f(\beta h_0)+\sinh^2 f(\beta h_0), \tag{31}$$

where q is the coordination number and we have used the Baxter's formula [26],

$$e^{-2\beta_c K}=\tan\left[\frac{\pi(q-2)}{4q}\right]. \tag{32}$$

Solving Eq. (31) for $f(\beta h_0)$, we obtain

$$f(\beta h_0)=\frac{1}{2}\ln\left(\frac{C}{2}+\sqrt{\frac{C^2}{4}-1}\right), \tag{33}$$

where

$$C(T)=\frac{4e^{-4\beta K}+2(1-e^{-4\beta_c K})}{1+e^{-4\beta_c K}}. \tag{34}$$

We can determine the functional form of f by considering two limits of Eq. (31).

In the limit $T\to\infty$ Eq. (31) approaches

$$1=\tan^2\left[\frac{\pi(q-2)}{4q}\right]\cosh^2 f(\beta h_0)+\sinh^2 f(\beta h_0). \tag{35}$$

This means $\beta h_0\to$const, which is consistent with the result obtained by Kurtze and Fisher [9],

$$\beta h_0\to\pi/2-(\beta K/z_0)^{1/2}+O(T^{-3/2}). \tag{36}$$

For the 1D Ising ferromagnet, Eq. (11) gives $z_0=1/4$.

In the limit $T\to T_c+$, expanding the left-hand side of Eq. (31) around β_c and the right-hand side around $f=0$, we obtain

$$f(\beta h_0)\to t^{1/2}\left[\frac{4K}{kT_c(1+e^{4\beta_c K})}\right]^{1/2}. \tag{37}$$

On the other hand, according to Eq. (5), $h_0\to|t|^{\beta+\gamma}$ in the limit $T\to T_c+$ (here β is the critical exponent).

We tried several functions and Taylor series for $f(\beta h_0)$ that would yield these two limits. We found that only the following functional form can satisfy the two limits (36) and (37):

$$f(\beta h_0)=A\,\sin[b_1(\beta h_0)^\lambda+b_2(\beta h_0)^{\lambda+1}+b_3(\beta h_0)^{\lambda+2}$$
$$+\cdots], \tag{38}$$

where we need to impose the condition

$$b_1\left(\frac{\pi}{2}\right)^\lambda+b_2\left(\frac{\pi}{2}\right)^{\lambda+1}+b_3\left(\frac{\pi}{2}\right)^{\lambda+2}+\cdots=\frac{\pi}{2}. \tag{39}$$

As $h_0\to 0$, $T\to T_c+$ and Eq. (38) approaches

$$f(\beta h_0)\to Ab_1(\beta h_0)^\lambda. \tag{40}$$

Comparing Eqs. (37) and (40) we obtain $h_0\to t^{1/2\lambda}$ and thus $\lambda=1/2(\beta+\gamma)$. Since for a 2D Ising ferromagnet, $\beta=1/8$ and $\gamma=7/4$, we obtain $\lambda=4/15$. In this way we obtain, as $T\to T_c+$,

$$h_0=t^{15/8}kT_c\left[(Ab_1)^{-2}\frac{4\beta_c K}{1+e^{4\beta_c K}}\right]^{15/8}. \tag{41}$$

As $T\to\infty$, let $\beta h_0\to\pi/2-y$ to obtain

$$f(\beta h_0)\to A-y^2B, \tag{42}$$

where

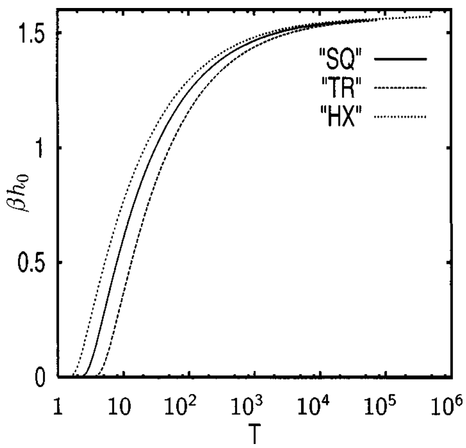

FIG. 2. The critical lines of 2D Ising ferromagnets. The unit of T is k/K.

$A = f(\pi/2)$ and

$$B = (A/2)[b_1\lambda(\pi/2)^{\lambda-1} + b_2(\lambda+1)(\pi/2)^{\lambda} + b_3(\lambda+2)(\pi/2)^{\lambda+1} + \cdots]^2. \tag{43}$$

Thus in this limit, $\sinh f(\beta h_0) \to \sinh A - y^2 B \cosh A$, and $\cosh f(\beta h_0) \to \cosh A - y^2 B \sinh A$. Substituting these results into Eq. (31) we obtain

$$y = \left[\frac{4\beta K}{B(\sinh 2A)(1+e^{-4\beta_c K})}\right]^{1/2}. \tag{44}$$

Therefore we obtain

$$z_0 = B(\sinh 2A)(1+e^{-4\beta_c K})/4. \tag{45}$$

In the following, we determine the coefficients b_n and thus obtain closed-form approximations to the critical lines.

B. Square lattice

Taking the high-temperature limit $\beta h_0 \to \pi/2$ of Eqs. (35) and (38) we obtain $A = \frac{1}{2}\ln(1+\sqrt{2}+\sqrt{2+2\sqrt{2}}) = 0.76429$. Making use of $z_0 = 0.088963$ computed in [13] and $h_0 = t^{15/8} 0.4425 kT_c$ in the limit $T \to T_c+$ and the normalization condition (39), we obtain $b_1 = 0.826\,23$, $b_2 = 0.42432$, and $b_3 = -0.04060$ with

$$f(\beta h_0) = A\ \sin[b_1(\beta h_0)^{\lambda} + b_2(\beta h_0)^{\lambda+1} + b_3(\beta h_0)^{\lambda+2}]. \tag{46}$$

The critical line is plotted in Fig. 2.

C. Triangular lattice

In this case we use $z_0 = 0.056\,076$ computed in [13] and $h_0 = t^{15/8} 0.4529 kT_c$ in the limit $T \to T_c+$. Following the same procedure, we obtain $A = \frac{1}{2}\ln(2+\sqrt{3}) = 0.658\,48$, $b_1 = 0.983\,07$, $b_2 = 0.276\,31$, and $b_3 = -0.009\,93$. The critical line is plotted in Fig. 2.

D. Honeycomb lattice

Kurtze and Fisher [13] calculated z_0 for some lattices: $z_0 = 0.088\,963$ (square, $q=4$); $z_0 = 0.056\,076$ (triangular, $q=6$); $z_0 = 0.052\,025$ (simple cubic, $q=6$); $z_0 = 0.037\,309$ (body centered cubic, $q=8$); $z_0 = 0.024\,224$ (face centered cubic, $q=12$). From these values we notice that z_0 is roughly proportional to $1/q$. Using these observations, we roughly estimate $z_0 = 0.12$ for the honeycomb lattice ($q=3$).

In Sec. II, we calculated the coefficients of Eq. (5) for Ising ferromagnets on many lattices. It is found that for a given dimension, the coefficients vary slightly and are approximately independent of lattice structures. Therefore we estimate that for an Ising ferromagnet on a honeycomb lattice, $h_0 = 0.45 t^{15/8} kT_c$ as $T \to T_c+$.

Using the above estimates and following the same procedure, we obtain $A = \frac{1}{2}\ln(1+\sqrt{3}+\sqrt{3+2\sqrt{3}}) = 0.831\,44$ and $b_1 = 0.62509$, $b_2 = 0.63654$ and $b_3 = -0.09418$. The critical line is plotted in Fig. 2.

VII. CONCLUSION

We have extended our recent approach exploiting the zeroes of Ising partition functions of the elementary cycles on square, triangular, and honeycomb lattices. The exact zero-field critical conditions are obtained as the zeroes of the transformed Ising partition functions. Making use of the critical condition of the Yang-Lee edge singularity, $(\partial h/\partial M)_T(T>T_c, h=ih_0)=0$, we extended the zero-field critical conditions to the case $T>T_c$ and $h=ih_0(T)$, and obtained the critical lines, $h=ih_0(T)$ of the Yang-Lee edge singularity of Ising ferromagnets on square, triangular, and honeycomb lattices.

We use the two limiting behaviors of h_0: (1) the property of the critical lines: $h_0 \to t^{15/8}$ as $T \to T_c+$, (2) the limiting form of the critical line as $T \to \infty$, $\beta h_0 \to \pi/2 - (\beta K/z_0)^{1/2} + O(T^{-3/2})$. The asymptotic behaviors of the critical line in the high-temperature limit and in the zero-field limit restrict the functional form of $f(\beta h_0)$ to a sine function. By using the numerical results obtained by Kurtze and Fisher, we obtained some constants A, b_1, b_2, and b_3 for the three lattices. Using Yang and Lee's exact solution of a square lattice Ising ferromagnet in a specific imaginary magnetic field $h = i(\pi/2)kT$ we obtained the exact critical field in the high-temperature limit.

In addition, we showed that the Yang-Lee edge singularity is a general aspect of critical phenomenon and any ferromagnet with the Curie point has the Yang-Lee edge singularity. We have not found numerical data for anisotropic lattices. If such data become available in the future, we will extend our method to the anisotropic case and obtain the critical lines on these lattices.

ACKNOWLEDGMENT

This work was funded by the Pohang University of Science & Technology.

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