Kink solitons of the double-quadratic model in the presence of an external spatially inhomogeneous force

J. A. Hołyst*
Institute of Physics, Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland
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Dynamics of kink solitons of the double-quadratic model in the presence of an external, spatially inhomogeneous force $F(x)$ is considered. Stability analysis of such solutions is performed. It is shown that similarly as for kinks of the ϕ^4 equation there are differences between a stability condition for the kink treated as a point particle and the kink treated as a spatially extended object. These differences occur when the force $F(x)$ possesses more than one zero and follow from interactions of kink wings (i.e., parts of kinks that are far from a kink center) with additional zeros of the force $F(x)$. [S1063-651X(98)08704-2]

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Understanding the behavior of kink solitons in the presence of external perturbations is a task that was a subject of several papers in the past, e.g., [1–7]. It is well known that in many cases the kinks can be treated as *structureless point particles* [1–4] and it is enough to consider the influence of external force on the *center* of such kinks. On the other hand, it has also been shown that the *internal structure* of kink solitons plays an important role for the kink-kink scattering [6], for the soliton stability in the presence of external, spatially extended forces [7], for the thermodynamical properties of a kink gas [8], and for the behavior of kinks in DNA chains [9]. A common manifestation of such an internal soliton structure is a spectrum of soliton-phonon [1,6–8] or soliton-magnon [10] *bound states* that can be treated as small localized oscillations around a kink. This spectrum determines directly the soliton stability, i.e., the soliton is stable provided that all eigenvalues of the corresponding stability operator are positive. In the paper [7] it has been shown that for kinks of the ϕ^4 model (with an external force included) this stability condition differs from the stability condition based on the simplified energetic approach assuming the kink as a *point particle*. The scope of the present paper is to study the stability of kinks in the perturbed one-dimensional double-quadratic (DQ) model that is an interesting example of nonlinear Klein-Gordon systems that have been used to describe properties of localized excitations in quasi-one-dimensional solids (e.g., domain walls in ferroelectrics and ferromagnets or dislocations in crystals [1,8]). Similarly to other nonlinear Klein-Gordon systems the DQ model makes use of a scalar classical field with *two degenerated vacua* [1,8] and its statistical mechanics has been considered in [8,12]. Special features of DQ models are as follows [8,12]: (i) the local field energy consists of two *harmonic* wells centered on positions corresponding to the field vacua; (ii) the kinks in this model are nontransparent to passing phonons; (iii) the effective scattering potential for kink-phonon interactions possesses a δ-like shape [8]; (iv) the model can be treated as a limiting case of a family of nonlinear Klein-Gordon models with Pöschel-Teller scattering potential [11].

*Electronic address: jholyst@if.pw.edu.pl

Let us consider the one-dimensional model of a scalar classical field $\phi(z,\tau)$ defined by the following Hamiltonian:

$$H=A\int\left\{\frac{1}{2}[(\phi_\tau)^2+c_0^2(\phi_z)^2]+\omega_0^2[V(\phi)-\phi F(z)]\right\}dz, \tag{1}$$

where ϕ_z and ϕ_τ mean corresponding partial derivatives, the constant A determines the energy scale, the constant c_0 is the system characteristic velocity, the constant ω_0 is the system characteristic frequency, the first part on the right-hand side of Eq. (1) is the field kinetic energy, the second one describes the elastic interactions (strain energy), the potential $V(\phi)=\frac{1}{2}(|\phi|-1)^2$ is the ''local'' potential depending only on the field ϕ, while the last term in Eq. (1) represents the potential of the external, space dependent force $F(z)$.

Writing the standard equation of motion for the field $\phi(z,\tau)$ and introducing the dimensionless time $t=\omega\tau$ and dimensionless distance $x=(\omega/c_0)z$ we get

$$\phi_{tt}=-(|\phi|-1)\,\mathrm{sgn}(\phi)+\phi_{xx}+F(x). \tag{2}$$

In the absence of the force $F(x)$ one gets immediately a kink solution of Eq. (2) in the form

$$\phi_{(k)}(x,t)=(1-e^{-|x|})\,\mathrm{sgn}(x) \tag{3}$$

that connects two degenerated minima $\phi=-1$ and $\phi=1$ of the local potential $V(\phi)$. Moving kinks can be simply obtained by a Lorentz-like transformation of the solution (3).

Now let us assume that the force $F(x)$ has the form

$$F(x)=[A-1+A(B^2-1)e^{-B|x|}]\,\mathrm{sgn}(Bx), \tag{4}$$

where $A>0$ and $B>0$ are parameters. The force (4) is an antisymmetric function of the position x and for $A=1$ it represents a localized impurity while for $B=1$ it corresponds to a steplike phase boundary (Fig. 1). Assuming that $\mathrm{sgn}(0)=0$ we see that the force $F(x)$ can possess one or three zeros. In fact for

$$(A-1)(AB^2-1)>0 \tag{5}$$

there is only one zero of $F(x)$, i.e., $x_0=0$ while for

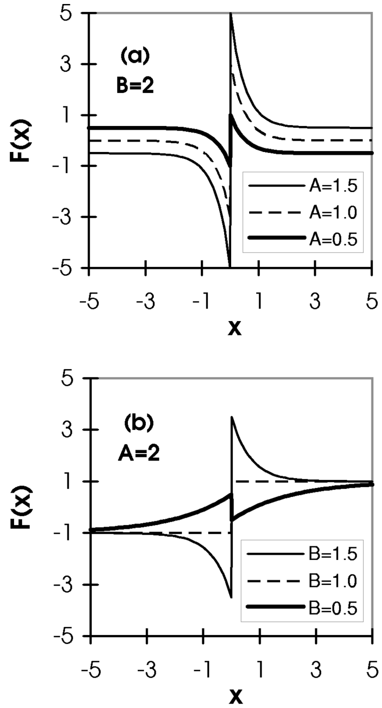

FIG. 1. Function $F(x)$ for various parameters A and B.

$$(A-1)(AB^2-1)<0 \tag{6}$$

we have three zeros of $F(x)$,

$$x_0=0, \quad x_+=B^{-1}\left|\ln\frac{1-A}{A(B^2-1)}\right|, \quad x_-=-x_+. \tag{7}$$

One can check by a simple algebra that for all values of the parameters A and B there is a kink solution of Eq. (2) in the form

$$\phi_{(k)}^{AB}(x)=A(1-e^{-B|x|})\ \mathrm{sgn}(Bx). \tag{8}$$

This solution represents a rescaled solution (3) centered on the point $x_0=0$, i.e., on the ''central'' zero of the function $F(x)$. The fact that the zero of the external force can determine the position of the kink soliton is well known [6,7].

Now let us consider the stability problem of the solution (8). Treating the kink as *a point particle* we can follow Ref. [6] and write the stability condition as

$$\left[\frac{dF(x)}{dx}\right]_{x=0}>0. \tag{9}$$

Putting Eq. (4) into Eq. (9) we get

$$AB^2>1. \tag{10}$$

On the other hand, we can perform a stability analysis of the kink $\phi_{(k)}^{AB}(x)$ treating this solution as *an extended object*, i.e., using the approach developed in [1,7]. Writing $\phi(x,t)=\phi_{(k)}^{AB}(x)+\eta(x,t)$ where $|\eta(x,t)|\ll|\phi_{(k)}^{AB}(x)|$ and assuming according to the linear stability theory [1,7,8,12] that the perturbation $\eta(x,t)$ is a spatially extended function with a harmonic time dependence, i.e., $\eta(x,t)=\exp(i\omega t)f(x)$ we get after the proper linearization of Eq. (2)

$$-f_{xx}(x)+\left[1-\frac{2}{AB}\delta(x)\right]f(x)=\omega^2 f(x). \tag{11}$$

This equation can be treated formally as an eigenvalue problem for a Schrödinger equation for a ''quantum'' particle in a ''δ-potential well'' described by the ''Hamiltonian''

$$\mathcal{H}=-\frac{d^2}{dx^2}+\left[1-\frac{2}{AB}\delta(x)\right]. \tag{12}$$

The stability condition of the solution $\phi_{(k)}^{AB}(x)$ is equivalent to

$$\omega^2>0 \tag{13}$$

for all ''eigenvalues'' of the problem (11). However, the Hamiltonian $\mathcal{H}$ can be treated as a limiting case of a generalized Pöschel-Teller problem [11,13]

$$\mathcal{H}^{\mathrm{PT}}=-\frac{d^2}{dx^2}+\left[1-\frac{\alpha^2\lambda(\lambda-1)}{\cosh^2(\alpha x)}\right], \tag{14}$$

where

$$\lambda=1+\frac{\epsilon}{AB}, \quad \alpha=\frac{1}{\epsilon}, \quad \epsilon\to 0. \tag{15}$$

Using this representation we find that the eigenvalue problem (11) possesses an infinite number of continuous eigenvalues $\omega_k^2>1$ and exactly one bound state

$$\omega_b^2=1-\frac{1}{A^2B^2}. \tag{16}$$

The stability condition (13) can now be written as

$$AB>1. \tag{17}$$

We see that this last condition is different from the stability condition (10) where the kink was treated as a point particle. However, if we combine the condition (10) with the condition (5) [being the condition for *a single zero* $x_0=0$ of the force $F(x)$] then we get $A>1$ and the condition (17) is fulfilled.

Putting the conditions (5), (6), (10), and (17) together we can divide the space of parameters A and B into six regions I–VI where depending on the shape of the force $F(x)$ the kink treated as a point particle (PP) or as an extended object (EO) behaves as follows (see Fig. 2).

I: $F(x)$ has a single zero, PP is stable, EO is stable.
II: $F(x)$ has three zeros, PP is stable, EO is stable.
III: $F(x)$ has three zeros, PP is stable, EO is unstable.
IV: $F(x)$ has a single zero, PP is unstable, EO is unstable.
V: $F(x)$ has three zeros, PP is unstable, EO is unstable.
VI: $F(x)$ has three zeros, PP is unstable, EO is stable.

We see that in region III the presence of additional zeros of the force $F(x)$ can *destabilize* the kink treated as an extended object in comparison to the point particle behavior.

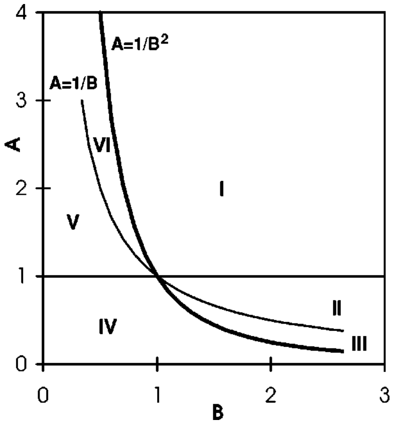

FIG. 2. Stability regions for the kink treated as a point particle and as an extended object.

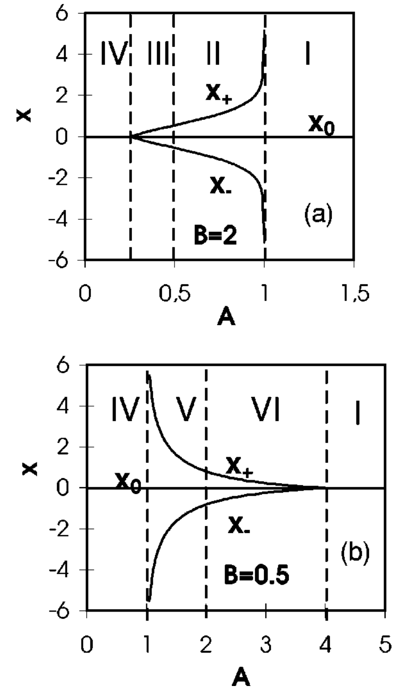

FIG. 3. Positions of zeros of the function $F(x)$.

On the other hand, in region VI these additional zeros can also *stabilize* the spatially extended object while the point particle is unstable. Such stabilization or destabilization effects follow from influence of the force $F(x)$ on the kink wings, i.e., on the parts of the kink that are far away from the kink center $x=0$. As the zeros of the force $F(x)$ determine boundaries of segments on the X axis where the corresponding parts of the kink are pushed by the force $F(x)$ along one of two opposite directions, the positions of the zeros x_+ and x_- are important for the soliton stability. In fact, in regions III and VI (Fig. 3) the zeros x_+ and x_- are close to the kink center $x_0=0$ and they influence essentially the soliton stability, while in regions II and V these additional zeros are far away from the kink center and their influence can be neglected. A similar phenomenon has been observed for perturbed kinks in the ϕ^4 model [7].

In conclusion, we have shown that the stability of kinks in the double-quadratic model in the presence of an additional, spatially dependent force depends on the existence and positions of zeros of such a force. The result demonstrates the spatial character of the kink soliton.

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