Electrical conductivity in a dilute gas far from equilibrium

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Electrical conductivity of a minor constituent of charged particles through a background of neutral particles is studied in the limit of small electrical fields. The neutral gas is in a steady state with both arbitrary velocity and temperature gradients. The results are obtained from the Gross-Krook model [Phys. Rev. **102**, 593 (1956)] of the Boltzmann equation for a binary mixture. The transport properties of the charged species are obtained from a perturbation expansion in powers of the electrical field. In the first order, the current density obeys a generalized Ohm's law where an electrical conductivity tensor is identified. The nonzero elements of this tensor are nonlinear functions of the shear rate, the mass ratio, and the force constant ratio. The results show that, in general, the presence of the shear produces an inhibition in the diffusion of charged particles. [S1063-651X(98)04704-7]

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I. INTRODUCTION

An interesting problem in nonequilibrium statistical mechanics is the description of transport properties of *charged* particles immersed in a bath of *neutral* particles and subjected to a constant electric field **E**. The system can be seen as a binary mixture of fluids, one of them is constituted by particles, e.g., ions, whose electric charge (positive or negative) is different from zero. Nevertheless, the general study of these systems is certainly much more complicated than that of ordinary fluids since now one needs to consider the Coulomb interaction between charged particles. This complexity leads one to analyze first more tractable situations for which a thorough description can be offered. A possible way to avoid the Coulomb interaction is to assume that the concentration of charged particles (labeled with the index 1) is much smaller than that of neutral particles (labeled with the index 2), so that charged particles are so dilute that their mutual interactions can be neglected. Furthermore, the state of the neutral fluid is not affected by the presence of the charged species. Under these conditions (*tracer* limit), one can only take into account elastic collisions of charged-neutral and neutral-neutral type. The analysis of transport processes occurring in such systems is an important problem in chemistry and physics [1] with applications to aeronomy, astrophysics, and other areas [2].

When the bath is at equilibrium and the electric field is weak, the current density $\mathbf{j}_1$ obeys the phenomenological Ohm's law, namely, $\mathbf{j}_1=\sigma_0\mathbf{E}$, σ_0 being the electrical conductivity coefficient. In the case of dilute gases, the expression of σ_0 can be obtained, for instance, from the conventional Chapman-Enskog expansion [3]. This situation has been widely investigated in the past few years in different contexts [4–6]. However, much less is known when the neutral gas is far from equilibrium. If the electric field is assumed to be small, one expects that Ohm's law still applies although a conductivity tensor σ_{ij} rather than a scalar may be identified. This tensor must be a function of the nonequilibrium parameters (hydrodynamic gradients) as well as of the parameters characterizing the mixture (mass and size ratios).

In order to capture the relevant aspects of such a nonlinear problem we choose a binary mixture in the low-density regime. In this case, the state of the mixture is characterized by the one-particle velocity distribution functions $f_i(\mathbf{r},\mathbf{v};t)$ $(i\equiv 1,2)$. As stated above, we consider a mixture in the tracer limit, namely, when the molar fraction of the charged particles 1 is much smaller than one. In this case, it is not necessary to consider the Coulomb interaction and the kinetic equations describing this situation reduce to a (closed) Boltzmann equation for the velocity distribution of neutral particles f_2, and a Boltzmann-Lorentz equation for the distribution f_1 of charged particles. In addition, the neutral gas is constituted by nonpolarizable molecules so that the corresponding polarization force is neglected. We also assume that the neutral gas is in steady planar Couette flow, namely, the system is enclosed between two parallel plates in relative motion and kept at different temperatures. Consequently, there are two parameters measuring the distance from equilibrium: the shear rate and the thermal gradient. Our goal is to evaluate the influence of such nonequilibrium parameters on the diffusion of charged particles in the limit of small electric fields.

Unfortunately, due to the mathematical difficulties embodied in the Couette flow problem, no analytic solution has been found to the Boltzmann equation, even for a single gas. For this reason, here we start from the well-known Gross-Krook (GK) kinetic model [7] for binary mixtures. The GK model is constructed in the same spirit as the Bhatnagar-Gross-Krook (BGK) model [8] of a single gas, for which an *exact* description of the planar Couette flow state has been given [9,10]. The reliability of the GK model has been assessed in several nonequilibrium problems [11] by comparison with exact results [12] and Monte Carlo simulations [13] of the Boltzmann equation.

Since the state of the neutral gas is well characterized, we solve the Boltzmann-Lorentz equation corresponding to the charged particles by performing a perturbation expansion in powers of the electric field. In contrast to the conventional Chapman-Enskog method, the zeroth-order approximation is not the local equilibrium but a nonequilibrium state with arbitrary values of the shear rate and the thermal gradient. To first order in the field, we get an explicit expression of the electrical conductivity tensor σ_{ij}. According to the geometry of the problem, there are five nonzero elements: σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, and σ_{yx}. They are highly nonlinear functions of the shear rate and the ratios of mass and force constants. In general, these elements decrease with the shear rate so that the Couette flow produces an inhibition on the current electrical density. The fact that all these elements are different shows clearly the anisotropy induced by the Couette flow in the transport of charge.

II. DESCRIPTION OF THE PROBLEM

Let us consider a low-density binary mixture composed of charged particles (of mass m_1, charge q, and number density n_1) and neutral particles (of mass m_2 and number density n_2). We assume that $n_1 \ll n_2$, so that the interactions of charged–charged type can be neglected in the kinetic equation of f_1. In addition, the smallness of the ratio n_1/n_2 allows us also to suppose that the state of the neutral gas is not influenced by the ion or electron motion. Consequently, only elastic interactions of type 1-2 and 2-2 will be considered in our description. We assume finally that on the mixture acts a uniform electric field $\mathbf{E}$ that generates a current of charged particles. Under these conditions, the set of coupled Boltzmann equations reads

$$\frac{\partial}{\partial t} f_1 + \mathbf{v} \cdot \nabla f_1 + \frac{q\mathbf{E}}{m_1} \cdot \frac{\partial}{\partial \mathbf{v}} f_1 = J_{12}[f_1, f_2], \tag{1}$$

$$\frac{\partial}{\partial t} f_2 + \mathbf{v} \cdot \nabla f_2 = J_{22}[f_2, f_2], \tag{2}$$

$f_i(\mathbf{r},\mathbf{v};t)$ being the one-particle velocity distribution function of species i ($i=1,2$) and $J_{ij}[f_i, f_j]$ is the Boltzmann collision operator. From the distribution f_i, one defines the number density and mean velocity of species i, respectively, as

$$n_i = \int d\mathbf{v} f_i, \tag{3}$$

$$\mathbf{u}_i = \frac{1}{n_i} \int d\mathbf{v} \mathbf{v} f_i. \tag{4}$$

It is also convenient to define a temperature T_i for each species, which is a measure of its mean kinetic energy per particle. It is given by

$$\frac{3}{2} n_i k_B T_i = \frac{m_i}{2} \int d\mathbf{v} (\mathbf{v} - \mathbf{u}_i)^2 f_i, \tag{5}$$

where k_B is the Boltzmann constant.

Due to the complicated mathematical structure of the operators J_{ij}, it is a very hard task to solve the set of equations (1) and (2), especially in far from equilibrium situations. In order to overcome such a problem, one possibility is to use a simplified kinetic model that retains the main qualitative aspects of the true Boltzmann collision operator. Here, we consider the well-known GK model [7], in which case the exact collision integrals J_{ij} are replaced by relaxation terms of the form

$$J_{ij}^{\mathrm{GK}} = -\nu_{ij}(f_i - f_{ij}), \tag{6}$$

where ν_{ij} is an effective collision frequency and the reference distribution functions f_{ij} are

$$f_{ij} = n_i \left(\frac{m_i}{2\pi k_B T_{ij}} \right)^{3/2} \exp\left(-\frac{m_i}{2 k_B T_{ij}} (\mathbf{v} - \mathbf{u}_{ij})^2 \right), \tag{7}$$

with

$$\mathbf{u}_{ij} = \frac{m_i \mathbf{u}_i + m_j \mathbf{u}_j}{m_i + m_j}, \tag{8}$$

$$T_{ij} = T_i + 2 \frac{m_i m_j}{(m_i + m_j)^2} \left[(T_j - T_i) + \frac{m_j}{6 k_B} (\mathbf{u}_i - \mathbf{u}_j)^2 \right]. \tag{9}$$

The above expressions are obtained by requiring that momentum and energy moments of J_{ij}^{GK} be the same as those of the Boltzmann operator for Maxwell molecules (i.e., an interaction potential of the form $\Phi_{ij} = \kappa_{ij} r^{-4}$). This allows one to identify ν_{ij} as [7]

$$\nu_{ij} = A n_j \left[\kappa_{ij} \frac{m_i + m_j}{m_i m_j} \right]^{1/2}, \tag{10}$$

where $A = 4\pi \times 0.422$. It must be remarked that the results derived in this paper could be in principle extended to more general potentials.

According to Eq. (6), the effect of the collisions on particles of species i is to produce a tendency toward a reference state characterized by the distribution f_{ij}. This function depends on space and time only through the density n_i, the mean velocities $\mathbf{u}_i$ and $\mathbf{u}_j$, and the partial temperatures T_i and T_j. All these quantities are moments of the velocity distribution functions f_i and f_j that must be determined self-consistently, so the GK kinetic model is actually *more* nonlinear than the original bilinear Boltzmann equation and consequently, it can be used to evaluate nonlinear transport properties. The results obtained in the past few years confirm the usefulness of this model in such nonequilibrium situations [11–14].

We describe now the problem we are interested in. Let us assume that the neutral gas is in steady planar Couette flow, namely, it is enclosed between two parallel plates (normal to the y axis) in relative motion (along the x axis) and kept at different temperatures. The Couette flow is not an idealized state since it can be generated in computer simulations by means of realistic boundary conditions [15]. These boundary conditions lead to combined heat and momentum transport. In this state, no explicit solution of the Boltzmann equation valid for arbitrary values of the velocity and temperature gradients is known, although a perturbation solution through super-Burnett order has been recently obtained [16]. Nevertheless, an exact description can be given if one uses the BGK approximation [8]. In this case, Eq. (1) becomes

$$v_y \frac{\partial}{\partial y} f_2 = -\nu_{22}(f_2 - f_{22}). \tag{11}$$

The solution is characterized by a uniform pressure p_2 and linear velocity and parabolic temperature profiles with respect to a scaled space variable, i.e.,

$$p_2 \equiv n_2 k_B T_2 = \text{const}, \tag{12}$$

$$\frac{1}{\nu_{22}(y)} \frac{\partial}{\partial y} u_{2,x} = a = \text{const}, \tag{13}$$

$$\left[\frac{1}{\nu_{22}(y)} \frac{\partial}{\partial y}\right]^2 T_2 = -\frac{2m_2}{k_B}\gamma(a) = \text{const}. \tag{14}$$

The dimensionless parameter $\gamma(a)$ is a nonlinear function of the *reduced* shear rate a through the implicit equation [9]

$$a^2 = \gamma \frac{2F_2(\gamma) + 3F_1(\gamma)}{F_1(\gamma)}, \tag{15}$$

where $F_r(x) \equiv [(d/dx)x]^r F_0(x)$ and

$$F_0(x) = \frac{2}{x}\int_0^\infty dt\, t e^{-t^2/2} K_0(2x^{-1/4}t^{1/2}), \tag{16}$$

K_0 being the zeroth-order modified Bessel function [17].

From the profiles (12)–(14), one may derive the expressions of the momentum and heat fluxes [9]. Furthermore, an explicit expression for the velocity distribution function f_2 has been also derived [10]. Now, our objective is to solve the kinetic equation for f_1 when an external field is applied.

III. ELECTRICAL CONDUCTIVITY TENSOR UNDER HEAT AND MOMENTUM TRANSPORT

Under the geometry established in the Couette flow problem and assuming that after a certain transient stage the charged particles reach a steady state, Eq. (1) becomes

$$v_y \frac{\partial}{\partial y} f_1 + \frac{q\mathbf{E}}{m_1} \cdot \frac{\partial}{\partial \mathbf{v}} f_1 = -\nu_{12}(f_1 - f_{12}), \tag{17}$$

where, according to Eq. (10),

$$\nu_{12} = \omega^2 \left(\frac{1+\mu}{2}\right)^{1/2} \nu_{22}. \tag{18}$$

Here, $\mu \equiv m_2/m_1$ is the mass ratio and $\omega \equiv (\kappa_{12}/\kappa_{22})^{1/4}$ is the force constant ratio [18]. We are interested in analyzing the influence of the Couette flow on the electrical current density. To this end, we shall follow a perturbation scheme in the same spirit as in the Chapman-Enskog method [3]. Assuming that the strength of the electric field is weak, we perform an expansion taking $\mathbf{E}$ as the perturbation parameter. The main feature of this expansion is that the zeroth-order approximation is a nonequilibrium state with arbitrarily large velocity and temperature gradients. As a consequence, the corresponding transport coefficients will be nonlinear functions of both gradients. In the same way as the Chapman-Enskog solution, it is expected that the expansion actually could be divergent although sufficiently asymptotic to be useful in the limit of small electric fields. Thus, we write f_1 in the form

$$f_1 = f_1^{(k)} + O(\mathbf{E}^{k+1}), \tag{19}$$

where the approximation $f_1^{(k)}$ contains all the contributions up to order k in $\mathbf{E}$, although it is a highly nonlinear function of the shear rate and the thermal gradient. The corresponding hydrodynamic fields $\mathbf{u}_{12}$ and T_{12} must also be expanded in a similar way. By substituting these expansions into Eq. (17), one gets a hierarchy of equations for the different distributions $f_1^{(k)}$. Here, we will restrict our calculations to the first order in the external field.

The zeroth-order approximation is concerned with a situation where no external field is applied on the system and, consequently, the current density vanishes. This reference state has been widely analyzed by Garzó and Santos [19] and now we offer a brief account of the main results. Thus, when $\mathbf{E} = \mathbf{0}$, $\mathbf{u}_1^{(0)} = \mathbf{u}_2$, and the state of the charged particles is characterized by the profiles

$$p_{12}^{(0)} \equiv n_2 k_B T_{12}^{(0)} = \text{const}, \tag{20}$$

$$\frac{1}{\nu_{12}(y)} \frac{\partial}{\partial y} u_{12,x}^{(0)} = \widetilde{a} = \text{const}, \tag{21}$$

$$\left[\frac{1}{\nu_{12}(y)} \frac{\partial}{\partial y}\right]^2 T_{12}^{(0)} = -\frac{2m_1}{k_B}\widetilde{\gamma} = \text{const}. \tag{22}$$

Here, $\mathbf{u}_{12}^{(0)} = \mathbf{u}_1^{(0)}$, $T_{12}^{(0)}/T_2 = \chi + 2M(1-\chi)$, $M \equiv \mu/(1+\mu)^2$, $\chi \equiv T_1^{(0)}/T_2$, and

$$\widetilde{a} \equiv \frac{a}{\omega^2\sqrt{(1+\mu)/2}}, \tag{23}$$

$$\widetilde{\gamma} \equiv \frac{2\mu}{1+\mu} \frac{\chi + 2M(1-\chi)}{\omega^4} \gamma. \tag{24}$$

The only unknown is the temperature ratio χ, which is the solution to the following implicit equation:

$$2\widetilde{F}_2 + \left(3 - \frac{\widetilde{a}^2}{\widetilde{\gamma}}\right)\widetilde{F}_1 = \frac{3}{\widetilde{\gamma}} \frac{M(1-\chi)}{\chi + 2M(1-\chi)}, \tag{25}$$

where $\widetilde{F}_r \equiv F_r(\widetilde{\gamma})$. The solution of Eq. (25) gives χ as a function of a, μ, and ω. The temperature ratio varies monotonically with the shear rate from 1 (when $a \to 0$) to $1/\mu$ (when $a \to \infty$). From this quantity, the partial contributions of the charged particles to the pressure tensor and the heat flux can be calculated. Their explicit expressions can be found in Ref. [19].

In addition, the velocity distribution function of the charged particles can also be obtained. This is one of the main advantages of using kinetic models. The distribution $f_1^{(0)}$ represents the reference state around which we carry out our expansion. It can be written as $f_1^{(0)} = n_1(m_1/2\pi k_B T_2)^{3/2}\Phi(\boldsymbol{\xi})$, where

$$\Phi(\boldsymbol{\xi})=\omega^2\left(\frac{1+\mu}{2\mu}\right)^{1/2}[\chi+2M(1-\chi)]^{-3/2}\frac{2\alpha(1+\alpha)^{3/2}}{\epsilon|\xi_y|}\int_{t_0}^{t_1}dt[2t-(1-\alpha)t^2]^{-5/2}$$
$$\times\exp\left\{-\omega^2\left(\frac{1+\mu}{2\mu}\right)^{1/2}\frac{2\alpha}{1+\alpha}\frac{1-t}{\epsilon\xi_y}-[\chi+2M(1-\chi)]^{-3/2}\frac{1+\alpha}{2t-(1-\alpha)t^2}\left[\left(\xi_x+\frac{2a\alpha}{1+\alpha}\frac{1-t}{\mu^{1/2}\epsilon}\right)^2+\xi_y^2+\xi_z^2\right]\right\}. \tag{26}$$

Here, $(t_0,t_1)=(0,1)$ if $\xi_y>0$ and $(t_0,t_1)=[1,2/(1-\alpha)]$ if $\xi_y<0$. Besides, $\boldsymbol{\xi}\equiv(m_1/2k_BT_2)^{1/2}(\mathbf{v}-\mathbf{u}_2)$,

$$\epsilon\equiv\frac{1}{\nu_{22}}\left(\frac{2k_BT_2}{m_2}\right)^{1/2}\frac{1}{T_2}\frac{\partial}{\partial y}T_2 \tag{27}$$

is a reduced thermal gradient, and

$$\alpha\equiv\frac{\epsilon}{(\epsilon^2+8\gamma)^{1/2}}. \tag{28}$$

The nonlinear dependence of $\Phi(\boldsymbol{\xi})$ on the reduced gradients a and ϵ and on the parameters of the mixture μ and ω is apparent. This implies that, in general, the GK distribution $f_1^{(0)}$ is highly distorted with respect to the local equilibrium Maxwellian [19].

Let us assume now that we disturb the above reference state by applying a weak electric field $\mathbf{E}$. Our goal is to get the electrical conductivity tensor when only terms up to first order in the field are retained. This tensor is identified from the electrical current density, which gives the mean velocity of the charged particles relative to the neutral particles. At first order, it is defined as

$$\mathbf{j}_1^{(1)}=q\int d\mathbf{v}\mathbf{V}f_1^{(1)}, \tag{29}$$

where $\mathbf{V}=\mathbf{v}-\mathbf{u}_2$. On physical grounds, since we are interested in a situation where the current of charged particles is only generated by the action of the external field, we also assume that the molar fraction $x_1=n_1/n_2$ is constant. As a consequence, no mutual diffusion due to a concentration gradient appears in the system. By substituting the expansion (19) into Eq. (17), one gets the kinetic equation

$$v_y\frac{\partial}{\partial y}f_1^{(1)}+\frac{q\mathbf{E}}{m_1}\cdot\frac{\partial}{\partial\mathbf{V}}f_1^{(0)}=-\nu_{12}(f_1^{(1)}-f_{12}^{(1)}), \tag{30}$$

where $f_{12}^{(1)}$ is given by

$$f_{12}^{(1)}=\left[1+\frac{m_1}{n_1k_BT_{12}^{(0)}}\frac{1}{q(1+\mu)}\mathbf{V}\cdot\mathbf{j}_1^{(1)}+\left(\frac{m_1V^2}{2k_BT_{12}^{(0)}}-\frac{3}{2}\right)\left(\frac{T_{12}^{(1)}}{T_{12}^{(0)}}-1\right)\right]f_{12}^{(0)} \tag{31}$$

and

$$f_{12}^{(0)}=n_1\left(\frac{m_1}{2\pi k_BT_{12}^{(0)}}\right)^{3/2}\exp\left(-\frac{m_1V^2}{2k_BT_{12}^{(0)}}\right). \tag{32}$$

The ratio $T_{12}^{(1)}/T_{12}^{(0)}$ is defined as

$$x_1\left(\frac{T_{12}^{(1)}}{T_{12}^{(0)}}-1\right)=(1-2M)\frac{p_1^{(1)}-p_1^{(0)}}{p_{12}^{(0)}}, \tag{33}$$

where

$$p_1^{(k)}=\frac{m_1}{3}\int d\mathbf{v}V^2f_1^{(k)} \tag{34}$$

is the partial pressure of charged species.

Before solving the general nonlinear problem, it is instructive to get the conductivity coefficient σ_0 when the bath is at equilibrium. In this case, the first term on the left-hand side of Eq. (30) vanishes (homogeneous state), $\mathbf{u}_2=0$, and $T_{12}=T_1=T_2$. A simple calculation yields the usual Ohm's law, i.e. $\mathbf{j}_1^{(1)}=\sigma_0\mathbf{E}$ with

$$\sigma_0=\frac{m_1+m_2}{m_1m_2}\frac{q^2n_1}{\nu_{12}}. \tag{35}$$

The mass balance equation associated with the charged particles implies that the current density is uniform. According to Eq. (35), this requires that the ratio $n_1(y)/\nu_{12}(y)=\text{const}$. For Maxwell molecules, $\nu_{12}(y)\propto n_2(y)$ so that σ_0 is a constant if and only if the molar fraction n_1/n_2 is also constant. This shows the consistency of the assumption previously established, at least in the absence of hydrodynamic gradients.

In order to evaluate the current density when the mixture is in steady Couette flow, let us consider the formal solution to Eq. (30) given by

$$f_1^{(1)}=\left(1+\frac{v_y}{\nu_{12}}\frac{\partial}{\partial y}\right)^{-1}\left(f_{12}^{(1)}-\frac{q}{m_1\nu_{12}}\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{V}}f_1^{(0)}\right). \tag{36}$$

Notice that when the operator $[1+(v_y/\nu_{12})(\partial/\partial y)]^{-1}$ acts on the quantities appearing on its right side, only terms through first order in the electric field need to be considered. On the other hand, the solution (36) is still formal since $f_{12}^{(1)}$ depends on the unknown moments $\mathbf{j}_1^{(1)}$ and $p_1^{(1)}$. The calculation of these moments is quite tedious and is carried out in the Appendix. According to these results, the electrical current density can be recast into the form of a generalized Ohm's law

$$j_{1,i}^{(1)}=\sigma_{ij}E_j. \tag{37}$$

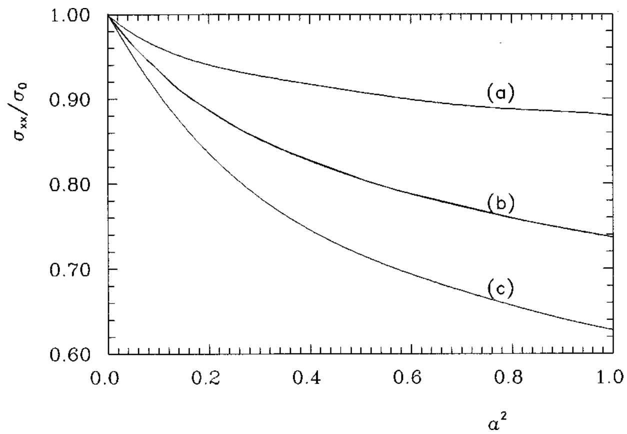

FIG. 1. Shear-rate dependence of σ_{xx}/σ_0 for $\omega=1$ and three values of the mass ratio $\mu\equiv m_2/m_1$: (a) $\mu=10$; (b) $\mu=1$; and (c) $\mu=0.5$.

The explicit expressions of the nonzero elements of the conductivity tensor σ_{ij} are given in the Appendix. Equation (37) describes the transport of a minor constituent of charged particles through a background of neutral species far from equilibrium in the limit of small electric fields. The nonzero elements of the conductivity tensor provide all the information about the physical mechanisms involved in this nonlinear problem. It must be also remarked that upon deriving Eq. (37) no restriction on the mass and force constant ratios have been considered.

According to the symmetry of the Couette flow, the relevant elements of the conductivity tensor are σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, and σ_{yx}. These elements are nonlinear functions of the shear rate a, the mass ratio μ, and the force constant ratio ω, but they do not depend explicitly on the thermal gradient. This is probably due to the particular interaction model considered. When $a=0$, $\sigma_{ij}=\sigma_0\delta_{ij}$, and one recovers the familiar Ohm law (35) for the current. The diagonal elements σ_{ii} can be interpreted as generalizations of the electrical conductivity coefficient σ_0 since they couple the ith component of the current density to the ith component of the electric field. The fact that these three elements are different clearly shows the high anisotropy induced in the system by the Couette flow. In Figs. 1, 2, and 3 we plot the reduced elements σ_{xx}/σ_0, σ_{yy}/σ_0, and σ_{zz}/σ_0, respectively, as a function of

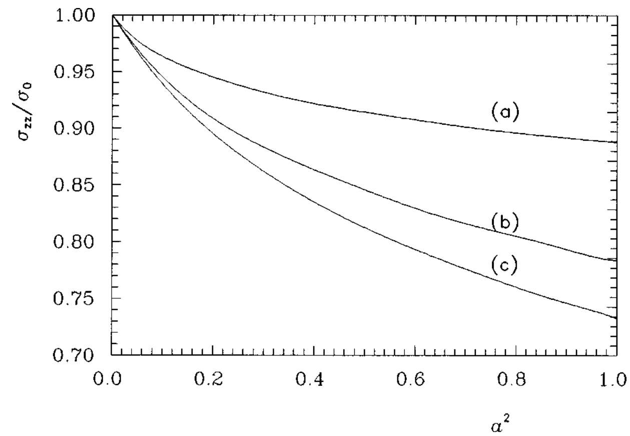

FIG. 3. Same as in Fig. 1, but for σ_{zz}/σ_0.

the reduced shear rate for $\omega=1$ and three values of the mass ratio. In general, we observe that the diagonal elements decrease as the shear rate increases whatever the mass ratio considered is. At a given value of the mass ratio, $\sigma_{zz}>\sigma_{xx}>\sigma_{yy}$. Further, for finite shear rates, the inhibition of the diagonal element σ_{ii} with a is more significant when the neutral particles are lighter than the charged particles. The nondiagonal elements σ_{xy} and σ_{yx} measure cross effects in the transport of charged particles. For instance, the x element of the electric field creates a current density parallel to the direction of velocity and temperature gradients (y axis). Since both off-diagonal elements are different, the conductivity tensor is nonsymmetric. While the xy element is negative, the yx element is positive although its value is very small for all the shear rates and mass ratios considered. This means that the current density along the y axis is practically generated by the component of the field parallel to the direction of the gradients. In Fig. 4, we show the shear-rate dependence of $-\sigma_{xy}/\sigma_0$. We see that the magnitude of this coefficient always increases with the shear rate. Once the behavior of the elements of the conductivity tensor has been analyzed, it is also interesting to study the effect of the shear rate on the current density $\mathbf{j}_1^{(1)}$. For the sake of simplicity, let us assume that $E_x=E_z=0$ so that the electric field is parallel to the direction of the gradients. In Fig. 5 we plot the magnitude of the current density relative to its linear value, i.e., $\Psi(a)\equiv|\mathbf{j}_1^{(1)}(a)|/|\mathbf{j}_1^{(1)}(0)|$ for the same cases as in the previ-

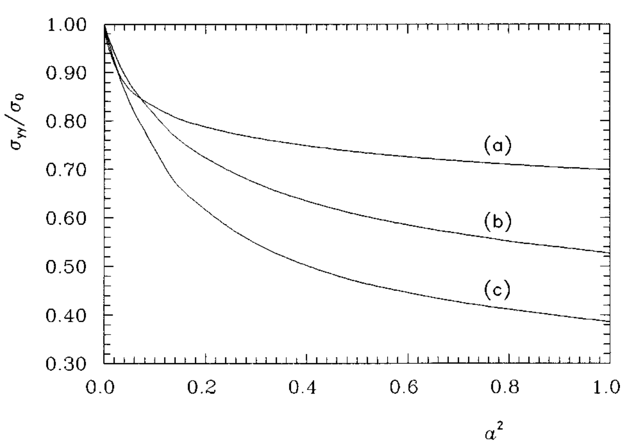

FIG. 2. Same as in Fig. 1, but for σ_{yy}/σ_0.

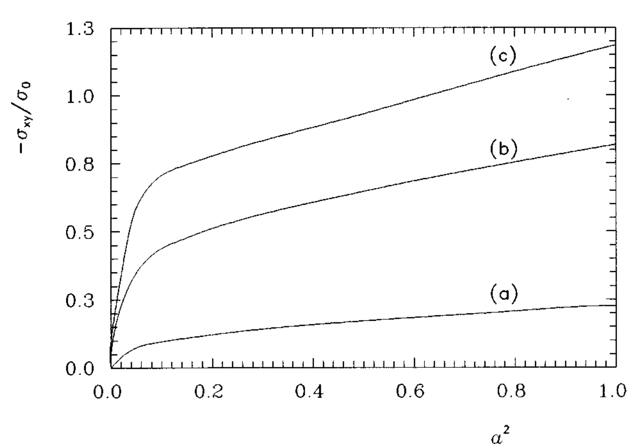

FIG. 4. Same as in Fig. 1, but for $-\sigma_{xy}/\sigma_0$.

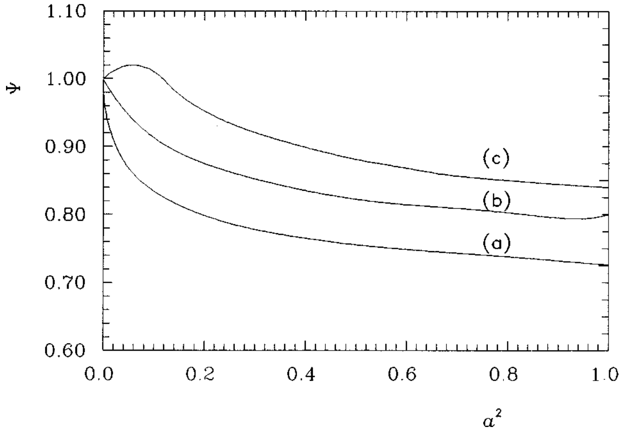

FIG. 5. Same as in Fig. 1, but for the ratio $\Psi(a) \equiv |\mathbf{j}_1^{(1)}(a)|/|\mathbf{j}_1^{(1)}(0)|$.

ous figures. In general, we observe that the shear rate inhibits the transport of charged particles. This inhibition is more significant when the particles of the neutral gas are heavier than that of ionized gas. Only in the case of mass ratio smaller than one is there a small region of values of a for which the relative current density is enhanced by the shear rate.

IV. DISCUSSION

In this paper we have addressed the study of transport properties of an ensemble of charged test particles diffusing in a background neutral gas under the action of a weak electric field. The molar fraction of the charged particles $x_1 = n_1/n_2$ is so small that the elastic interactions of charged-neutral and neutral-neutral type are the dominant ones. This assumption simplifies the description as we do have not to consider the complex long-range nature of the Coulomb force. In these conditions, the current density $\mathbf{j}_1$ induced by the electric field $\mathbf{E}$ is the main transport property of the problem. Usually, this current is measured in a steady homogeneous situation, namely, when the neutral gas is at equilibrium. In this case, the phenomenological Ohm law establishes a linear relation between $\mathbf{j}_1$ and $\mathbf{E}$ through the electrical conductivity coefficient σ_0. Now we have generalized the above description to the case in which the neutral gas is far from equilibrium. Specifically, we have considered a rarefied neutral gas under steady Couette flow. Our purpose has been to get the current density for arbitrarily large velocity and temperature gradients in the limit of small electric fields. Furthermore, charged and neutral particles are mechanically different. To the best of our knowledge, this is the first derivation of an explicit expression of the flux of charged particles under such extreme conditions.

Due to the mathematical complexity of the Boltzmann collision operators, we have used the nonlinear GK model for binary mixtures of Maxwell molecules. In the tracer limit ($x_1 \ll 1$), the kinetic equation for the distribution function f_2 of the excess gas reduces to the BGK equation of a single gas for which an exact solution in the steady Couette flow is known [9,10]. The knowledge of f_2 allows us to solve the corresponding Boltzmann-Lorentz equation for the velocity distribution f_1 of the charged particles by means of a perturbation expansion in powers of the electric field. The zeroth-order approximation corresponds to a nonequilibrium state characterized by the absence of diffusion but with *arbitrary* values of the velocity and temperature gradients and the parameters of the mixture [19]. In the first order of the expansion, we obtain that the current density obeys a generalized Ohm's law where an electric conductivity tensor σ_{ij} can be identified. The nonzero elements of this tensor are highly nonlinear functions of the shear rate, the mass ratio, and the force constant ratio. Since we have taken the electric field to lie along the three axes, according to the symmetry of the Couette flow, there exist five relevant (different) elements of σ_{ij}: three diagonals and two off-diagonals (xy and yx). The shear-rate dependence of the diagonal elements is quite similar, namely, they decrease as the shear rate increases. This decrease is more significant as the mass of the neutral gas is lighter than that of charged gas. With respect to the off-diagonal elements, the magnitude of σ_{xy} increases with the shear rate while σ_{yx} is practically zero. In general, the net consequence of the presence of the Couette flow is to produce an inhibition on the transport of charge.

Although the kinetics of charge carriers in metals and semiconductors is of greater practical value than that of in gases, it is evident that the latter is one of the most interesting applications of the Boltzmann equation. In this context, the situation studied here may be useful for understanding processes ocurring in gas discharges or in the high-latitude ionosphere. On the other hand, we are fully aware that the tracer limit considered here is certainly a restriction that one would like to get rid of. Beyond this limit, it is evident that the main difficulty is to incorporate the long range nature of the Coulomb interactions in our kinetic model. Perhaps, a first possibility could be to assume that the Coulomb potential acts on every charged particle as if it were due to an external field (Vlassov approximation). Under these conditions, one could take the results reported here as the reference state and introduce the Coulomb interaction in a perturbation way.

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APPENDIX

In this appendix we derive the expressions of the nonzero elements of the conductivity tensor σ_{ij}. Introducing the operator $\partial_s \equiv (1/\nu_{12})\partial/\partial y$, the formal solution (36) can be written as

$$\begin{aligned} f_1^{(1)} &= (1+V_y\partial_s)^{-1}\left(f_{12}^{(1)} - \frac{q}{m_1\nu_{12}} E_i \frac{\partial}{\partial V_i} f_1^{(0)} \right) \\ &= \sum_{k=0}^{\infty} (-\partial_s)^k V_y^k \left(f_{12}^{(1)} - \frac{q}{m_1\nu_{12}} E_i \frac{\partial}{\partial V_i} f_1^{(0)} \right) \\ &\equiv \Lambda^{\mathrm{I}} + \Lambda^{\mathrm{II}}. \end{aligned} \tag{A1}$$

The first term of the right-hand side of Eq. (A1), Λ^{I}, is identical to the one appearing in the tracer diffusion problem under Couette flow [20]. In that problem, the transport of tracer particles is generated by a concentration gradient (parallel to the y axis) instead of a constant electric field. Consequently, the contribution of $\Lambda^{\rm I}$ to $j_{1,x}^{(1)}$, $j_{1,y}^{(1)}$, and $p_1^{(1)}$ is formally the same as the one obtained in the Appendix of Ref. [20], except that now the molar fraction x_1 is constant. This implies that all the terms proportional to $\partial_s x_1$ in Eqs. (A6), (A8), and (A14) of Ref. [20] are now zero. Therefore, in this paper we have to explicitly compute the extra contributions to $j_{1,x}^{(1)}$, $j_{1,y}^{(1)}$, and $p_1^{(1)}$ coming from the second term of the right-hand side $\Lambda^{\rm II}$, as well as the total contribution (coming from $\Lambda^{\rm I}+\Lambda^{\rm II}$) to $j_{1,z}^{(1)}$, since this component was zero in the related diffusion problem. To this end, in the same way as in Ref. [20], we assume (to be verified later) that $\partial_s^2(T_{12}^{(1)}/T_{12}^{(0)})=0$.

Let us start with the calculation of the extra contributions to the current density and the partial pressure coming from $\Lambda^{\rm II}$. This term can be decomposed into the form

$$\Lambda^{\rm II}=-\frac{qn_2k_B}{m_1\nu_{12}p_{12}^{(0)}}(\Lambda_x^{\rm II}+\Lambda_y^{\rm II}+\Lambda_z^{\rm II}), \tag{A2}$$

where

$$\Lambda_i^{\rm II}=E_i\sum_{k=0}^{\infty}(-\partial_s)^k T_{12}^{(0)}V_y^k\frac{\partial}{\partial V_i}f_1^{(0)}. \tag{A3}$$

The more involved contribution to the fluxes corresponds to $\Lambda_y^{\rm II}$. Let us focus on its contribution. First, taking into account that $T_{12}^{(0)}$ is a quadratic polynomial in s, $\Lambda_y^{\rm II}$ can be rewritten as

$$\Lambda_y^{\rm II}=T_{12}^{(0)}\sum_{k=0}^{\infty}(-\partial_s)^kV_y^k\frac{\partial}{\partial V_y}f_1^{(0)}-(\partial_sT_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)(-\partial_s)^kV_y^{k+1}\frac{\partial}{\partial V_y}f_1^{(0)}+\frac{1}{2}(\partial_s^2T_{12}^{(0)})$$
$$\times\sum_{k=0}^{\infty}(k+1)(k+2)(-\partial_s)^kV_y^{k+2}\frac{\partial}{\partial V_y}f_1^{(0)}\equiv\Psi_1+\Psi_2+\Psi_3, \tag{A4}$$

where the $\Psi_{1,2,3}$ can be easily identified.

The contribution to $j_{1,x}^{(1)}$ is proportional to the integral

$$\int d\mathbf{v}V_x(\Psi_1+\Psi_2+\Psi_3). \tag{A5}$$

Let us evaluate each contribution separately. The first one is

$$\begin{aligned}\int d\mathbf{v}V_x\Psi_1&=-T_{12}^{(0)}\sum_{k=0}^{\infty}(k+1)\int d\mathbf{v}V_x(-\partial_s)^{k+1}V_y^kf_1^{(0)}\\ &=-T_{12}^{(0)}\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}\int d\mathbf{v}V_x(-\partial_s)^{k+1}V_y^kf_{12}^{(0)}\\ &=T_{12}^{(0)}\widetilde{a}\sum_{k=0}^{\infty}\frac{(k+1)^2(k+2)}{2}\int d\mathbf{v}(-\partial_s)^kV_y^kf_{12}^{(0)}=n_1T_{12}^{(0)}\widetilde{a},\end{aligned} \tag{A6}$$

where use has been made of the formal solution of $f_1^{(0)}$, namely,

$$f_1^{(0)}=\sum_{k=0}^{\infty}(-\partial_s)^kV_y^kf_{12}^{(0)}. \tag{A7}$$

The second contribution is given by

$$\begin{aligned}\int d\mathbf{v}V_x\Psi_2&=(\partial_sT_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)^2\int d\mathbf{v}V_x(-\partial_s)^kV_y^kf_1^{(0)}=(\partial_sT_{12}^{(0)})\sum_{k=0}^{\infty}\frac{(k+1)(2k^2+7k+6)}{6}\int d\mathbf{v}V_x(-\partial_s)^kV_y^kf_{12}^{(0)}\\ &=-(\partial_sT_{12}^{(0)})\widetilde{a}\sum_{k=0}^{\infty}\frac{(k+1)(k+2)(2k^2+11k+15)}{6}\int d\mathbf{v}(-\partial_s)^kV_y^{k+1}f_{12}^{(0)}=0,\end{aligned} \tag{A8}$$

since, according to the s dependence of the hydrodynamic fields, the order of the derivative ∂_s increases faster than the power of s. The third contribution is

$$\int d\mathbf{v}V_x\Psi_3=-\frac{1}{2}(\partial_s^2T_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)(k+2)^2\int d\mathbf{v}V_x(-\partial_s)^kV_y^{k+1}f_1^{(0)}$$
$$=-\frac{1}{2}(\partial_s^2T_{12}^{(0)})\sum_{k=0}^{\infty}\frac{(k+1)(3k^3+23k^2+58k+48)}{12}\int d\mathbf{v}V_x(-\partial_s)^kV_y^{k+1}f_{12}^{(0)}$$
$$=\frac{1}{2}(\partial_s^2T_{12}^{(0)})\widetilde{a}\sum_{k=0}^{\infty}\frac{(k+1)(k+2)(3k^3+32k^2+113k+132)}{12}\int d\mathbf{v}(-\partial_s)^kV_y^{k+2}f_{12}^{(0)}$$
$$=\frac{1}{6}x_1(\partial_s^2T_{12}^{(0)})\frac{p_{12}^{(0)}\widetilde{a}}{m_1}\sum_{k=0}^{\infty}(k+1)(12k^3+64k^2+113k+66)(2k+1)!(2k+1)!!(-\widetilde{\gamma})^k$$
$$=\frac{1}{6}x_1(\partial_s^2T_{12}^{(0)})\frac{p_{12}^{(0)}\widetilde{a}}{m_1}(12\widetilde{F}_4+28\widetilde{F}_3+21\widetilde{F}_2+5\widetilde{F}_1), \tag{A9}$$

where in the last step use has been made of the asymptotic form of the functions $\widetilde{F}_r$, i.e. [9],

$$\widetilde{F}_r\equiv F_r(\widetilde{\gamma})=\sum_{k=0}^{\infty}(k+1)^r(2k+1)!(2k+1)!!(-\widetilde{\gamma})^k. \tag{A10}$$

By collecting all the contributions, Eqs. (A5), (A8), and (A9), one gets that

$$\int d\mathbf{v}V_x\Lambda_y^{\mathrm{II}}=n_1T_{12}^{(0)}\widetilde{a}\left[1-\widetilde{\gamma}\left(4\widetilde{F}_4+\frac{28}{3}\widetilde{F}_3+7\widetilde{F}_2+\frac{5}{3}\widetilde{F}_1\right)\right]E_y. \tag{A11}$$

In the case of $j_{1,y}^{(1)}$, one needs the results

$$\int d\mathbf{v}V_y\Psi_1=-T_{12}^{(0)}\sum_{k=0}^{\infty}(k+1)\int d\mathbf{v}V_y(-\partial_s)^kV_y^kf_1^{(0)}=-T_{12}^{(0)}\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}\int d\mathbf{v}(-\partial_s)^kV_y^kf_{12}^{(0)}=-n_1T_{12}^{(0)}, \tag{A12}$$

$$\int d\mathbf{v}V_y\Psi_2=0, \tag{A13}$$

$$\int d\mathbf{v}V_y\Psi_3=-\frac{1}{2}(\partial_s^2T_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)\int d\mathbf{v}(-\partial_s)^kV_y^{k+2}f_1^{(0)}$$
$$=-\frac{1}{2}(\partial_s^2T_{12}^{(0)})\sum_{k=0}^{\infty}\frac{(k+1)(k^3+9k^2+26k+24)}{4}\int d\mathbf{v}(-\partial_s)^kV_y^{k+2}f_{12}^{(0)}$$
$$=-\frac{1}{2}x_1(\partial_s^2T_{12}^{(0)})\frac{p_{12}^{(0)}}{m_1}\sum_{k=0}^{\infty}(2k^3+9k^2+13k+6)(2k+1)!(2k+1)!!(-\widetilde{\gamma})^k$$
$$=-\frac{1}{2}x_1(\partial_s^2T_{12}^{(0)})\frac{p_{12}^{(0)}}{m_1}(2\widetilde{F}_3+3\widetilde{F}_2+\widetilde{F}_1). \tag{A14}$$

Consequently, the contribution to $j_{1,y}^{(1)}$ from Λ_y^{II} is

$$\int d\mathbf{v}V_y\Lambda_y^{II}=n_1T_{12}^{(0)}[\widetilde{\gamma}(2\widetilde{F}_3+3\widetilde{F}_2+\widetilde{F}_1)-1]E_y. \tag{A15}$$

In the case of $j_{1,z}^{(1)}$, it is easy to see that

$$\int d\mathbf{v}V_z\Lambda_y^{\mathrm{II}}=0. \tag{A16}$$

Let us consider now the contribution of Λ_y^{II} to the partial pressure $p_1^{(1)}$. It is proportional to the integral

$$\int d\mathbf{v}V^2(\Psi_1+\Psi_2+\Psi_3). \tag{A17}$$

The first contribution is

$$\int d\mathbf{v}V^2\Psi_1=0 \tag{A18}$$

when one takes into account the operator identity

$$\partial_s^k V^2=V^2\partial_s^k-2\widetilde{a}k\partial_s^{k-1}V_x-\widetilde{a}^2k(k-1)\partial_s^{k-2}. \tag{A19}$$

The second contribution is

$$\begin{aligned}\int d\mathbf{v}V^2\Psi_2&=(\partial_s T_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)\int d\mathbf{v}[2V_y^{k+2}+(k+1)V^2V_y^k](-\partial_s)^k f_1^{(0)}\\&=(\partial_s T_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)(k+2)(-\partial_s)^k\int d\mathbf{v}V_y^{k+2}f_{12}^{(0)}+(\partial_s T_{12}^{(0)})\sum_{k=0}^{\infty}\frac{(k+1)(2k^2+7k+6)}{6}\int d\mathbf{v}V^2V_y^k(-\partial_s)^k f_{12}^{(0)}\\&=\frac{x_1p_{12}^{(0)}}{m_1}(\partial_s T_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)(2k+1)!(2k+1)!!(-\widetilde{\gamma})^k+2\frac{x_1p_{12}^{(0)}}{m_1}(\partial_s T_{12}^{(0)})\\&\quad\times\sum_{k=0}^{\infty}(4k^2+7k+3)(2k)!(2k+3)!!(-\widetilde{\gamma})^k+\frac{1}{3}\frac{x_1p_{12}^{(0)}}{m_1}(\partial_s T_{12}^{(0)})\widetilde{a}^2\\&\quad\times\sum_{k=0}^{\infty}(2k+3)(8k^2+30k+28)(k+1)(2k+1)!(2k+1)!!(-\widetilde{\gamma})^k\\&=\frac{x_1p_{12}^{(0)}}{m_1}(\partial_s T_{12}^{(0)})\left[3+2\widetilde{F}_1-\widetilde{\gamma}\left(\frac{32}{3}\widetilde{F}_5+40\widetilde{F}_4+\frac{160}{3}\widetilde{F}_3+30\widetilde{F}_2+6\widetilde{F}_1\right)+\frac{2}{3}\widetilde{a}^2(8\widetilde{F}_4+18\widetilde{F}_3+13\widetilde{F}_2+3\widetilde{F}_1)\right].\end{aligned} \tag{A20}$$

The third contribution is

$$\begin{aligned}\int d\mathbf{v}V^2\Psi_3&=-\frac{1}{2}(\partial_s^2 T_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)(k+2)\int d\mathbf{v}[2V_y^{k+3}+(k+2)V^2V_y^{k+1}](-\partial_s)^k f_1^{(0)}\\&=-\frac{1}{3}(\partial_s^2 T_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)(k^2+5k+6)(-\partial_s)^k\int d\mathbf{v}V_y^{k+3}f_{12}^{(0)}-\frac{1}{24}(\partial_s^2 T_{12}^{(0)})\\&\quad\times\sum_{k=0}^{\infty}(k+1)(3k^3+23k^2+58k+48)\int d\mathbf{v}V^2V_y^{k+1}(-\partial_s)^k f_{12}^{(0)}\\&=\frac{8}{3}\frac{x_1p_{12}^{(0)}}{m_1}\widetilde{\gamma}(\partial_s T_{12}^{(0)})\sum_{k=0}^{\infty}(k+1)^2(2k+3)(2k^2+7k+6)(2k+1)!(2k+1)!!(-\widetilde{\gamma})^k-\frac{1}{3}\frac{x_1p_{12}^{(0)}}{m_1}\widetilde{\gamma}(\partial_s T_{12}^{(0)})\\&\quad\times\sum_{k=0}^{\infty}(k+1)^2(2k+5)(12k^3+64k^2+113k+66)(2k+1)!(2k+1)!!(-\widetilde{\gamma})^k-\frac{2}{3}\frac{x_1p_{12}^{(0)}}{m_1}\widetilde{\gamma}(\partial_s T_{12}^{(0)})\widetilde{a}^2\\&\quad\times\sum_{k=0}^{\infty}(k+1)^2(2k+3)^2(k+2)(12k^3+100k^2+277k+255)(2k+1)!(2k+1)!!(-\widetilde{\gamma})^k\\&=-\frac{1}{3}\frac{x_1p_{12}^{(0)}}{m_1}\widetilde{\gamma}(\partial_s T_{12}^{(0)})[23\widetilde{F}_2+113\widetilde{F}_3+2(95\widetilde{F}_4+62\widetilde{F}_5+12\widetilde{F}_6)\\&\quad+2\widetilde{a}^2(48\widetilde{F}_8+352\widetilde{F}_7+1024\widetilde{F}_6+1500\widetilde{F}_5+1157\widetilde{F}_4+443\widetilde{F}_3+66\widetilde{F}_2)],\end{aligned} \tag{A21}$$

where use has been made of the identity

$$\partial_s^{2k+1}T_{12}^{(0)}=(k+1)(2k+1)(\partial_s T_{12}^{(0)})\partial_s^{2k}T_{12}^{(0)k}. \quad \text{(A22)}$$

The total contribution to the partial pressure coming from $\Lambda_y^{\rm II}$ can be easily obtained from Eqs. (A18), (A20), and (A21).

The remaining contributions to the fluxes coming from the terms $\Lambda_{x,z}^{\rm II}$ and $\Lambda^{\rm I}$ (in the case of $j_{1,z}^{(1)}$) can be obtained by following similar mathematical steps as the ones made with $\Lambda_y^{\rm II}$. The algebra is very tedious and here we only quote the final results:

$$\frac{1}{E_x}\int d\mathbf{v}V_x\Lambda_x^{\rm II}=\frac{1}{E_z}\int d\mathbf{v}V_z\Lambda_z^{\rm II}=n_1T_{12}^{(0)}\left[\frac{2}{3}\widetilde{\gamma}(2\widetilde{F}_2+\widetilde{F}_1)-1\right], \quad \text{(A23)}$$

$$\int d\mathbf{v}V_{y,z}\Lambda_x^{\rm II}=\int d\mathbf{v}V_{x,y}\Lambda_z^{\rm II}=0, \quad \text{(A24)}$$

$$q\int d\mathbf{v}V_z\Lambda^{\rm I}=\frac{1}{1+\mu}(1-2\widetilde{\gamma}\widetilde{F}_1), \quad \text{(A25)}$$

$$\int d\mathbf{v}V^2\Lambda_z^{\rm II}=0, \quad \text{(A26)}$$

$$\int d\mathbf{v}V^2\Lambda_x^{\rm II}=2\frac{x_1p_{12}^{(0)}}{m_1}(\partial_sT_{12}^{(0)})\widetilde{a}\left[\frac{4}{3}\widetilde{\gamma}(8\widetilde{F}_6+28\widetilde{F}_5+34\widetilde{F}_4+17\widetilde{F}_3+3\widetilde{F}_2)-(2\widetilde{F}_2+\widetilde{F}_1)\right]E_x. \quad \text{(A27)}$$

Now we are in conditions to write the explicit expressions of $\mathbf{j}_1^{(1)}$ and $p_1^{(1)}$. To close the problem one needs to know the derivative $\partial_s(T_{12}^{(1)}/T_{12}^{(0)})$. This can be obtained by applying the operator ∂_s on both sides of the relation which defines the partial pressure. The calculation leads to $\partial_s^2(T_{12}^{(1)}/T_{12}^{(0)})=0$, which confirms the assumption previously established. The explicit expression of $\partial_s(T_{12}^{(1)}/T_{12}^{(0)})$ is not very illuminating and will be omitted here. Therefore, by taking into account the results derived in Ref. [20] and using all the relations obtained in this Appendix, one may finally recast the current density into the form of a generalized Ohm's law, Eq. (37), where the nonzero elements of the conductivity tensor are given by

$$\sigma_{zz}=\sigma_0\frac{\mu}{3}\frac{3-2\widetilde{\gamma}(\widetilde{F}_1+2\widetilde{F}_2)}{2\widetilde{F}_1\widetilde{\gamma}+\mu}, \quad \text{(A28)}$$

$$\sigma_{yy}=\sigma_0\mu\frac{N_1}{N_2}, \quad \text{(A29)}$$

$$\sigma_{xy}=\sigma_0\frac{\sigma_{yy}N_4+\mu N_5}{N_3}\widetilde{a}, \quad \text{(A30)}$$

$$\sigma_{xx}=\sigma_0\frac{\mu}{3}\frac{N_6}{N_2}, \quad \text{(A31)}$$

$$\sigma_{yx}=\sigma_0\frac{2\sigma_{xx}A_5+\mu A_6\widetilde{a}^2}{\widetilde{a}N_7}, \quad \text{(A32)}$$

where σ_0 is given by Eq. (35) and

$$N_1=\frac{4}{3}\widetilde{\gamma}\lambda A_5[\widetilde{F}_1(5\widetilde{F}_1\widetilde{\gamma}+15\widetilde{F}_2\widetilde{\gamma}+28\widetilde{F}_3\widetilde{\gamma}+12\widetilde{F}_4\widetilde{\gamma}-3)-6\widetilde{F}_2(3\widetilde{F}_2\widetilde{\gamma}+2\widetilde{F}_3\widetilde{\gamma}-1)]+(2\widetilde{F}_1\widetilde{\gamma}+\mu)[2A_1\widetilde{F}_1\widetilde{\gamma}\lambda+(2A_3\lambda-4A_4\widetilde{\gamma}\lambda+1)(\widetilde{F}_1\widetilde{\gamma}+3\widetilde{F}_2\widetilde{\gamma}+2\widetilde{F}_3\widetilde{\gamma}-1)], \quad \text{(A33)}$$

$$N_2=4A_5\widetilde{\gamma}\lambda[\widetilde{F}_0\widetilde{F}_1+2\widetilde{F}_2(2\widetilde{F}_1\widetilde{\gamma}+4\widetilde{F}_2\widetilde{\gamma}+\mu)]+(2\widetilde{F}_1\widetilde{\gamma}+\mu)[4A_2\widetilde{F}_1\widetilde{\gamma}\lambda-(2A_3\lambda-4A_4\widetilde{\gamma}\lambda+1)(2\widetilde{F}_1\widetilde{\gamma}+4\widetilde{F}_2\widetilde{\gamma}+\mu)], \quad \text{(A34)}$$

$$N_3=8A_5\widetilde{F}_2\widetilde{\gamma}-(2\widetilde{F}_1\widetilde{\gamma}+\mu)(2A_3\lambda-4A_4\widetilde{\gamma}\lambda+1), \quad \text{(A35)}$$

$$N_4=8A_2\widetilde{F}_2\widetilde{\gamma}\lambda+\widetilde{F}_0(2A_3\lambda-4A_4\widetilde{\gamma}\lambda+1), \quad \text{(A36)}$$

$$N_5=[1-\tfrac{1}{3}\widetilde{\gamma}(5\widetilde{F}_1+21\widetilde{F}_2+28\widetilde{F}_3+12\widetilde{F}_4)](2A_3\lambda-4A_4\widetilde{\gamma}\lambda+1)-4A_1\widetilde{F}_2\widetilde{\gamma}\lambda, \quad \text{(A37)}$$

$$N_6=(2\widetilde{F}_1\widetilde{\gamma}+4\widetilde{F}_2\widetilde{\gamma}-3)[(2\widetilde{F}_1\widetilde{\gamma}+4\widetilde{F}_2\widetilde{\gamma}+\mu)(2A_3\lambda-4A_4\widetilde{\gamma}\lambda+1)-4A_2\widetilde{F}_1\widetilde{\gamma}\lambda]-6\widetilde{a}^2A_6\widetilde{\gamma}\lambda[\widetilde{F}_0\widetilde{F}_1+2\widetilde{F}_2(2\widetilde{F}_1\widetilde{\gamma}+4\widetilde{F}_2\widetilde{\gamma}+\mu)], \quad \text{(A38)}$$

$$N_7=\frac{4A_2\widetilde{F}_1\widetilde{\gamma}\lambda-(2\widetilde{F}_1\widetilde{\gamma}+4\widetilde{F}_2\widetilde{\gamma}+\mu)(2A_3\lambda-4A_4\widetilde{\gamma}\lambda+1)}{2\widetilde{F}_1\widetilde{\gamma}\lambda}. \quad \text{(A39)}$$

In the above expressions we have introduced the coefficients

$$A_1=-1-\frac{2}{3}\widetilde{F}_1+\frac{\widetilde{\gamma}}{9}[18\widetilde{F}_1+113\widetilde{F}_2+273\widetilde{F}_3+310\widetilde{F}_4+12(13\widetilde{F}_5+2\widetilde{F}_6)]+\frac{2}{9\widetilde{F}_1}[3\widetilde{F}_1+\widetilde{F}_2(13-66\widetilde{\gamma})+\widetilde{F}_3(18-443\widetilde{\gamma}) +\widetilde{F}_4(8-1157\widetilde{\gamma})-4\widetilde{\gamma}(375\widetilde{F}_5+256\widetilde{F}_6+88\widetilde{F}_7+12\widetilde{F}_8)][3\beta-\widetilde{\gamma}(3\widetilde{F}_1+2\widetilde{F}_2)], \tag{A40}$$

$$A_2=\frac{\widetilde{F}_0}{6\widetilde{F}_1}(3\widetilde{F}_1+2\widetilde{F}_2)-\frac{2}{\widetilde{F}_1\widetilde{\gamma}}(\widetilde{F}_0-\widetilde{F}_1)\beta, \tag{A41}$$

$$A_3=\frac{1}{12}(\widetilde{F}_0+5)-\frac{\widetilde{\gamma}}{6\widetilde{F}_1}[6\widetilde{F}_0\widetilde{F}_1-(\widetilde{F}_1^2+4\widetilde{F}_2^2)]-\frac{\widetilde{F}_2}{\widetilde{F}_1}\beta, \tag{A42}$$

$$A_4=\frac{1}{12\widetilde{\gamma}\widetilde{F}_1}[8\widetilde{\gamma}\widetilde{F}_2(3\widetilde{F}_1+2\widetilde{F}_2)+\widetilde{F}_0(\widetilde{F}_1+\widetilde{F}_2)-\widetilde{F}_1^2-\widetilde{F}_2]-\frac{1}{8\widetilde{\gamma}^2\widetilde{F}_1}(\widetilde{F}_0+2\widetilde{F}_1\widetilde{\gamma}+16\widetilde{F}_2\widetilde{\gamma}-1)\beta, \tag{A43}$$

$$A_5=\frac{2}{3}\frac{\widetilde{F}_2}{\widetilde{F}_1}\widetilde{\gamma}(2\widetilde{F}_2+3\widetilde{F}_1)-2\frac{\widetilde{F}_2}{\widetilde{F}_1}\beta, \tag{A44}$$

$$A_6=\tfrac{2}{3}[\widetilde{F}_1+2\widetilde{F}_2-\tfrac{4}{3}\widetilde{\gamma}(8\widetilde{F}_6+28\widetilde{F}_5+34\widetilde{F}_4+17\widetilde{F}_3+3\widetilde{F}_2)]. \tag{A45}$$

Here, $\lambda=2M-1=-(1+\mu^2)/(1+\mu)^2$, and

$$\beta\equiv\frac{M(1-\chi)}{\chi+2M(1-\chi)}. \tag{A46}$$

In the limit of small shear rates, the elements of the conductivity tensor behave as

$$\frac{\sigma_{yy}}{\sigma_0}\approx 1-\frac{2}{15}\frac{5\mu^2+18\mu+5}{\omega^4\mu}a^2, \tag{A47}$$

$$\frac{\sigma_{xy}}{\sigma_0}\approx -\frac{1}{\omega^2}\left(\frac{2}{1+\mu}\right)^{1/2}\frac{1+\mu}{\mu}a, \tag{A48}$$

$$\frac{\sigma_{yx}}{\sigma_0}\approx\frac{2}{15\omega^6}\frac{(2+3\mu)(1+\mu^2)}{[(1+\mu)/2]^{3/2}\mu}a^3, \tag{A49}$$

$$\frac{\sigma_{xx}}{\sigma_0}\approx\frac{\sigma_{zz}}{\sigma_0}\approx 1-\frac{4}{5\omega^4}a^2. \tag{A50}$$

From the knowledge of the current density, the partial pressure can also be obtained. It can be written in the form

$$p_1^{(1)}-p_1^{(0)}=-\frac{5}{6}\frac{(1+\mu)^3}{\mu^2}\frac{qk_Bn_1}{m_1\nu_{12}^2}(\Omega_xE_x+\Omega_yE_y)\frac{\partial T_2}{\partial y}, \tag{A51}$$

where

$$\Omega_x=-\frac{6}{5}\frac{\mu}{(1+\mu)^3}[\chi+2M(1-\chi)] \times\frac{1}{N_8}\left(\mu A_6-2A_2\sigma_{yx}+2\frac{A_5\sigma_{xx}}{\widetilde{a}}\right), \tag{A52}$$

$$\Omega_y=-\frac{6}{5}\frac{\mu}{(1+\mu)^3}[\chi+2M(1-\chi)] \times\frac{1}{N_8}\left(\mu A_1-2A_2\sigma_{yy}+2\frac{A_5\sigma_{xy}}{\widetilde{a}}\right), \tag{A53}$$

and

$$N_8=1+2A_3\lambda-4A_4\widetilde{\gamma}\lambda. \tag{A54}$$

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