Dynamical quantum chaos as fluid turbulence

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A quantum particle subject to a time-dependent force appears as if it were a collisionless turbulent fluid with a tensorial pressure given by a known equation of state. Such a fluid may possess topological singularities such as line vortices and sheet vortices, which are frozen in the fluid. Creation and destruction of these vortices are only possible when the forcing potential is singular. In addition, when the initial data are of large scale, the quantum fluctuations have a tendency to become steepened, characteristic of the classical compressible fluid in forming shock waves, and the nonlinear steepening is halted by the wave dispersion in generating an abundance of short waves. Chaotic quantum dynamics is expected to be governed by the interplay between wave steepening and vortex interactions. [S1063-651X(98)05702-X]

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I. INTRODUCTION

Quantum mechanics (QM) has long been thought to be a matured methodology for investigating atomic energy levels, through which transitions between different levels can be theoretically computed. Since the energies of the emitted photons are experimentally detectable at high precision, the agreement between theories and experiments is often thought to verify QM.

The domain of applications for quantum mechanics of this type is in fact rather restricted from the dynamical viewpoint. The fact that atoms have different energy levels only indicates that they can be at different eigenstates or, from the dynamical perspective, different stationary states. It is conceivable that the time-dependent Schrödinger equation can admit a much wider domain of solutions than the mere stationary eigenstates. Specifically, when the system is subject to (external) time-dependent forces, the stationary states can no longer exist. In classical mechanics, this situation is often nonintegrable if the forces are nonlinear, and the system exhibits chaos. The quantum-mechanical counterpart of the time-dependent, nonintegrable Hamiltonian is expected to exhibit peculiar behaviors, which we shall call the dynamical quantum chaos to differentiate it from the other type of quantum chaos where the Hamiltonian is time independent but nonintegrable. For the latter, the energy levels still exist [1,2], but for the former, energy levels have no meaning and one is to understand such a system only through investigations of the evolving wave functions [3].

In fact, systems with time-dependent Hamiltonians are not unfamiliar. However, in the standard treatments, one usually considers the situations where the time-dependent force is either a short pulse or adiabatic. In both cases, dynamical quantum chaos can hardly occur, due either to having no sufficient time for the system to respond or to the existence of an adiabatically invariant action. The true dynamical quantum chaos can occur when the time scale of the external force is comparable to $\hbar/E$, where E is the typical energy scale of the system. For a slowly varying external force, this situation can only occur at the transition from a free particle to a trapped particle in the presence of a potential well, i.e., the resonant interactions. In this paper, a fluid description for such dynamical quantum chaos will be given. This approach makes extensive use of the analogy of a quantum particle to the classical fluid. Coupling of the quantum particle to radiation in the time-dependent potential, which yields energy loss, is not accounted for in this work. Hence only dissipationless quantum chaos is considered. A conceivable example for such a system can be a quantum particle experiencing a rapidly varying gravitational field.

A quantum system subject to a time-dependent potential rarely has been analyzed in the literature partly because of the complexity associated with it. However, with the present fluid approach, it is possible to understand the important dynamical features, such as topological singularities, of the chaotic quantum system from its analogy to the better-understood classical fluid. In this regard, the fluid description of QM can be superior to the Schrödinger representation. However, the formulation of the conventional Schrödinger equation also has its own great merits. In particular, the Schrödinger equation is a linear equation, in contrast to the nonlinear fluid equations. A seemingly turbulent quantum fluid described by the fluid equations can actually be decomposed into many dynamically independent modes when the Schrödinger representation is used. Thus the complicated nonlinearities in the fluid equations turn out to be the nonlinear mixtures of these linear modes. From a mathematical perspective, it has long been suspected that the fluid turbulence may possess certain underlying hidden integrals that permit turbulence to exhibit intermittency [4–6] and coherent structures [7]. In this regard, the Schrödinger representation of QM already has offered a successful example that uncovers the hidden symmetry (or integrability) inherent to the nonlinear quantum fluid. It is therefore the dual purpose of this work on the one hand, to utilize the fluid representation for extracting information about the dynamics of quantum chaos and, on the other hand, to bring about how the Schrödinger representation leads to uncovering the hidden symmetry in the turbulent quantum fluid.

Section II derives a complete set of fluid equations, including the equation of state, from the Schrödinger equation;

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they can be used to evolve the quantum fluids. Section III focuses on various dynamical properties of the singular vortices. Dynamical quantum chaos engages not only interactions of singular vortices but also nonlinear steepening of compressional fluctuations. These issues are addressed in Sec. IV. A discussion is given in Sec. V.

II. CONSERVATION EQUATIONS OF MASS, MOMENTUM, AND ENERGY

The quantum dynamics of a single particle is governed by the local conservation laws of energy and momentum. Much like the classical fluids, these conservation laws must also be constrained by the conservation of mass. The Schrödinger equation reads

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2} - \phi(\mathbf{x},t) \right] \psi = 0, \tag{1}$$

where ϕ is the potential of the external force and the particle mass has been set to unity. The conservation of mass, or the continuity equation, is obtained by multiplying ψ^* by Eq. (1) and keeping only the imaginary part. This yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2}$$

where we have decomposed the wave function into a real phase S and real amplitude f, with $\rho \equiv f^2$; in addition, we have set $\psi \equiv f e^{iS/\hbar}$ and $\mathbf{v} \equiv \nabla S$. Equation (2) warrants that the quantity $\int \rho \, d^3x$ always remains a constant during the dynamical evolution. This quantity is defined to be unity in order for ρ to be interpreted as the probability density of the particle.

The real part of the above operation yields

$$-\frac{\partial S}{\partial t} = -\frac{\hbar^2 \nabla^2 f}{2f} + \frac{(\nabla S)^2}{2} + \phi. \tag{3}$$

Derivations for Eqs. (2) and (3) are well known and are often given in any standard textbook of quantum mechanics. However, less well known is that Eq. (3) is familiar in fluid mechanics, known as the Bernoulli equation, except for the difference where the enthalpy $H(\rho) \equiv \int (dP/\rho)$ for classical fluids is now replaced by the quantum enthalpy, the first term on the right-hand side, for quantum ''fluids.'' In the classical fluids, the pressure of an isentropic fluid is a function of density $P = P(\rho)$ and an alternative form of the equation of state $H = H(\rho)$ for classical fluids has a quantum counterpart $H_q \equiv -\hbar^2 (\nabla^2 \sqrt{\rho})/2\sqrt{\rho}$. The quantum enthalpy H_q stems from the quantum fluctuations of the quantum particle in contrast to the thermal fluctuations of a collection of classical particles.

The momentum equation can be easily derived from Eq. (3) by taking a gradient on both sides of it to obtain

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \rho \nabla \left(\frac{\hbar^2 \nabla^2 f}{2f} \right) - \rho \nabla \phi. \tag{4}$$

Note that the pressure force for the quantum fluid satisfies $-\nabla P = \rho \nabla (\hbar^2 \nabla^2 f / 2f)$. To write the pressure force in a conserved form, one finds that the quantum pressure is in fact not a scalar but a tensor:

$$P_{ij} = \hbar^2 \left[(\partial_i f)(\partial_j f) - \frac{\nabla^2 \rho}{4} \delta_{ij} \right]. \tag{5}$$

Tensorial stress is a common feature for the classical collisionless fluid and hence the quantum fluid for a single quantum particle is closely analogous to a classical collisionless fluid.

To make a further analogy to fluid mechanics, one may construct the evolution equation for energy by taking the innerproduct of Eq. (4) with $\mathbf{v}$. For notational simplicity, we set $\mathbf{w} \equiv \hbar \nabla \ln(f)$ and it follows that

$$\frac{\partial}{\partial t} \left[\frac{\rho}{2} (v^2 + w^2 + 2\phi) \right] + \nabla \cdot \left(\frac{\rho}{2} [\mathbf{v}(v^2 - w^2 + \phi) + \mathbf{v} \cdot (2\mathbf{ww}) + (\nabla \cdot \mathbf{v})\mathbf{w} - (\nabla \cdot \mathbf{w})\mathbf{v}] \right) = \rho \left(\frac{\partial \phi}{\partial t} \right), \tag{6}$$

where $\rho w^2/2$ is the internal energy of the quantum fluid and the second term on the left-hand side is a tensorial energy flux. With the help of Eqs. (2) and (5), Eq. (4) and (6) may be turned into the conservation of momentum and energy ordinarily described by $\partial T_{\alpha\beta} / \partial x_\beta = -\rho \partial \phi / \partial x_\alpha$ in the nonrelativistic limit, where $T_{\alpha\beta}$ is the energy-momentum tensor and the indices α and β run from 0 to 3 in the Minkowsky metrics. Without the continuity equation, these equations themselves may fully evolve the quantum particle as if it were a classical fluid. Alternatively, one may choose Eqs. (2) and (4) to evolve the quantum fluid.

III. GENERATION OF VORTICES

In the following discussion, we will show that new vortices in the quantum fluids are *not* allowed to be generated or destroyed when the external potential ϕ is a smooth and regular function of space. Only when the external potential becomes singular or a multivalued function of space may the vortices possibly be generated or destroyed in quantum fluids.

From Eq. (4) it is clear that the quantum ''flow'' can only be a potential flow, unless $\nabla \times \nabla S \neq \mathbf{0}$, which holds only when S is a multivalued function of space. When this is so, S must have branch lines [8] and other singularities. Familiar examples of a multivalued S can be found for the stationary bound states with finite angular momenta. In these cases, the wave function can be expressed as

$$\psi = f(r,\theta) e^{in\chi}, \tag{7}$$

where r is the distance to the force center, θ the poloidal angle, χ the toroidal angle, and n an integer. In this example, the phase $S/\hbar = n\chi$. Since χ is a multivalued function, S must also be a multivalued function and the branch line is located at $\theta = 0$. The construct of QM is such that at the singularity the probability density vanishes, $\rho = 0$, so as to suppress the contribution of this singularity to the probability of finding a quantum particle in this state [8]. For example, if the density vanishes as $f \sim r$, Eq. (3) yields that $\nabla^2 f/f \sim r^{-2}$, where r refers to the radius in the cylindrical coordinate. If all terms in Eq. (3) are comparable, it follows that $v \sim r^{-1}$, corresponding to a singular line of finite angular momentum, i.e., a vortex line. A special case for the line vortex is the ring vortex of zero size, corresponding a point vortex. Both the line vortex and point vortex have finite energies associated with them [cf. Eq. (6) for the definition of energy density] and therefore excitation of these objects is *not* forbidable from the energetic viewpoint.

In addition, there can be surface singularities: sheet vortices. They occur when the velocity is expressed, for example, as $(\partial S/\partial x)+i(\partial S/\partial y)=[\xi(1-\xi)]^{-1/2}$, where ξ ($\equiv x+iy$) is a complex coordinate and hence S is a nonanalytical function of space. The total angular momentum in the sheet vortex is quantized in units of $\hbar$ to ensure a single-valued wave function ψ away from the vortex sheet. In this complex plane, a branch line must exist, running between $\xi=0$ and 1 and defining a vortex sheet in the (x,y,z) space. Although the trajectory of the branch line in the complex plane can be arbitrary according to the above expression for the velocity, it is in fact defined by a line (or surface in three dimensions) of density void. The reason for associating a density void with a sheet vortex is the same as that for the line vortex [8]. One may also understand this requirement by examining the quantum pressure given in Eq. (5); only when the density vanishes can the pressure be finite at the vortex sheet. This is a singular line connecting the branch points $\xi=0$ and 1. The density profile near the sheet vortex can be $f=|y|[1+c(x)y^2]$, where y is the direction perpendicular to the branch line and $c(x)$ must be adjusted so as to match satisfy Eq. (3). The sheet vortex also contains a finite energy and it is a much less singular object than the line vortex.

The equation of motion for the singular vortices can be obtained by taking a curl on Eq. (4) after dividing both sides by ρ. The resulting equation becomes

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\right)\left(\frac{\boldsymbol{\omega}}{\rho}\right)-\left(\frac{\boldsymbol{\omega}}{\rho}\right)\cdot\boldsymbol{\nabla}\mathbf{v}=\frac{-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi}{\rho}=\mathbf{0}, \qquad (8)$$

where $\boldsymbol{\omega}\equiv\boldsymbol{\nabla}\times\boldsymbol{\nabla}S$. This equation simply describes conservation of angular momentum in the fluid. The evolution of fluids described by Eqs. (2), (4), and (6) contains no dissipation and hence the system is an ideal fluid. For a classical ideal fluid, one often questions whether the fluid can develop vortex singularities within a finite time from an initially smooth data of ρ and $\mathbf{v}$ [9]. According to the frozen-in condition of vortices [Eq. (8)], it appears that no new vortex could be generated.

What happens when a pair of vortices of opposite signs are created at the same site and break up afterward? This situation is only possible when a quantum fluid, initially containing no density void, generates a pair of singular density voids at the same location. Development of a singular density void requires the existence of a singular (external) force core, which expels the fluid from the core. Hence creation of paired vortices requires the appearance of a pair of coalesced singular potential barriers, which later move apart from each other. On the other hand, when the forcing potential is everywhere smooth and regular, then creation or destruction of vortices will become impossible. These vortex-generating singular potentials include those that are multivalued functions of space.

A rotating potential is an example of the multivalued function of space, for which $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi=\mathbf{0}$ over an extended region; it exerts a torque on the fluid, thereby generating vorticity. In this case, the right-hand side of Eq. (8) survives and smooth vorticity can therefore be generated in the quantum fluid. The effects of a rotating potential are the same as those of a magnetic field. In the presence of a magnetic field, the fluid velocity consists of a part that is irrotational and another that is rotational and proportional to the vector potential. These smooth vorticity distributions must coalesce into singular vortices after the rotating potential is turned off.

Finally, we remark further on the conventionally underexplored sheet vortices. A stationary sheet vortex described above theoretically can be constructed in a quantum system containing an infinitely thin plate of finite size located in a region detached from the boundaries. The presence of the impenetrable plate yields a vanishing density at the plate, and if circulation takes place around the plate, the plate will be a natural site to house the sheet vortex. The strength of the local vorticity is not constant along the sheet; however, the circulation around the sheet must still be quantized. Once the thin plate is removed and the system becomes force-free, the vortex sheet will no longer be stationary. Although the sheet vortex can be decomposed into many independent plane-wave solutions in a force-free system, the sheet must remain intact in accordance with Eq. (8). The survival of the sheet vortex arises primarily from the strong initial correlations among the plane-wave solutions, which do not disperse and lead to the subsequent phase lock in the dynamics. This aspect of the dynamics can only be revealed in the framework of the fluid formulation and can hardly be detected directly from the Schrödinger formulation.

IV. GENERIC PICTURE OF TIME-DEPENDENT QUANTUM CHAOS

It is instructive to question how the vortex lines or sheets behave when they collide under the condition that the forcing potential remains smooth and regular. Since the fluid elements cannot overlap as a result of conservation of mass, if the two singular vortices cross each other, the angular momentum for the fluid elements at the crossing will have to change. This will violate the frozen-in condition of vortices and hence such a process is not permissible. It thus follows that the vortices must rebound upon collision. The three-dimensional topological characteristics of the vortex lines, such as links and knots [10–13] given by the initial condition, must therefore always persist as long as the frozen-in condition is respected. Likewise, the three-dimensional topology of vortex sheets, such as toroidal twists, must persist over the evolution. Interactions for singular vortices described above represent important dynamical features in quantum chaos.

Despite the inability of the quantum fluid to generate new singularities if no singular forcing potential is present, the quantum fluid does, however, have the tendency to develop sharp boundaries with large velocity gradients. This is due primarily to the nonlinear steepening effects of Eq. (2). Suppose that the initial velocity and density are both smooth and vary on the spatial scale L; in addition, the time-dependent external forces are also of scale L. Defining $R \equiv (|\nabla S|)L/\hbar$, we may consider the regime where $R \gg 1$. In this regime, the ''pressure'' force, the first term on the right-hand side of Eq. (4), is much smaller than the inertial force, the second term on the left. It immediately becomes obvious that Eq. (4) is identical to the Euler equation of ideal classical fluids in the limit of zero pressure. An ideal classical fluid without pressure is bound to steepen rapidly at the leading edges of disturbances to form surface discontinuities [14]. The nonlinear steepening has been understood to be caused by the fluid elements of higher velocities tending to run faster and catch up with the slower fluid elements originally located in front of them. This interpretation also holds for the quantum fluids. Indeed, the short-wavelength matter waves have higher speeds and run faster than the long-wavelength waves, and naturally the wave steepening arises.

In reality, such steepening in classical fluids is stalled by the small viscosity in yielding shock waves; similarly, wave steepening in quantum fluids can also be halted by the dissipationless quantum fluctuations. The dissipationless quantum fluctuations give rise to wave dispersion, which serves to counteract the wave steepening in the same way as dissipation does for classical fluids. In fact, a dispersive classical fluid, such as the shallow-water wave, can also avoid the shock formation by converting the flow of kinetic energy into solitons or solitary wave trains. The quantum fluids likewise permit solitary wave trains excited by the steepening of large-scale flows. This will be shown below. To make a final connection between the classical and quantum fluids, the quantity R defined above may be regarded as the Reynolds number in the classical fluids in this context.

To illustrate the solitary wave solutions, Eqs. (2) and (3) can be rearranged to become

$$\frac{\partial \mathbf{u}_+}{\partial t} + \mathbf{u}_+ \cdot \nabla \mathbf{u}_+ + \frac{\hbar}{2} \nabla^2 \mathbf{u}_- = -\nabla \phi, \tag{9}$$

$$\frac{\partial \mathbf{u}_-}{\partial t} + \mathbf{u}_- \cdot \nabla \mathbf{u}_- - \frac{\hbar}{2} \nabla^2 \mathbf{u}_+ = -\nabla \phi, \tag{10}$$

where $\mathbf{u}_\pm \equiv \mathbf{v} \pm \mathbf{w}$. For a force-free, static solution where $\phi = 0$ and $\mathbf{v} = \mathbf{0}$, the two equations coincide and become the stationary Burger equation

$$\mathbf{w} \cdot \nabla \mathbf{w} - \frac{\hbar}{2} \nabla^2 \mathbf{w} = \mathbf{0}. \tag{11}$$

In one dimension, the Burger equation is known to contain soliton solutions, which satisfy

$$\frac{dw}{dx} = \frac{\hbar}{2} (w^2 - C), \tag{12}$$

where C is an integration constant; the solution is a kink soliton $w = \sqrt{C} \tanh(2x/\hbar)$ when C is positive. However, this kink solution cannot be a physical solution since $w \propto d[\ln(\rho)]/dx$ and this kink solution yields an infinite density at $|x| \to \infty$. Another possible solution can be obtained from Eq. (12) by using the variable f, and Eq. (12) is re-expressed as

$$\frac{1}{f} \frac{d^2 f}{d^2 x} = -\frac{C}{2\hbar^2}. \tag{13}$$

When $C < 0$, $f = \cos(\sqrt{C}x/2\hbar)$, recovering the linear standing-wave solution of the Schrödinger equation. Thus the nonlinear steepening in quantum fluids is halted by the wave dispersion that converts the available flow of kinetic energy into the internal energy, with the generation of an abundance of short waves. Here C is approximately the large-scale flow of kinetic energy.

Thus the second important aspect of the chaotic quantum dynamics can be pictured as follows. Large-scale fluctuations are injected into the system by the large-scale forces and eventually develop sharp boundaries of large gradients through nonlinear steepening. Engulfed within the sharp boundaries is an abundance of short waves. Collisions among these boundaries are possible, yielding even more chaotic structures. In classical fluids, collisions of shock waves are the effective mechanisms for generating arrays of line vortices in the postshock regions [14]. However, in quantum fluids, the singular vortices cannot be so generated.

V. DISCUSSION

The present formulation for quantum mechanics as an initial-value problem provides a useful perspective to view chaotic quantum dynamics. In particular, the constraints on the evolution of singular vortices can clearly reveal themselves through the nonlinear equation describing the conservation of angular momentum. This aspect of quantum dynamics may indeed be difficult to detect when the linear superposition of evolving plane-wave solutions is adopted for analysis. On the other hand, this fluid formulation, when closely compared with the Schrödinger formulation, provides a suggestion regarding how the long-sought hidden integrals in fluid turbulence can be constructed.

In the absence of a time-dependent potential, Eqs. (2), (4), and (6) resemble the Navier-Stokes equations of the classical fluids. It is well known that the solutions to the Navier-Stokes equations are notoriously complicated in the regime of fully developped turbulence. Likewise, when the initial data of the quantum fluid is sufficiently complicated, containing permissible (finite L^2 measure) singularities of all sorts, the solution at any later time must also appear complicated and one would have concluded that such a quantum fluid is turbulent. Yet this quantum system is in fact integrable using the Schrödinger formulation. In this regard, one may attribute the integrability of the quantum fluid equations (2), (4), and (6) to some kind of hidden symmetry, which disguises itself behind the deceptive complexity of the nonlinear fluid equations. Without any *a priori* knowledge about the relation of this set of nonlinear fluid equations to the linear Schrödinger equation, it is inconceivable that one is able detect this hidden symmetry, which only reveals itself by a proper combination of the variables $\mathbf{v}$ and ρ to form the wave function ψ.

With this observation, one may turn the problem around and ask whether the Navier-Stokes equation may also contain any similar hidden symmetry that leads to certain degrees of integrability in yielding the intermittency [4–6] and coherent structures [7] often observed in fluid turbulence. The expectation is also reinforced by the fact that fluid turbulence may proceed without external forces. In this sense the classical fluids may share similar symmetries with the quantum fluids in the absence of external forces. At the present time, there is no systematic way to detect hidden symmetries, if any, in three-dimensional Navier-Stokes systems. A possible way to seek symmetries may be based on the Lagrangian of the Navier-Stokes equation [14]. However, inspired by the similarity between the quantum particles and classical fluids, one may perhaps gain penetrating insights for fluid turbulence if deeper connections between the fluid formulation of QM and the Schrödinger equation are well understood.

In sum, we have presented an unconventional perspective of the dissipationless chaotic quantum dynamics for a quantum particle subject to a time-dependent force. This perspective of the quantum dynamics makes an extensive analogy to a classical ideal fluid, where the quantum fluctuations replace the thermal fluctuations. The quantum fluctuations, resulting in wave dispersion, can play a similar role to the fluid viscosity in stopping the formation of surface discontinuities resulting from nonlinear wave steepening. Furthermore, the singular vortices in quantum fluids cannot be created or destroyed if the potential is a smooth function of space. However, once initially given, these vortices must evolve in a specific way without changing their topological characteristics. The chaotic quantum dynamics is therefore expected to be chiefly governed by the interplay between wave steepening and vortex interactions.

Dynamics of line vortices in three dimensions has been a subject under intense investigations in recent years [15,16]. The possible potential applications of this subject area focus on the high-T_c superconductivity.

Finally, we shall comment on how one may proceed to numerically investigate the dynamical quantum chaos within the fluid framework. One may first single out the vortex singularity, which is located where the density $\rho = 0$. One then use the Biot-Savart law to calculate the incompressible part of the potential flow. The compressional component of the potential flow is governed by an equation that results from taking a Laplacian of the Bernoulli equation (3). In this step, one must replace $(\nabla S)^2$ by $\mathbf{v}^2$ to distinguish the compressional and incompressional velocity components. The density evolves according to the continuity equation (2) as usual. Finally, the singular vortices obey Eq. (8), which is nothing more than the conservation of circulation per unit mass. In the case of line vortices, the strength of circulation associated with each line is quantized and hence one may simply trace of the locations of the frozen lines [17] or trace the locations of density voids. In the case of sheet vortices, the situation can be more complex. Since the local vorticity on the sheet is not necessarily uniform, one not only needs to trace the location of the vortex sheet but also records the change of vorticity distribution on the sheet. Nonetheless, one may handle this complication by considering the vortex sheet to be composed of an array of many vortex lines of given strengths; the frozen lines can be evolved independently.

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