# Appearance and nonappearance of self-organized criticality in sandpiles

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Self-organized criticality (SOC) often features in sandpile-type modeling of complex systems such as magnetic fusion plasmas, but its observed role in real sandpiles is equivocal. Here, a probabilistic model displaying some of the observed phenomenology of real sandpiles is proposed. It generates avalanches involving particle loss that, like most experiments, do not exhibit SOC, but its internal energy avalanches resemble recent experiments by showing SOC. This suggests that the absence of SOC in the flow of matter from such systems can be consistent with internal dynamics governed by SOC. [S1063-651X(98)10903-0]

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The concept of self-organized criticality (SOC) [1,2] and the physics of sandpiles [3–23] are of great interest, both intrinsically and as a paradigm for more complex macroscopic systems the global properties of which may arise from the growth, saturation, and nearest-neighbor couplings of a discrete sequence of spatially localized modes. An example of the latter is provided by sandpile modeling of the transport properties of magnetically confined plasmas, some of which are claimed to display features of SOC [21–23]. In this paper, we construct a theoretical sandpile model to help address an apparently paradoxical feature of sandpile physics: while the statistics of avalanches in most experimental sandpiles do not display SOC, the reverse is true for mathematical sandpile algorithms.

SOC was introduced by Bak and co-workers [1] who showed that certain systems of cellular automata evolve to a critical state through a self-organized process. The state is termed critical because it has no characteristic length or time scales; it is self-organized since it is insensitive to initial conditions. The simplest such systems are mathematical sandpiles, the states of which are specified by an array of integers $h_n$ representing the height of the pile (number of sand grains) at each position. At each time step a grain of sand is added at a random position. If the local slope of the pile exceeds some critical value, sand is redistributed so as to reduce the local slope; this may result in the critical gradient being exceeded at neighboring positions, leading to redistribution there. The process may then spread, becoming an avalanche. Kadanoff and co-workers [2] have investigated the distribution of avalanches in mathematical sandpiles, which typically follows a simple scaling law (for example, a power law) over a range of several orders of magnitude, implying SOC. The overall behavior is typically independent of the critical gradient and of the finer details of the redistribution algorithm.

It is apparently accepted [3–6] that experimental sandpiles [6–11] do not usually (with one notable exception [6,11]) exhibit SOC avalanches. Experiments have been carried out using slowly rotating drums partially filled with sand [7,9]; conical sandpiles with grains fed to the apex [8,10]; sandpiles fed by random sprinkling of particles over the active surface [7]; and piles of rice grains [6,11]. Reanalysis in Ref. [4] of the data of Refs. [7–10] suggests the absence of SOC, while the occurrence of SOC in Ref. [6] depends on the aspect ratio [(length)/(diameter)] of the rice grains. The experimental data includes the following: amplitude and time separation of avalanches involving mass loss from the sandpile, for example, Fig. 1 of Ref. [7]; time evolution of sandpile mass, for example Fig. 2 of Ref. [8] and Fig. 1 of Ref. [10]; the confinement time of tracer particles, for example Fig. 3 of Ref. [11]; avalanches involving rearrangement rather than mass loss, for example, Fig. 2 of Ref. [6] and Fig. 2 of Ref. [9]; and avalanche size distributions [4,6–11]. The existing experimental database on avalanches is weighted towards the subset of avalanches that result in particle loss from the edge of the sandpile. However, it can be argued [6] that avalanches involving internal reorganization reflect a more direct response to the driving than do avalanches involving loss. It has also been pointed out [4] that energy dissipation in avalanches (see Ref. [6]) is a fundamental physical quantity that has not yet been subjected to the same level of scrutiny as sand movement. The observation that avalanches in real sandpile experiments, with exceptions [6,11], tend not to replicate the SOC behavior that emerges from idealized mathematical sandpile models poses several challenges. One of the most immediate, which we now address, is to construct a theoretical model that yields more realistic results.

We begin with a few remarks. First, the internal dynamics of most sandpile models is completely deterministic: randomness is usually introduced only through the fueling process, for instance by adding grains of sand from above at random positions. However, in a real sandpile the variations among individual sand grains and their stacking properties are expected to introduce an element of randomness in the pile itself. The random internal distribution of stress has been illuminated by several recent advances [13–16]. Our choice of sandpile rules will therefore be probabilistic rather than deterministic. In this, we follow a trend set in the models of Ref. [11] and of Refs. [17–19], where an element of randomness has been introduced in, for example, the critical slope or the relaxation rule. The question of which classes of sandpile model will best replicate experimental results, and the origin of differences between classes of model, is highly topical [19] and not yet resolved. Second, we note that in order to model experiments such as those done with a rotating drum [7,9], where the avalanches are caused by tilting the pile rather than adding sand to it, the slope should be a continuous variable. In contrast, the slope is constrained to integer values in most sandpile algorithms appearing in the literature. Third, experiments support the idea of relaxation to an angle of repose. This means that if an avalanche occurs somewhere in the pile, the slope is reduced, at least locally, to the value defined by this angle.

To proceed with the model, we discretize the pile into a series of nodes labeled by $n=1,\ldots,N$ each with a slope $z_n=h_n-h_{n+1}$, as is conventional in sandpile models. The height $h_n$ and the slope $z_n$ are normalized to zero at the angle of repose, since the material in the pile below the angle of repose plays no explicit role in our model. Redistribution of sand occurs as a result of instability that arises if the pile is too steep, in the following sense. Consider an initially stable sandpile, and suppose the slope somewhere increases by an infinitesimal amount $dz$, either by external interference (like tilting the pile or adding sand) or by internal redistribution. That position may then become unstable, and we assign a probability $dp$ to that possibility. We expect $dp$ to depend on the local slope $z$, so that

$$dp/dz=f(z).$$

There is no longer, in our model, a single critical gradient at which instability occurs; instead there is a probability of instability $dp$ associated with each increment in slope $dz$. This models the randomness inherent in real sandpiles and produces a probabilistic spreading of criticality, which is now distributed across a range of gradients whose width is controlled by the function $f(z)$. Since instability becomes increasingly likely as the pile becomes steeper, $f(z)$ should be a monotonically increasing function, vanishing at the angle of repose $z=0$. The simplest model is

$$f(z)=Az^y, \tag{1}$$

with $A$ and $y>0$ adjustable parameters. We can choose $A=1$ by an appropriate normalization of the slope $z$, since this leaves the redistribution rules (yet to be specified) unchanged. The remaining parameter $y$ defines the sharpness of the critical gradient. In the limit $y\to\infty$ instability always occurs exactly when $z=1$, which then becomes an exactly defined critical gradient, as in conventional sandpile models. For finite $y$, the slope at which instability occurs is a stochastic variable. Its distribution function $F(z)$, i.e., the probability that any particular position becomes unstable before the slope there exceeds $z$ is given by

$$F(z)=1-\prod_{z'<z}[1-f(z')]dz',$$

where the product represents the probability that the position remains stable when the slope $z$ is reached. Taking logarithms we find

$$\ln[1-F(z)]=\sum_{z'<z}\ln[1-f(z')]dz'=-\int_0^z f(z')dz',$$

whence

$$F(z)=1-\exp\left[\frac{-z^{1+y}}{1+y}\right].$$

Note that when $y\to\infty$ this function approaches a step function at $z=1$, where instability then always occurs.

Once instability occurs somewhere in the pile, the slope relaxes locally to the angle of repose by redistributing sand. The first step of this process is constructed in the conventional way by the rule $z_n\to 0$, $z_{n\pm 1}\to z_{n\pm 1}+z_n/2$, which conserves the total amount of sand in the pile. At the boundary of the pile, $n=N$, the redistribution rule is, as usual, slightly different to allow sand to fall off the edge: we expect $h_{N+1}=0$ and therefore $z_N\to 0$, $z_{N-1}\to z_{N-1}+z_N$. We also make the conventional assumption that the redistribution is instantaneous. However, in contrast to conventional sandpile models, we allow for the possibility that several adjacent positions be *simultaneously* unstable. This is clearly the case in real sandpiles where, during avalanches, many grains of sand are simultaneously in motion over an extended spatial region. Therefore, if as a result of redistribution from one position, one or both of the neighboring positions become unstable, the pile is flattened over the entire unstable region. This process continues, in an avalanche that may or may not spread over the entire pile, until all positions are stable again. To work out the redistribution rule during an ongoing avalanches, suppose that, after a series of redistribution events, the positions $n$, $n+1,\ldots,n+k-1$ are unstable. The pile relaxes to the angle of repose in this region, $z_i\to z_i^*=0$ for $n\leqslant i\leqslant n+k-1$, where an asterisk denotes the configuration after relaxation at this step. As a result of redistribution, the pile is locally flattened so that $z_n^*=z_{n+1}^*=\cdots=z_{n+k-1}^*=0$. It follows from the requirement that sand be conserved that after redistribution the slopes become

$$z_{n-1}^*=\frac{1}{k+1}\sum_{i=1}^{k+1} z_{n+k-i}i, \quad z_{n+k}^*=\frac{1}{k+1}\sum_{i=1}^{k+1} z_{n+i-1}i,$$

if no sand leaves the pile $(n+k-1<N)$, and otherwise

$$z_{n-1}^*=\sum_{i=n-1}^{N} z_i .$$

Sand can be added in different ways, and we explore the following possibilities. (1) Central fueling—adding single grains at the top of the pile: $z_1\to z_1+g$ with $g$ the grain size. (2) Sprinkling—adding single grains at random positions, uniformly distributed over the pile: $z_n\to z_n+g$, $z_{n-1}\to z_{n-1}-g$, with $n$ random and uniformly distributed over $1\leqslant n\leqslant N$. (3) Tilting—increasing the slope continuously by equal amounts everywhere: $z_n\to z_n+t$ for all $n$, where $t$ denotes dimensionless time. In each case the slope is increased by one of these external means until some position becomes unstable. The redistribution rules are then applied until the pile has relaxed to a new stable state, after which the procedure is repeated.

Using this model in numerical simulations of a sandpile with discrete central fueling, the amount of sand above the angle of repose typically varies with the total amount of sand added as shown in Fig. 1. It looks much like the corresponding graph obtained for a continuously tilted pile in Fig. 2. The amount of sand in the pile increases linearly until a major avalanche occurs. The pile then relaxes to the angle of repose, and all excess material leaves the system. Small ava-

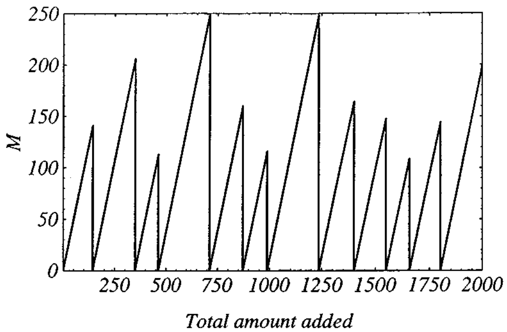


FIG. 1. The total amount of sand $M$ (above the angle of repose) in the pile versus the total amount of sand added to the pile by central fueling of particles. The length of the pile is $N=50$, the exponent in Eq. (1) is $y=1$, and the size of the sand grains is $g=0.01$.

lanches, resulting in some, but not all, sand above the angle of repose leaving the pile are rare. All avalanches are of the same order of magnitude, and no power law or other broad distribution of avalanche sizes is observed. This is in agreement with most experiments on real sandpiles, see, for example, Fig. 2(d) of Ref. [8] and Fig. 1 of Ref. [10]. Changing the parameter $y$ does not influence the results much, regardless of the way the pile is fueled. If $y \gg 1$, instability is very unlikely unless $z \gg 1$. For large $y$, there is therefore a lower limit to the avalanche sizes.

The situation is only slightly different if the pile is fueled by random sprinkling by small grains, $g \ll 1$, as shown in Fig. 3. Since the sand is now added discretely but, on average, evenly over the pile, the slope only builds up at the last position, $n=N$. When this position becomes unstable, a small avalanche occurs involving only one, or sometimes a few, positions at the end of the pile. The latter is thus slightly eroded. As the sprinkling continues, the slope again increases at the edge. The next time it becomes unstable, a large avalanche spreading over the entire pile usually develops. Most of the material above the angle of repose thus leaves the pile in major avalanches.

Thus, SOC is not observed in the material that leaves the sandpile for the models we have considered. However, if one studies the *internal* dynamics of our model sandpile growing by central fueling, a rather different picture emerges, which *does* involve SOC. Let us follow recent suggestions [4,20] that what should be measured is the amount of potential energy,

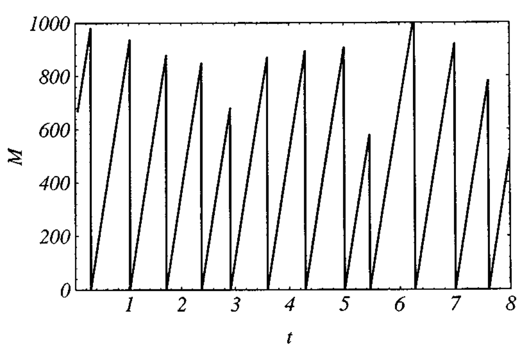


FIG. 2. As Fig. 1, for a continuously tilted pile with dimensionless time $t$ as independent variable; $N=50$, $y=1$.

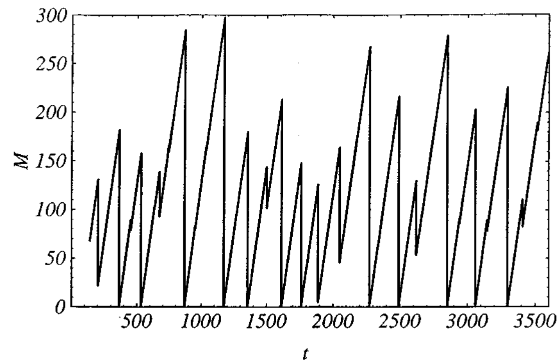


FIG. 3. As Fig. 1, where sand is added to the pile by random sprinkling across the surface; $N=50, y=1, g=0.01$.

$$E=\sum_{n=1}^{N} h_n^2,$$

dissipated in avalanches, and pursue the conjecture that the statistics associated with the energy $\Delta E$ dissipated in avalanches may be different from the drop numbers measured in most experiments. Figure 4 shows the evolution of $E$ in a growing sandpile fueled at the center. The energy increases quadratically with the amount of sand added, but is frequently interrupted by minor avalanches. In this respect, central fueling is different in our model from random sprinkling or tilting, which, as we have seen, tend to make the pile grow uninterrupted until it is emptied by a major avalanche. In contrast, when the pile is at the angle of repose and the fueling begins from the center, a small excess is first established near the center since this is the only place where sand is added. The pile then grows and spreads by a series of avalanches propagating outwards. These events occur on all scales until the front of he pile reaches the edge, and a major avalanche soon occurs, reducing the entire pile to the angle of repose. The time series of avalanches, shown in Fig. 5, is a strongly reminiscent of Fig. 2(c) in Ref. [6], which appears to be the only report of an experiment where energy dissipation has been recorded. Figure 6 shows a logarithmic plot of the avalanche energy distribution. Most of the distribution is well approximated by a power law, shown as a solid line; only the very largest events are exceptional since they involve the entire pile and thus possess a characteristic scale.

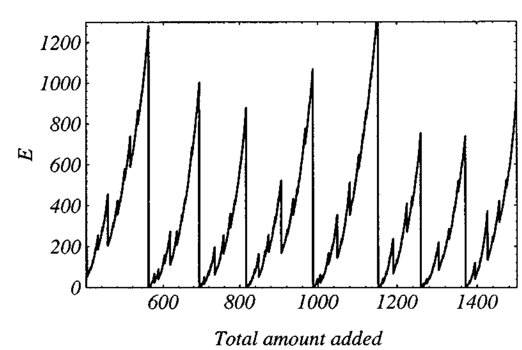


FIG. 4. Potential energy $E$ stored in the sandpile versus the total amount of sand added by fueling at the center; $y=1, N=50, g=0.001$.

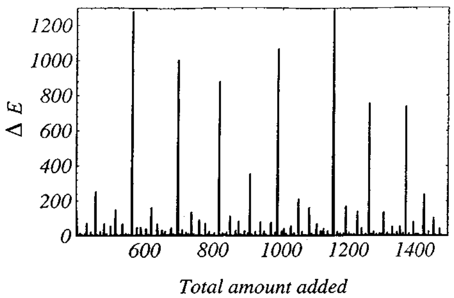


FIG. 5. Potential energy released in the avalanches in Fig. 4.

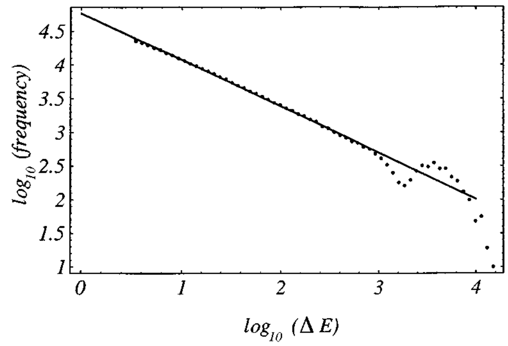


FIG. 6. Distribution of avalanche energies (dots) in a centrally fueled pile; $y=1, N=100, g=0.001$.

The power law exponent is about $\tau=-0.65$. We find that the behavior is highly insensitive to the size $N$ of the sandpile, the grain size $g$ (as long as it is small), and the exponent $y$. From these observations it is clear that the internal dynamics of the growing sandpile is indeed self-organized and critical, but that this manifestation of SOC is invisible to an observer watching only what leaves the system.

In conclusion, we have proposed a simple new theoretical model for sandpile, which appears consistent with many experimentally observed features. Furthermore, the model provides a conceptually appealing step towards resolving the role of SOC in sandpile experiments: for a growing, centrally fueled sandpile, the statistics for energy release in internal avalanches display SOC, whereas the statistics for material lost from the pile due to avalanches do not. This suggests that nonobservation of SOC in the flow of matter from experimental sandpile-type systems can be consistent with internal dynamics that are governed by SOC.

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