Chaos and exponentially localized eigenstates in smooth Hamiltonian systems

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We present numerical evidence to show that the wave functions of smooth classically chaotic Hamiltonian systems scarred by certain simple periodic orbits are exponentially localized in the space of unperturbed basis states. The degree of localization, as measured by the information entropy, is shown to be correlated with the local phase-space structure around the scarring orbit. Sharp localization is observed when the orbit scarring the wave function undergoes a pitchfork bifurcation and loses stability. [S1063-651X(98)00501-7]

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It is now well known that classical periodic orbits have an enduring influence on the quantum mechanics and the semiclassics of classically chaotic quantum systems. The knowledge of all the isolated periodic orbits of such a system allows us to estimate semiclassically the eigenvalues through the application of Gutzwiller's trace formula [1]. Heller [2] has given theoretical arguments to show that for the classically chaotic quantum systems the eigenfunctions will show density enhancements, called scars, along the least unstable periodic orbits. Such localized probability density structures of the wave functions and their correspondence to the underlying periodic orbits are widely reported for many classically chaotic quantum systems, such as the hydrogen atom in a magnetic field [3]. The effect of localized states in a quantum system with predominant classical chaos has been observed experimentally using tunnel-current spectroscopy in semiconductor heterostructures [4]. The role played by the bifurcation properties of periodic orbits is also of considerable importance. The effect of orbit bifurcation is observed experimentally in the spectra of atoms in external fields [5,6].

Numerical evidence for wave-function localization in systems with classical chaos came from the studies of Bunimovich billiards [2,7]. For the kicked rotor, Grempel *et al.* [8] have shown that the localization in momentum space is exponential, similar to Anderson localization in the case of charged-particle dynamics in a series of potential wells with random depths. In smooth potentials such as coupled oscillators, adiabatic methods have been widely applied to predict some eigenvalues, but the construction of adiabatic wave functions still remains an open problem. de Polavieja *et al.* [9] have used wave-packet propagation techniques to construct wave functions highly localized on a given classical periodic orbit. A qualitative study of the effect of pitchfork bifurcation of simple orbits on the eigenfunctions has also been reported [10]. In this paper we explore the connection between certain simple classical periodic orbits, their stability, and the localization of the quantum wave functions scarred by them, using coupled nonlinear oscillators. In such systems, even in the regions of large-scale classical chaos, certain simple periodic orbits with short-time periods and high stability are known to scar an infinite series of WKB-like states in the spectrum [11] and recently it has been observed that these states affect the eigenvalue spacing distributions [12]. In particular, our numerical evidence for smooth potentials indicates that such states, scarred by simple periodic orbits, are exponentially localized in the space of unperturbed basis states. Thus, for such states, the unperturbed Hamiltonian basis is the preferred one even in the case of large coupling.

In the analysis of wave-function scarring [13] stability of the periodic orbits is shown to affect the magnitude of scarring significantly. The analysis remains limited to averaged Wigner functions and is not valid at the points of bifurcations where Gutzwiller's density of states formula breaks down, leading to predictions of either infinite scar weights or scar amplitudes. In this context we will show that gross measures of individual wave-function localization, such as entropy, are strongly correlated with the periodic orbit's stability oscillations (with a parameter). When the orbit loses stability we will find that the entropy is also low, with the point of bifurcation being approximately the point of a local minimum in the entropy (again as a function of a parameter). We will note that this does not always coincide with the points at which the Berry formula [13] predicts an infinite scar amplitude.

We study smooth Hamiltonian systems of the form

$$H(x,y,p_x,p_y;\alpha)=\frac{p_x^2}{2m}+\frac{p_y^2}{2m}+V(x,y;\alpha), \tag{1}$$

whose classical dynamics can be regular or chaotic depending on the value of the parameter α. The coupled quartic oscillator given by the homogeneous potential $V(x,y;\alpha)=x^4+y^4+\alpha x^2y^2$ is used below with $m=1/2$. This system is integrable for α=0, 2, and 6 and exhibits increasing chaos as α is increased beyond 6. The presence of a ''channel'' in this potential leads to trapping of the particle in a motion along the channel periodic orbit $(x,y=0;\ \ p_x,p_y=0)$, which has a short-time period and interesting stability properties. That these initial conditions determine a periodic orbit is evident, as the equations of motion imply that there is zero velocity and acceleration in the y direction, thereby restricting the motion to the x direction alone. The stability of the channel periodic orbit as measured by $\mathrm{Tr}\ J(\alpha)$, where $J(\alpha)$ is the monodromy matrix, displays bounded oscillations as a

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function of α and for this simple periodic orbit of the Hamiltonian in Eq. (1) the analytical expression for $\mathrm{Tr}\ J(\alpha)$, due to Yoshida [14], is given by

$$\mathrm{Tr}\quad J(\alpha)=2\sqrt{2}\cos\left(\frac{\pi}{4}\sqrt{1+4\alpha}\right). \tag{2}$$

The monodromy matrix refers to the ''half Poincaré map'' [5], which is the usual Poincaré map without the restriction of positive momentum on the intersection. We note that this is the appropriate monodromy matrix in the semiclassical analysis as well due to the fact that we are restricting the quantum spectrum to the A_1 irreducible representation of the C_{4v} point group that the Hamiltonian possesses. The important inference from Eq. (2) is that this channel orbit changes stability when $\alpha=n(n+1)$, where n is an integer, through a pitchfork bifurcation. This follows because at these parameter values $|\mathrm{Tr}[J(\alpha)]|=2$. Another interesting property we have found recently is that the Poincaré section around the channel periodic orbit locally scales with respect to the parameter α for all one-parameter homogeneous two-dimensional Hamiltonian systems. The scaling exponents depend simply on the degree of homogeneity of the potential [15]. In this paper, we will focus our attention on this channel periodic orbit and its influence on the quantum wave functions.

The quantization of the Hamiltonian in Eq. (1) is by numerically solving the Schrödinger equation in the basis of the eigenfunctions of the corresponding unperturbed problem, namely, the $\alpha=0$ case in Hamiltonian (1). The actual basis states $\psi_j(x,y)$ employed are the symmetrized linear combination of the wave functions for the unperturbed problem [16]. The wave function for the mth state is given by $\Psi_m(x,y;\alpha)=\Sigma_{i=1}^{N}a_{m,j}\psi_j(x,y)$, where j represents the pair of even integers (n_1,n_2) corresponding to the quantum numbers of two one-dimensional states that makes up the basis state and $a_{m,j}$ are the expansion coefficients. The information entropy measure, for the mth state with M components, where M is the dimension of the Hamiltonian matrix, defined as

$$\mathcal{S}_m^{\alpha}=-\sum_{j=1}^{M}|a_{m,j}^{\alpha}|^2\ln|a_{m,j}^{\alpha}|^2, \tag{3}$$

is applied to the wave-function spectrum and we have shown that the localized states scarred by the channel periodic orbit are identified by a pronounced dip in the information entropy curve and they were also visually identified [11].

The Berry-Voros conjecture that the Wigner function of a typical eigenstate of a quantized chaotic system is a microcanonical distribution on the energy shell is of course violated by many scarred states and especially by the channel localized ones. The channel states are in fact so localized that, as we discuss below, they are in a sense even exponentially localized.

In Fig. 1(a) we show $a_{m,j}$ plotted as a function of j for one such highly excited state at $\alpha=90.0$. In Fig. 1(b) $a_{m,j}$ corresponding to a channel localized state at the same value of α is plotted. We immediately recognize that for the channel localized state very few basis states contribute to the building up of the wave function. The largest peak is made

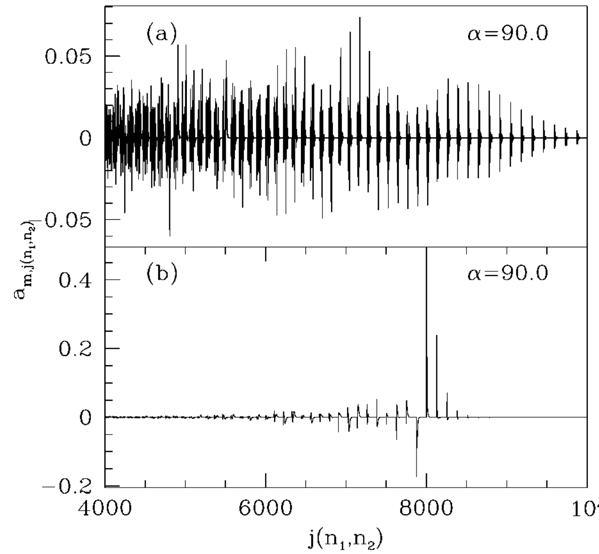

FIG. 1. Significant expansion coefficients of two states from the same spectrum for the quartic oscillator: (a) a typical state, number 1971 from the ground state, and (b) a channel localized state, number 1774.

of basis states whose quantum numbers are of the form $(N,0),(N,2),(N,4), \ldots$, in decreasing order of their contribution. N refers to the number of quanta of excitation for the motion along the channel. We observed that the pattern in Fig. 1(b) is qualitatively generic for all the channel localized states. For instance, in Fig. 2 we have the wave functions of localized states for the parameters $\alpha=88.0$ and 96.0.

The discontinuous manner in which the peaks rise is due to the ordering of the pair of quantum numbers (n_1,n_2) along the one-dimensional array j. For instance, the pair of quantum numbers immediately preceding the largest peak at, say, $(N,0)$ would have large excitations in *both* the x and y directions and therefore have vanishing influence on the channel localized states. The exponential falloff is therefore

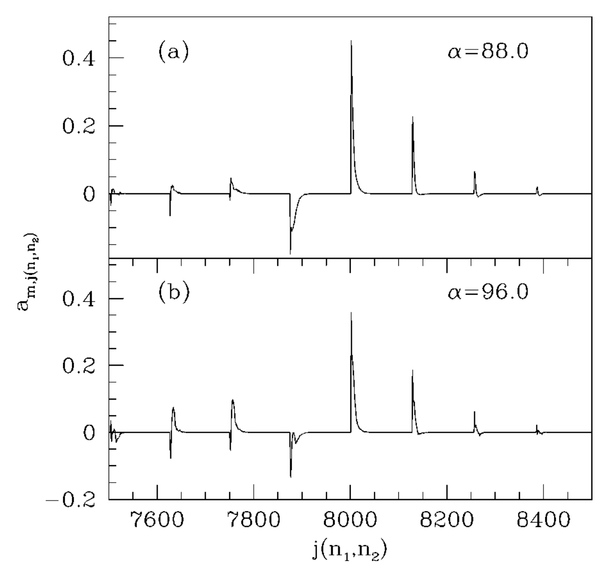

FIG. 2. Significant expansion coefficients of two channel localized states at different parameter values: (a) state number 1786 and (b) state number 1740. The principal peak $N=252$ (see the text) for these states and those in Fig. 3.

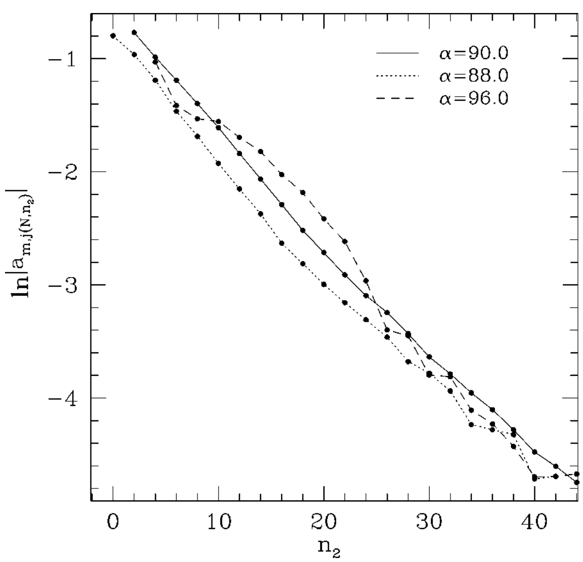

FIG. 3. Logarithm of the expansion coefficients in the principal peak of the states 1786, 1774, and 1740 at α=88.0, 90.0, and 96.0, respectively. The values of n_2 have been shifted to the right by 2 and 4 for the states at $\alpha=90$ and 96, respectively, otherwise the lines practically fall on top of each other.

one sided. The falloff within a peak could have alternating signs, as indeed is the case for the strongly perturbed harmonic oscillator presented below, in which case the modulus is taken before analysis. This is largely a matter of sign conventions adopted for the one-dimensional states forming the two-dimensional basis. Even the gross features of such states are not yet quantitatively understood. As a subclass of wave functions of a chaotic system, however, they appear to be the most accessible.

Significantly, the quanta N also enters an adiabatic formula, with constants $b_0 \approx b_1/3$ and b_1,

$$E_N(\alpha)=b_0\sqrt{\alpha}\left(N+\frac{1}{2}\right)^{1/3}+b_1\left(N+\frac{1}{2}\right)^{4/3}, \quad (4)$$

to estimate the eigenvalues E_N of the channel localized states. The value of $b_1=2.185\,069$ coincides with that obtained from the WKB approximation to the one-dimensional quartic oscillator. The existence of such a formula for a subclass of states in a chaotic spectrum is indicative of the special nature of these states as no systematics has yet been uncovered for scarring. Although this formula does not include the stability of the orbit as a parameter, we have found that this is most accurate when the orbit is about to lose stability and the wave function is highly localized as noted below. We derived Eq. (4) as a consequence of adiabaticity as developed in [17].

Our numerical results show that the channel localized states in the unperturbed basis are dominated by exponentially falling peaks in the quantum number of the motion perpendicular to the direction of the channel and further that the degree of localization is related to the stability of the channel periodic orbit. In Fig. 3 we plot $\ln(a_{m,j(N,n_2)})$ as a function of n_2 for the principal peaks corresponding to the states shown in Figs. 1(b) and 2 and the fairly straight lines obtained show that the fall is indeed exponential. The next dominant peak with contributions coming mainly from $(N+2,0),(N+2,2),(N+2,4),\ldots$ basis states also provides evidence (not shown here) of exponential localization. However, for the third peak, corresponding to $(N+4,0)$, the values of $a_{m,j}$ fall within our accuracy of our calculation and hence, although unequivocal conclusions cannot be drawn, we expect the fall to be exponential. A localized A_1 eigenfunction with a dominant peak from $(N,0)$ may thus be approximated as

$$\Psi_N(l,n)=\sum_k A_k \exp(-n/\xi_k)\,\delta_{N+k,l}, \quad (5)$$

where (l,n) is a pair of even integers and k is an even integer index. A_k are the amplitudes of the peaks and only very few of them are appreciably different from zero. The localization lengths ξ_k are the inverses of the slopes of the straight lines as in Fig. 3.

It has been established that the eigenstates of time-dependent systems, such as the kicked rotor, are exponentially localized, but the above result is an observation of exponential localization in smooth chaotic systems. Why is there no exponential localization for other scarred states? We believe that the answer lies in the fact that (a) the basis in which there is exponential localization belongs to the Hamiltonian, namely, Eq. (1) with $\alpha=0$, for which the scarring orbit is also a valid orbit and (b) the stability of such an orbit is high and the channel orbit never becomes very unstable, however large the nonlinearity. For instance, the 45° straight-line periodic orbits could be exponentially localized in the 45° rotated unperturbed basis, but this orbit becomes

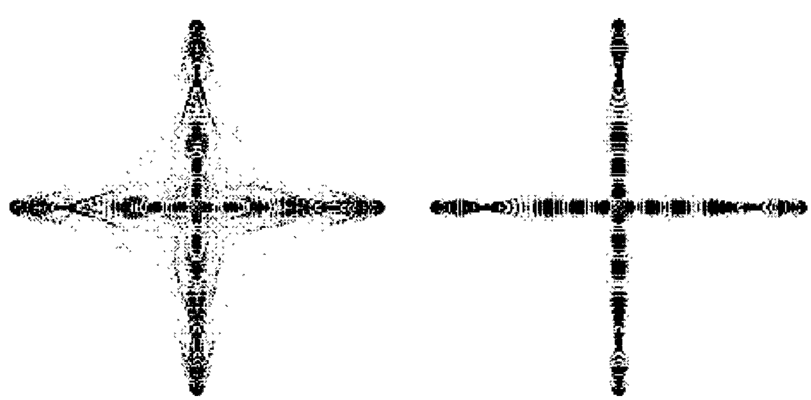

FIG. 4. Configuration-space channel localized wave-function densities for states 1740 (α=96.0, left) and 1774 (α=90.0, right).

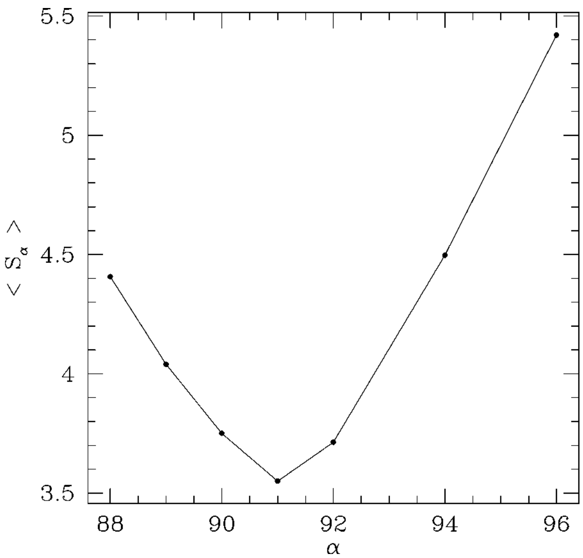

FIG. 5. Averaged information entropy plotted against α for the quartic oscillator.

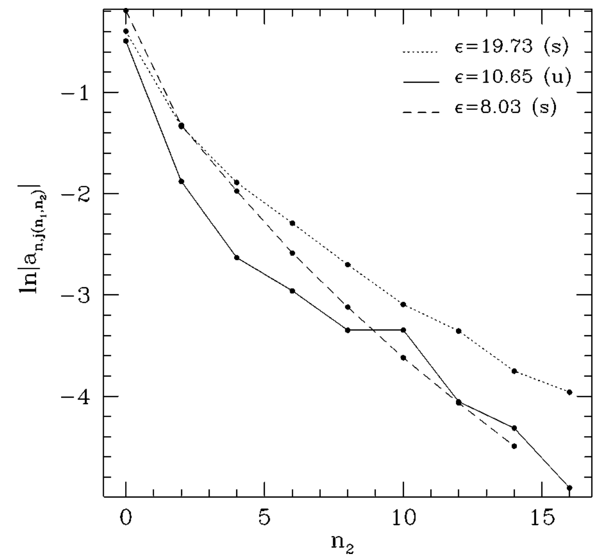

FIG. 6. Exponential localization for three channel localized states of the perturbed oscillator. u and s represent unstable and stable channel orbits. The n_1 for the three curves are not the same.

extremely unstable, thereby creating complex states in which other orbits contribute to the scarred state.

The channel periodic orbit loses stability through a pitchfork bifurcation [Tr $J(\alpha)=-2$] at $\alpha=90.0$ while creating two other stable orbits. We notice from Fig. 3 that we find the best exponential behavior at $\alpha=90.0$ and it progressively becomes less exponential as we explore the parameter regions in which the channel orbit also becomes progressively unstable. In Fig. 4 we have striking visual evidence for two wave functions at two different α values at which the channel orbit is stable and unstable. The wave function at $\alpha=90.0$ is compact and has almost collapsed onto the periodic orbit in comparison to the wave function at $\alpha=96.0$.

We calculated the average information entropy for a particular α by taking the mean of information entropies $\langle S_\alpha\rangle=\Sigma_\sigma S_\sigma^\alpha/k$ for a group of k localized states represented by σ within some energy range. The plot of $\langle S_\alpha\rangle$ in Fig. 5 shows that even this averaged measure reflects the trend observed in the stability oscillations of the channel orbit in the vicinity of $\alpha=90.0$, though the exact minimum of the entropy seems to be slightly removed from this point of bifurcation. A tentative explanation of this is provided in the observation that exactly at the point of bifurcation the region around the channel orbit is locally flat, while just after the bifurcation, even though the channel orbit has lost stability and has become hyperbolic, there are two neighboring newly created stable orbits that provide the region with enhanced overall stability. Thus an initial wave packet launched in the channel would spread out more slowly immediately after the bifurcation, thereby enabling sharper localization. Moving away from the point of bifurcation the stable orbits move away from the channel orbit and thus their stability is of no consequence and the wave functions relatively delocalize. Further evidence of entropy oscillations is seen in the perturbed oscillator system we present below.

In order to check the validity of our finding on other similar systems, we studied the perturbed oscillator, given by the potential $V(x,y)=x^2/2+y^2/2+\beta x^2y^2$, where β is the parameter; we have taken $m=1$ and $\beta=0.1$ below. This is a nonhomogeneous system and the scaled parameter is $\epsilon=\beta E$, where E is the energy of the system. The advantage of nonhomogeneity in this system is that now the stability oscillations of the channel orbit are with respect to ϵ and hence with a fixed value of β we can study how the wave-function localization is affected by these oscillations.

The results presented below pertain to the perturbed oscillator, unless otherwise specified. The structure of channel localized wave functions in the unperturbed space has a generic pattern similar to Fig. 1(b) for the case of the coupled quartic oscillator. The approximate exponential falloff from the principal peak for three states, with different scaled pa-

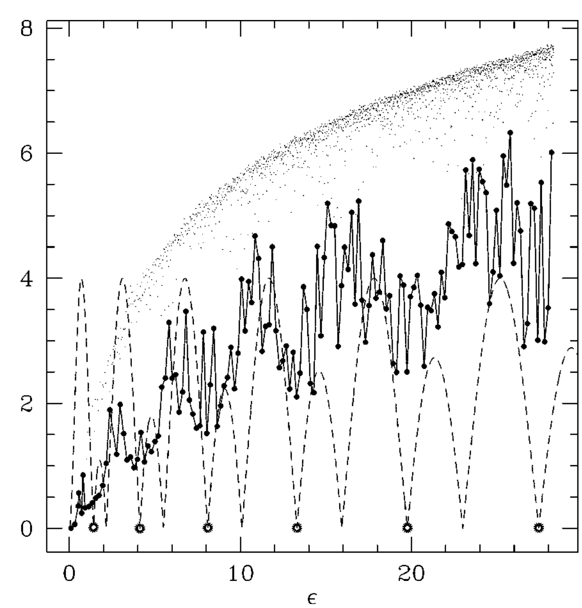

FIG. 7. Entropy of the first 2000 states of the perturbed oscillator. The solid line connects the channel localized states only, while the dashed line is $|\mathrm{Tr}[J(\epsilon)]-2|$. See the text for details. (The scale of the stability curve coincides with the entropy scale shown only at zero.)

rameters ϵ=8.03, 19.73, and 10.65 corresponding to stable and unstable motion of the channel orbit, is shown in Fig. 6. Shown is a plot of $\ln(|a_{m,j}|)$ against j, illustrating that falloff is more exponential when the orbit is stable. Since the scaled parameter ϵ is a function of the energy E, a global picture of the stability of the channel orbit and its influence on the degree of localization, as measured by the information entropy, can be obtained in this system. In Fig. 7 the quantity $|\mathrm{Tr}[J(\epsilon)]-2|$, an indicator of the stability of the channel orbit, and the information entropy for the first 2000 states are plotted against ϵ (here the monodromy matrix is for the full Poincaré map). The tiny dots represent the entropies of states that are not channel localized and they roughly follow random matrix theory predictions [11]. The entropy of channel localized states, however, show remarkable oscillations that strongly correlate with the stability oscillations of the channel orbit.

We note from Fig. 7 that the open circles corresponding to $\mathrm{Tr}[J(\epsilon)]=2$ are points of pitchfork bifurcation at which channel orbit loses stability, which correlate strongly with the entropy minima of the channel states. We may note that when the orbit gains stability the entropy is not a minimum, although Gutzwiller's trace formula breaks down here as well and Berry's scarring amplitude formula [13] may diverge. Thus the entropy minima must also have to do with the local structure around the periodic orbit and not depend on only the stability matrix of the scarring orbit.

The study of the simplest scarred states, the channel localized ones, have thus shown interesting exponential localization properties as well as strong correlation of the degree of localization with the local structure of the scarring orbit including its stability. Other states that share some of these properties are being actively studied. The scarred state noted in the experiment in Ref. [4] also has been shown to obey an underlying approximate WKB-like eigenvalue formula [4] and it may be possible to experimentally observe some of the phenomena noted in this paper either by such experiments or possible microwave cavity experiments [18].

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