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Liénard systems, limit cycles, and Melnikov theory

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Liénard systems constitute a general class of two-dimensional autonomous systems, among which the van der Pol equation is found. Recently Giacomini and Neukirch [Phys. Rev. E **57**, 3809 (1997)] introduced a sequence of polynomials whose roots are related to the number and location of limit cycles of Liénard systems. We show that in the limit of these sequences, the same information is given by a polynomial which Melnikov theory associates with a given Liénard system, and discuss the relationship existing among them. [S1063-651X(98)00201-3]

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I. INTRODUCTION

The interest of limit cycles, as isolated periodic orbits, is ubiquitous in the natural sciences and technology. The general problem of finding the number of limit cycles for a specific dynamical system is a rather complicated problem, which has some connections to the already unsolved Hilbert's 16th problem. However, many results are well known for some specific systems. In particular, there exists a theory which establishes a relationship between the number of limit cycles in terms of the zeros of a function, which is the Melnikov function for subharmonics. Among the differential systems possessing limit cycles, Liénard systems, which may be written as

$$\ddot{x}+f(x)\dot{x}+g(x)=0, \tag{1}$$

are well known. They were first studied by Liénard in the context of self-sustained oscillations. The applications of these dynamical systems to natural sciences is enormous. A special case of a Liénard system is the van der Pol equation

$$\ddot{x}+\varepsilon(x^2-1)\dot{x}+x=0, \tag{2}$$

which is probably the best known example of a system possessing a limit cycle. Equation (1) may be written as a two-dimensional autonomous dynamical system in the following ways:

$$\dot{x}=y-\varepsilon F(x),$$

$$\dot{y}=-g(x), \tag{3}$$

where $F(x)=\int_0^x f(\tau)d\tau$.

Liénard systems have received special attention recently by Giacomini and Neukirch (GN) [1]. They introduced a sequence of polynomials associated to a Liénard system, with the particularity that the number and position of the limit cycles is given by these polynomials. For a given Liénard system, the subharmonic Melnikov theory [2,3] provides a polynomial whose zeros give information about the number of limit cycles and their approximate radii. Here we address the problem of the number of limit cycles as viewed from the GN polynomials and from Melnikov theory, and discuss the relationship among them. The main observation is that the limit of the roots of the sequence of the GN polynomials converges to the roots of the Melnikov polynomials.

II. LIMIT CYCLES AND MELNIKOV THEORY

The main idea is that the number, positions, and multiplicities of the limit cycles that bifurcate under perturbations are related to the number, positions, and multiplicities of the zeros of the subharmonic Melnikov function for the system. Melnikov analyses for perturbed systems have occurred in numerous works [3,4] since the pioneering work of Poincaré [5], mostly for the bifurcation of homoclinic and heteroclinic orbits. A Melnikov theory of subharmonics and their bifurcations in forced oscillations was recently considered by Yagasaki [6]. The theory associates a polynomial to a given dynamical system. With the previous perspective in mind, a global bifurcation problem is thus reduced to a computational problem in terms of polynomials. In short, a Melnikov polynomial gives the appropriate information about the number and location of limit cycles.

A. General theory

A general equation of the type

$$\dot{x}=f(x)+\varepsilon g(x,\varepsilon,\mu) \tag{4}$$

is considered, where $f(x)$ and $g(x,\varepsilon,\mu)$ are analytical in R^2, and $\varepsilon \ll 1$, $x\in R^2$ and $\mu \in R^n$. The theory assumes that the unperturbed system, i.e., $\varepsilon=0$, has a one-parameter family of periodic orbits $\Gamma_\alpha:\gamma_\alpha(t), \alpha\in(0,\infty)$ and period T_α.

Definition. *The Melnikov function for system (4) along the cycle $\Gamma_\alpha:\gamma_\alpha(t)$ of period T_α of the unperturbed system is given by*

$$M(\alpha,\mu)=\int_0^{T_\alpha} e^{-\int_0^t \nabla\cdot f(\gamma_\alpha(s))ds} f\wedge g(\gamma_\alpha(t),0,\mu)dt, \tag{5}$$

where the wedge product of two vectors $x,y\in R^2$ is defined as $x\wedge y=x_1y_2-y_1x_2$.

With the previous definition and assuming the existence of periodic orbits for the unperturbed system, the following theorems are proved in [2].

Theorem 2.1. *If there exists an $\alpha_0 \in I$ and a $\mu_0 \in R^n$ such that*

$$M(\alpha_0,\mu_0)=0 \quad \text{and} \quad M_\alpha(\alpha_0,\mu_0)\neq 0,$$

then for all sufficiently small $\varepsilon\neq 0$, system (4) has a unique hyperbolic limit cycle in an $O(\varepsilon)$ neighborhood of Γ_{α_0}. Furthermore if $M(\alpha_0,\mu_0)\neq 0$, then for all sufficiently small $\varepsilon \neq 0$, system (4) has no cycle in an $O(\epsilon)$ neighborhood of Γ_{α_0}.

Theorem 2.2. *If the equation $M(\alpha_0,\mu_0)=0$ has exactly k solutions $\alpha_{1,}\alpha_{2,\ldots,}\alpha_k \in I$ with $M_\alpha(\alpha_0,\mu_0)\neq 0$ for $j=1,\ldots,k$, then for all sufficiently small $\varepsilon\neq 0$, exactly k one-parameter families of hyperbolic limit cycles of system (4) bifurcate from the period annulus a of the unperturbed system at cycles through the points $\alpha_{1,}\alpha_{2,\ldots,}\alpha_k$ on the Poincaré section Σ normal to the one-parameter family of periodic orbits Γ_α. If $M(\alpha_0,\mu_0)$ has no zeros for $\alpha \in I$, then no one-parameter families of limit cycles of system (4) bifurcate from a for $\varepsilon\neq 0$.*

Theorem 2.3. *Under the same assumptions as the preceding theorems, if there exists an $\alpha_0 \in I$ and a $\mu_0 \in R^n$ such that*

$$M(\alpha_0,\mu_0)=M_\alpha(\alpha_0,\mu_0)=\cdots=M_\alpha^{(m-1)}(\alpha_0,\mu_0)=0,$$

$$M_\alpha^{(m)}(\alpha_0,\mu_0)=0 \quad \text{and} \quad M_{\mu_j}(\alpha_0,\mu_0)\neq 0$$

for some $j=1,\ldots,n$, then, for all sufficiently small ε, there is an analytic function $\mu(\varepsilon)=\mu_0+O(\varepsilon)$ such that for small $\varepsilon\neq 0$ system (4) has a unique limit cycle of multiplicity m in an $O(\varepsilon)$ neighborhood of the cycle of Γ_{α_0}.

B. Melnikov theory applied to Liénard systems

In this section we will consider Liénard systems of the types

$$\dot{x}=y-\varepsilon F(x),$$

$$\dot{y}=-g(x), \tag{6}$$

where $g(x)=x$ and $F(x)$ is a polynomial of odd degree. Notice that when $F(x)=(x^3/3)-x$, we have the van der Pol equation. For the Liénard system we have a bifurcation from the center $f(x)=(y,-x)$ and consequently $\nabla\cdot f(x)=0$, so that the Melnikov function is given by

$$M(\alpha,\mu)=\int_0^{T_\alpha} f\wedge g(\gamma_\alpha(t),0,\mu)dt. \tag{7}$$

The unperturbed system has a one-parameter family of periodic orbits $\Gamma_\alpha:\gamma_\alpha(t)=(\alpha\ \cos t,\alpha \sin t),\alpha\in(0,\infty)$, and $T_\alpha=2\pi$, $\forall\alpha$.

The perturbed Liénard system for a $F(x)$ of odd degree, containing all powers in x, may be written as

$$\dot{x}=y-\varepsilon(a_1x+a_2x^2+a_3x^3+\cdots+a_{2n+1}x^{2n+1})$$

$$\dot{y}=-x, \tag{8}$$

which for $\mu=(a_1,\ldots,a_{2n+1})$ has the form of system (4). The Melnikov function for this system is

$$M(\alpha,\mu)=-\int_0^{2\pi}\{a_1x(t)+a_2x^2(t)+a_3x^3(t)+\cdots + a_{2n+1}x^{2n+1}(t)\}x(t)dt. \tag{9}$$

After substituting $x(t)$ from the periodic orbit $\gamma_\alpha(t)$, we obtain

$$M(\alpha,\mu)=-\int_0^{2\pi}\{a_1\alpha^2\cos^2t+a_2\alpha^3\cos^3t+a_3\alpha^4\cos^4t+\cdots + a_{2n+1}\alpha^{2n+2}\cos^{2n+2}t\}dt. \tag{10}$$

Clearly all odd terms are zero, and the even terms give the result

$$\int_0^{2\pi}\cos^{2n+2}t\,dt=\binom{2n+2}{n+1}\frac{a_{2n+1}}{2^{2n+1}}\pi, \tag{11}$$

from which, as a consequence, it is derived that

$$M(\alpha,\mu)=-2\pi\alpha^2\left\{\frac{a_1}{2}+\frac{3}{8}a_3\alpha^2+\cdots + \binom{2n+2}{n+1}\frac{a_{2n+1}}{2^{2n+2}}\alpha^{2n}\right\}. \tag{12}$$

Based on the preceding theorems 2.1–2.3, and the result obtained for the Melnikov function associated with the Liénard system, the following result due to Perko and co-workers provides the necessary information about the number of limit cycles and their radii [2,3].

Proposition. *The Liénard system, Eq. (8), for sufficiently small $\epsilon\neq 0$, has at most n limit cycles. Furthermore it has exactly n hyperbolic limit cycles asymptotic to circles of radii r_j, $j=1,2,\ldots,n$ as $\varepsilon\to 0$ if and only if the nth degree polynomial in r^2,*

$$P(r^2,n)=\frac{a_1}{2}+\frac{3}{8}a_3r^2+\cdots+\binom{2n+2}{n+1}\frac{a_{2n+1}}{2^{2n+2}}r^{2n},$$

has n positive roots $r^2=r_j^2$, $j=1,\ldots,n$.

This result allows us to construct Liénard systems with the exact number of limit cycles and radii we wish. All we need to do is to write a polynomial with the chosen roots, and afterwards find the coefficients a_i associated to the polynomial $P(r^2,n)$. In the following we will call this the *Melnikov polynomial*, since it is related to the Melnikov function through the expression

$$P(r^2,n)=-\frac{1}{2\pi r^2}M(r,\mu). \tag{13}$$

C. Applications to some specific Liénard systems

Using the previous results, we compute all the cases of Liénard systems considered by GN [1].

(1) The first case is for $F(x)=a_1x+a_3x^3$. The polynomial $P(r^2,1)$ we obtain is given by

$$P(r^2,1)=\frac{a_1}{2}+\frac{3a_3}{8}r^2. \tag{14}$$

This polynomial has one solution given by $r^2=-4a_1/3a_3$. Then if $a_1a_3<0$ the polynomial $P(r^2,1)$ has a unique positive root and hence a unique limit cycle, and if $a_1a_3>0$ it has no real roots and hence no limit cycles. This case has been also considered in Ref. [7].

(2) van der Pol equation, with $F(x)=(x^3/3)-x$. Then $a_1=-1$ and $a_3=\frac{1}{3}$. This is a subcase of the previous one with the property that $a_1a_3=-\frac{1}{3}<0$, and consequently the corresponding $P(r^2,1)$ polynomial has a unique positive root, and hence a unique limit cycle of radius $r=2$. GN [1] obtained an approximate numerical value of 2.001 for the maximum value of x, using their method.

(3) In this case we consider $F(x)=0.8x-\frac{4}{3}x^3+0.32x^5$. Then $a_1=0.8$, $a_3=-\frac{4}{3}$, and $a_5=0.32$. We obtain the Melnikov polynomial

$$P(r^2,2)=0.4-0.5r^2+0.1r^4. \tag{15}$$

The polynomial has two different roots in r^2, with solutions of 4 and 1, respectively, providing as positive roots the radii 2 and 1. The results obtained by GN [1] were 1.9992 and 1.0034, respectively, for the maximum value of x.

(4) The following case is $F(x)=x^5-\mu x^3+x$. Then $a_1=1$, $a_3=-\mu$, and $a_5=1$. We obtain the polynomial

$$P(r^2,2)=\frac{1}{2}-\frac{3\mu}{8}r^2+\frac{5}{16}r^4.$$

The polynomial has two different roots in r^2 given by $(3\mu/5)\pm\frac{1}{5}\sqrt{9\mu^2-40}$. If $9\mu^2-40>0$, i.e., if $\mu>\sqrt{40/9}=2.1082$, there are two roots and consequently the system has two limit cycles. This is the limit value μ^*, which apparently GN get in their calculations shown in Table 3 of Ref. [1]. Figure 1 shows the two limit cycles of the Liénard system with $\mu=2.5$, for $\varepsilon=0.01$.

(5) Finally, for $F(x)=x(x^2-1.6^2)(x^2-4)(x^2-9)$. Expanding the terms in powers of x, $F(x)=x^7-15.56x^5+69.28x^3-92.16x$ is obtained, from where we easily obtain the a_i coefficients. Then the Melnikov polynomial is given by

$$P(r^2,3)=\frac{-92.16}{2}-\frac{3\times 69.28}{8}r^2-\frac{5\times 15.56}{16}r^4+\frac{35}{128}r^6. \tag{16}$$

This has a unique positive root in r^2 given by $r^2=9.9134$, and then a unique limit cycle of radius $r\approx 3.1486$. Figure 2 shows this unique limit cycle for $\varepsilon=0.01$.

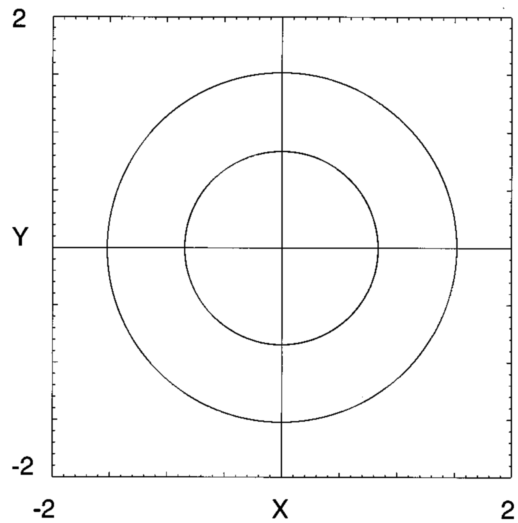

FIG. 1. This figure shows the two limit cycles which possess the Liénard system with $F(x)=x-\mu x^3+x^5$, for $\mu=2.5$ and $\varepsilon=0.01$.

III. RELATIONSHIP BETWEEN MELNIKOV AND GN POLYNOMIALS

GN [1] introduced a sequence of polynomials $h_n(x,y)$ in x and y, which is associated to every Liénard system of the type given by Eq. (6), with the following expression:

$$h_n(x,y)=y^n+g_{n-1,n}(x)y^{n-1}+\cdots+g_{1,n}(x)y+g_{0,n}(x), \tag{17}$$

and where $g_{j,n}(x)$, $0\leqslant j\leqslant n-1$ are polynomials in x. The time derivative of the polynomic function $h_n(x,y)$ is given by

$$\dot{h}_n(x,y)=\frac{\partial h_n(x,y)}{\partial x}(y-F(x))-\frac{\partial h_n(x,y)}{\partial y}x, \tag{18}$$

where

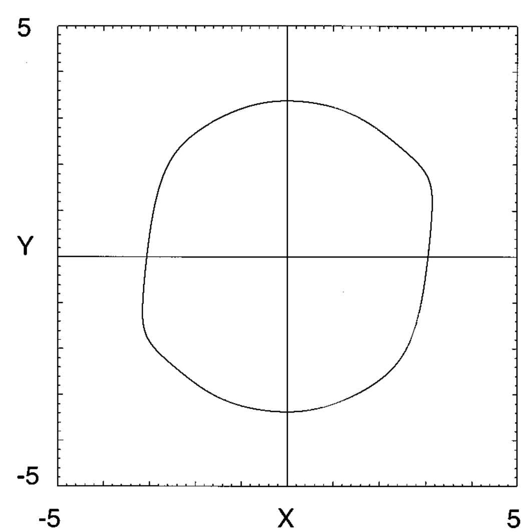

FIG. 2. In this figure the unique limit cycle of the Liénard system with a polynomial function $F(x)=x(x^2-1.6^2)(x^2-4)(x^2-9)$ for $\varepsilon=0.01$ is shown.

$$\frac{\partial h_n(x,y)}{\partial x} = g'_{n-1,n}(x)y^{n-1} + \cdots + g'_{1,n}(x)y + g'_{0,n}(x) \tag{19}$$

and

$$\frac{\partial h_n(x,y)}{\partial y} = ny^{n-1} + (n-1)g_{n-1,n}(x)y^{n-2} + \cdots + g_{1,n}(x), \tag{20}$$

where $g'_{j,n}(x), 0 \leqslant j \leqslant n-1$, stands for the derivative with respect to x of the polynomial $g_{j,n}(x)$. Introducing Eqs. (19) and (20) into Eq. (18), the following relation holds:

$$\begin{aligned}\dot{h}_n(x,y) = {} & g'_{n-1,n}(x)y^n + (g'_{n-2,n}(x) \\ & - F(x)g'_{n-1,n}(x) - nx)y^{n-1} + \cdots + (g'_{0,n}(x) \\ & - F(x)g'_{1,n}(x) - 2xg_{2,n}(x))y \\ & + g'_{0,n}(x)F(x) - xg_{1,n}(x).\end{aligned} \tag{21}$$

Imposing that every coefficient of y^n be zero in $\dot{h}_n(x,y)$ gives rise to a set of ordinary differential equations for the polynomials $g_{j,n}(x), 0 \leqslant j \leqslant n-1$. Moreover, the symmetry $(x,y) \to (-x,-y)$, which is shared by $h_n(x,y)$, imposes certain conditions for the polynomials $g_{j,n}(x)$, as well. The result of this construction defines the polynomials $R_n(x) = \dot{h}_n(x,y)$, which may be written as

$$R_n(x) = -g'_{0,n}(x)F(x) - xg_{1,n}(x). \tag{22}$$

Application to the van der Pol equation

Following GN [1], for the van der Pol equation, the following polynomial is introduced:

$$h_2(x,y) = y^2 + g_{1,2}(x)y + g_{0,2}(x). \tag{23}$$

The symmetry $(x,y) \to (-x,-y)$ of the limit cycle solutions imposes that $g_{1,2}(x) = 0$. On the other hand, the condition that the coefficients of y^2 and y be zero in $\dot{h}_2(x,y)$, implies that $g_{0,2}(x) = x^2 + k$. Consequently the following polynomial is obtained:

$$h_2(x,y) = y^2 + x^2 + k. \tag{24}$$

For $\varepsilon = 0$, system (6) is Hamiltonian, with Hamiltonian function $H(x,y) = (x^2/2) + (y^2/2)$. This shows that the polynomial $h_2(x,y)$ is physically related to the energy. For the time derivative we have

$$\dot{h}_2(x,y) = R_2(x) = -g'_{0,2}(x)F(x) - xg'_{1,2}(x) = -2xF(x). \tag{25}$$

The root of this polynomial $R_2(x)$ is exactly $\sqrt{3}$. For the iteration $n = 4$, it is derived from Eq. (22) that

$$R_4(x) = -g'_{0,4}(x)F(x) - xg_{1,4}(x), \tag{26}$$

where $g'_{0,4}(x) = 4x^3 - 4x[F(x)]^2$ and $g_{1,4}(x) = \frac{4}{3}x^3\{(x^2/5) - 1\}$. Consequently, we obtain a polynomial of degree 10 whose terms are

$$R_4(x) = \tfrac{4}{27}x^{10} - \tfrac{4}{3}x^8 + \tfrac{28}{5}x^6 - \tfrac{28}{3}x^4. \tag{27}$$

This polynomial has a unique positive root which is given by $r = 1.8248 > \sqrt{3}$. GN gave results for the roots of $R_n(x)$ until $n = 20$, and it seems that they converge toward 2, *which is precisely the root of the Melnikov polynomial* $P(r^2,1)$. On the one hand, the root of the odd multiplicity of $g_{1,4}(x)$ is given by $\sqrt{5}$. The roots of $g_{1,n}(x)$ for $n \geqslant 4$ [1] seem to converge toward 2 as n increases, as well.

Another interesting point of this research is the integration along the limit cycle, since the integrand should change sign in the region where limit cycles are expected. We can integrate $\dot{h}_n(x,y) = R_n(x)$ along a limit cycle $\Gamma_\alpha : \gamma_\alpha(t)$ of period $T^\alpha = 2\pi$. For the first iteration, the result is

$$\int_0^{2\pi} \dot{h}_2(x,y)dt = \int_0^{2\pi} R_2(x(t))dt = 2\alpha^2\pi\left\{1 - \frac{\alpha^2}{4}\right\}. \tag{28}$$

The result of the integral is another polynomial, whose root gives precisely the radius of the van der Pol limit cycle $\alpha = 2$. Note that according to the definition of $R_2(x)$ from Eq. (25) the last integral coincides with twice the Melnikov function.

We can repeat the calculations with higher order polynomials $R_n(x)$ and the result is

$$\begin{aligned}\int_0^{2\pi} \dot{h}_4(x,y)dt &= \int_0^{2\pi} R_4(x(t))dt \\ &= \alpha^4\pi\{-\tfrac{28}{4} + \tfrac{28}{8}\alpha^2 - \tfrac{35}{48}\alpha^4 + \tfrac{7}{96}\alpha^6\}.\end{aligned} \tag{29}$$

This new polynomial, obtained after the integration, has a unique positive root $\alpha = 2$, which is precisely the unique root obtained through the Melnikov theory, and which coincides with the radius of the limit cycle. However the calculations for $R_n(x), n \geqslant 6$ do not seem to give the same result. There is no similar result for the $g_{j,n}(x)$ polynomials.

IV. DISCUSSION AND CONCLUSIONS

We have analyzed all the Liénard cases calculating the Melnikov polynomial and comparing with the calculations carried out by GN [1]. Based on that, the following observations are established

Observation 1. *For each Liénard system we obtain the polynomial $P(r^2,n)$, which is associated with the subharmonic Melnikov function. For the van der Pol equation, this polynomial has a unique positive root of value $r = 2$. Associated to the van der Pol equation we have a polynomial $R_n(x)$ of even degree $3(n-2)+4$, $n = 2,4,6,\ldots$. This polynomial has a unique positive root $\alpha_n < \alpha$, starting from $\alpha_2 = \sqrt{3}$. Accordingly, as n increases, $R_n(x)$ increases its degree and the unique root it possesses α_n seems to converge to the root α of the Melnikov polynomial.*

Since the Melnikov polynomial $P(r^2,n)$ provides the same information as the GN polynomials $R_n(x)$, it is derived that the number of limit cycles of the Liénard system to which they are associated is less than or equal to the number of roots in r^2 of $P(r^2,n)$. Moreover, since $P(r^2,n)$ is of degree n in r^2, its roots come in pairs, which at least is consistent with the GN conjecture [1].

Observation 2. *Analogously for each system we obtain the polynomial $g_{1,n}(x)$, which by construction is a polynomial of odd degree $3(n-3)+2$, $n=4,6,\ldots$. According to GN [1] the unique root it possesses is $\beta_n>\alpha$, whose starting value is $\beta_4=\sqrt{5}$, for the van der Pol system. As n increases the degree of $g_{1,n}(x)$ increases, and it is conjectured that it has a unique positive root β_n, which converges toward the root of the Melnikov potential α.*

Finally we may conclude all this with the following conjecture

Conjecture. *For a given Liénard system, there are associated a Melnikov polynomial $P(r^2,n)$, and two sequence of polynomials $R_n(x)$, $n=2,4,6,\ldots$ and $g_{1,n}(x),n=4,6,8,\ldots$. For a fixed given value of $n\geqslant 4$, the three polynomials possess a unique positive root given by α,α_n, and β_n respectively, $\alpha_n<\alpha<\beta_n$, and with the property that as n increases $\alpha_n\to\alpha$ and $\beta_n\to\alpha$.*

Summarizing, a comparative analysis of Liénard systems using different methods is carried out. One of the main conclusions is that, in the limit, the GN sequence of polynomials provides the same information concerning the number of limit cycles and their radii as the polynomial associated with the Liénard system through the subharmonic Melnikov theory. The Melnikov polynomial is of nth degree in r^2, and its number of zeros is less than or equal to the number of limit cycles. Furthermore, since it is of odd degree, the roots come in pairs. This gives an explanation for the GN conjecture relative to the number and position of limit cycles. Finally, a relationship among them is conjectured in the limit as n increases.

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